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NEGATIVE SOLUTION FOR THE SYSTEM OF THE NONLINEAR WAVE EQUATIONS WITH CRITICAL GROWTH

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Abstract. We show the existence of a negative solution for the system of the following nonlinear wave equations with critical growth, under Dirichlet boundary condition and periodic condition
$u_{tt} - u_{xx} = au + bv + \frac{2\alpha}{\alpha+\beta} u_+^{\alpha-1} v_+^{\beta} + s\phi_{00} + f,$
$v_{tt} - v_{xx} = cu + dv + \frac{2\beta}{\alpha+\beta} u_+^{\alpha} v_+^{\beta-1} + t\phi_{00} + g,$
where α, $\beta > 1$ are real constants, $u_+ = \max\{u, 0\}$, s, $t \in R$, ϕ_{00} is the eigenfunction corresponding to the positive eigenvalue λ_{00} of the wave operator and f, g are π-periodic, even in x and t and bounded functions.

1. Introduction and main result

In this paper we show the existence of the negative solution for the system of the following nonlinear wave equations with critical growth

$$
(1.1) \qquad \begin{cases}
u_{tt} - u_{xx} = au + bv + \dfrac{2\alpha}{\alpha+\beta} u_+^{\alpha-1} v_+^{\beta} + s\phi_{00} + f, \\
v_{tt} - v_{xx} = cu + dv + \dfrac{2\beta}{\alpha+\beta} u_+^{\alpha} v_+^{\beta-1} + t\phi_{00} + g, \\
u(\pm\dfrac{\pi}{2}, t) = v(\pm\dfrac{\pi}{2}, t) = 0, \\
u(x, t+\pi) = u(x, t) = u(-x, t) = u(x, -t), \\
v(x, t+\pi) = v(x, t) = v(-x, t) = v(x, -t),
\end{cases}
$$

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where α, $\beta > 1$ are real constants, $u_+ = \max\{u, 0\}$, s, $t \in R$, ϕ_{00} is the eigenfunction corresponding to the positive eigenvalue $\lambda_{00} = 1$ of the eigenvalue problem $u_{tt} - u_{xx} = \lambda_{mn} u$ with $u(\pm\frac{\pi}{2}, t) = 0$, $u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t)$. We assume that f, g are π-periodic, even in x and t and bounded functions in $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\phi_{00} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\phi_{00} = 0$.

In [6] Lazer and McKenna point out that this kind of nonlinearity bu^+ can furnish a model to study travelling waves in suspension bridges. So the nonlinear equation with jumping nonlinearity have been extensively studied by many authors. For fourth elliptic equation Tarantello [11] , Micheletti and Pistoia [8,9] proved the existence of nontrivial solutions used degree theory and critical points theory separately. For one-dimensional case Lazer and McKenna [7] proved the existence of nontrivial solution by the global bifurcation method. For this jumping nonlinearity we are interest in the multiple nontrivial solutions of the equation.

System (1.1) can be rewritten by

$$(1.2)\quad \begin{cases} U_{tt} - U_{xx} = \nabla(\frac{1}{2}(AU, U) + \frac{2}{\alpha+\beta} u_+^{\alpha} v_+^{\beta}) + \binom{s}{t}\phi_{00} + \binom{f}{g} \\ U(\pm\frac{\pi}{2}, t) = 0, \\ U(x, t+\pi) = U(x,t) = U(-x,t) = U(x,-t), \end{cases}$$

where ∇ is the gradient operator, $U = \binom{u}{v}$, $U_{tt} - U_{xx} = \binom{u_{tt}-u_{xx}}{v_{tt}-v_{xx}}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(R)$. Let us denote r_1 and r_2 the eigenvalues of the matrix A when $a = b$. Let us define the Hilbert space spanned by eigenfunctions as follows:
The eigenvalue problem for $u(x,t)$,

$$u_{tt} - u_{xx} = \lambda u \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R,$$

$$u(\pm\frac{\pi}{2}, t) = 0, \qquad u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t)$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2 \qquad (m, n = 0, 1, 2, \ldots)$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m, n \geq 0)$ given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi} \cos(2n+1)x \qquad \text{for} \quad n \geq 0,$$

$$\phi_{mn} = \frac{2}{\pi} \cos 2mt \cdot \cos(2n+1)x \qquad \text{for} \quad m > 0, n \geq 0.$$

Let n be a fixed integer and define

$$\lambda_n^+ = \inf_m \{\lambda_{mn} : \lambda_{mn} > 0\} = 4n+1,$$

$$\lambda_n^- = \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -4n-3.$$

Letting $n \to \infty$, we obtain that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$. We can check easily that the eigenvalues in the interval (-15,9) are given by

$$\lambda_{32} = -11 < \lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) |\ u \text{ is even in } x \text{ and } t\}.$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in H_0.

Let us denote an element u, in H_0, by

$$u = \sum h_{mn} \phi_{mn}.$$

We define a Hilbert space H as follows

$$H = \{u \in H_0 : \ \sum |\lambda_{mn}| h_{mn}^2 < \infty\}.$$

Then this space is a Banach space with norm

$$\|u\|^2 = [\sum |\lambda_{mn} h_{mn}^2|]^{\frac{1}{2}}.$$

Let us set $E = H \times H$. We endow the Hilbert E with the norm

$$\|(u,v)\|_E^2 = \|u\|^2 + \|v\|^2.$$

We are looking for the weak solutions of (1.1) in H, that is, (u, v) satisfying the equation

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (u_{tt} - u_{xx}) z + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (v_{tt} - v_{xx}) w - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (A(u,v), (z,w))$$
$$- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\tfrac{2\alpha}{\alpha+\beta} u_+^{\alpha-1} v_+^{\beta} + s\phi_{00} + f] z - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\tfrac{2\beta}{\alpha+\beta} u_+^{\alpha} v_+^{\beta-1} + t\phi_{00} + g] w = 0$$

for all $(z, w) \in E$, where $u = \sum c_{mn} \phi_{mn}$, $v = \sum d_{mn} \phi_{mn}$ with $u_{tt} - u_{xx} = \sum \lambda_{mn} c_{mn} \phi_{mn} \in H$, $v_{tt} - v_{xx} = \sum \lambda_{mn} d_{mn} \phi_{mn} \in H$ i.e., with $\sum c_{mn}^2 \lambda_{mn}^2 < \infty$, $\sum d_{mn}^2 \lambda_{mn}^2 < \infty$, which implies u, $v \in H$.

Now we state the main result:

Theorem 1.1. *(Existence of a negative solution) Assume that*

$$\lambda_{p+10} < r_1, r_2 < \lambda_{p0} < \lambda_{10} = -3 < \lambda_{00} = 1 \text{ for } p > 1 \tag{1.3}$$

$$(\lambda_{mn} - a)(\lambda_{mn} - d) - bc \neq 0, \qquad \text{for all } m, n \text{ with } (m, n) \neq (0, 0), \tag{1.4}$$

$$b, c, \lambda_{00} - a, \lambda_{00} - d > 0. \tag{1.5}$$

Then, for each f, $g \in H$ such that f and g are π-periodic, even in x and t and bounded functions with $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\phi_{00} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\phi_{00} = 0$, the system (1.1) has a negative solution.

2. Proof of Theorem 1.1

We have some properties. Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

Lemma 2.1. *(i) $\|u\| \geq \|u\|_{L^2(Q)}$, where $\|u\|_{L^2(Q)}$ denotes the L^2 norm of u.*
(ii) $\|u\| = 0$ if and only if $\|u\|_{L^2(Q)} = 0$.
(iii) $u_{tt} - u_{xx} \in H$ implies $u \in H$.

Lemma 2.2. *Suppose that c is not an eigenvalue of L, $Lu = u_{tt} - u_{xx}$, and let $u \in H_0$. Then we have $(L - c)^{-1}u \in H$.*

Proof. When n is fixed, λ_n^+ and λ_n^- were defined in section 1:

$$\lambda_n^+ = 4n + 1,$$
$$\lambda_n^- = -4n - 3.$$

We see that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$ as $n \to \infty$. Hence the number of elements in the set $\{\lambda_{mn} : |\lambda_{mn}| < |c|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$u = \sum h_{mn}\phi_{mn}.$$

Then

$$(L - c)^{-1}u = \sum \frac{1}{\lambda_{mn} + c} h_{mn}\phi_{mn}.$$

Hence we have the inequality

$$\|(L - c)^{-1}u\| = \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn} + c)^2} h_{mn}^2 \leq C \sum h_{mn}^2$$

for some C, which means that

$$\|(L - c)^{-1}u\| \leq C_1\|u\|_{L^2(Q)}, \qquad C_1 = \sqrt{C}.$$

□

Lemma 2.3. *Assume that the conditions (1.3), (1.4) and (1.5) hold and f, $g \in H$ are π-periodic, even in x and t and bounded functions with $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\phi_{00} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\phi_{00} = 0$. Then the system*

$$(2.1) \qquad \begin{cases} u_{tt} - u_{xx} = au + bv + s\phi_{00}, \\ v_{tt} - v_{xx} = cu + dv + t\phi_{00}, \\ u(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) = v(x,t) = v(-x,t) = v(x,-t), \end{cases}$$

has a unique solution $(u_*, v_*) \in E$.

Proof. We note that (u_*, v_*) with

$$u_* = [\frac{b}{\lambda_{00} - a}(\frac{cs + t(\lambda_{00} - a)}{(\lambda_{00} - a)(\lambda_{00} - d) - bc}) + \frac{s}{\lambda_{00} - a}]\phi_{00},$$

$$v_* = [\frac{cs + t(\lambda_{00} - a)}{(\lambda_{00} - a)(\lambda_{00} - d) - bc}]\phi_{00}$$

is a solution of the system (2.1). □

Lemma 2.4. *Assume that the conditions (1.3), (1.4), (1.5) hold. Then the system*

$$(2.2) \qquad U_{tt} - U_{xx} = AU, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix} \in E,$$

$$U(\pm\frac{\pi}{2}, t) = 0,$$

$$U(x, t+\pi) = U(x,t) = U(-x,t) = U(x,-t)$$

has only a trivial solution $U(x,t) = \binom{0}{0}$.

Proof. We assume that there exists a nontrivial solution $U = (u, v) \in E$ of (2.2) of the form $u = \phi_{mn}$ and $v = \phi_{m'n'}$. The equation

$$L\begin{pmatrix} \phi_{mn} \\ \phi_{m'n'} \end{pmatrix} = A\begin{pmatrix} \phi_{mn} \\ \phi_{m'n'} \end{pmatrix}$$

is equivalent to the equation

$$\begin{pmatrix} \lambda_{mn}\phi_{mn} \\ \lambda_{m'n'}\phi_{m'n'} \end{pmatrix} = \begin{pmatrix} a\phi_{mn} + b\phi_{m'n'} \\ c\phi_{mn} + d\phi_{m'n'} \end{pmatrix}.$$

Thus when $mn \neq m'n'$, we have $\lambda_{mn} = a$, $b = 0$, $\lambda_{mn} = c$ and $d = 0$, which means that $(\lambda_{mn} - a)(\lambda_{mn} - c) - bd = 0$. When $mn = m'n'$, we have $\lambda_{mn} = a + b$ and $\lambda_{mn} = c + d$, which means that $(\lambda_{mn} - a)(\lambda_{mn} - c) - bd = 0$. These contradict the assumption (1.4). □

LEMMA 2.5. *Assume that the conditions (1.4) and (1.5) hold and $f, g \in H$ are π-periodic, even in x and t and bounded functions with $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\phi_{00} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\phi_{00} = 0$, and $\alpha > 0$ be given. Then there exists $R_0 > 0$ (depending on α) such that for all r_1 and r_2 with $\lambda_{p+10} + \alpha \leq r_1, r_2 \leq \lambda_{p0} - \alpha < \lambda_{10} = -3 < \lambda_{00} = 1$, $p > 1$, the solutions $U = \binom{u}{v}$ of the equation*

$$U_{tt} - U_{xx} = AU + \begin{pmatrix} f \\ g \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix},$$

$$U(\pm\frac{\pi}{2}, t) = 0,$$

$$U(x, t + \pi) = U(x, t) = U(-x, t) = U(x, -t)$$

satisfy $\|U\| \leq R_0$.

Proof. Let $LU = U_{tt} - U_{xx}$, $U = \binom{u}{v}$. We suppose that the conclusion does not hold. Then there exists a sequence $(a_n, b_n, c_n, d_n, r_{1n}, r_{2n}, u_n, v_n, U_n)$, $U_n = \binom{u_n}{v_n}$ such that the eigenvalues (r_{1n}), (r_{2n}) of the matrix $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ lie in the interval $[\lambda_{p+10} + \alpha, \lambda_{p0} - \alpha]$, $\|U_n\| \to \infty$ and

$$U_n = L^{-1}(A_n U_n + \begin{pmatrix} s\phi_{00} \\ t\phi_{00} \end{pmatrix}).$$

We note that $W_n = \frac{1}{\|U_n\|} U_n$ satisfy the equation

$$W_n = L^{-1}(A_n W_n + \begin{pmatrix} s\phi_{00} \\ t\phi_{00} \end{pmatrix} \frac{1}{\|U_n\|}).$$

Now L^{-1} is a compact operator. Therefore we may assume that $W_n \to W_0$ and $a_n \to a_0$, $b_n \to b_0$, $c_n \to c_0$, $d_n \to d_0$, $(r_{1n}) \to r_{10}$, $(r_{2n}) \to r_{20}$, where r_{10} and r_{20} are the eigenvalues of the matrix $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ and r_{10}, $r_{20} \in (\lambda_{p+10,\lambda_{p0}})$. Since $\|W_n\| = 1$ it follows that $\|W_0\| = 1$ and

$$W_0 = L^{-1}(A_0 W_0).$$

This contradicts Lemma 2.4 and proves the lemma. □

LEMMA 2.6. *Assume that the conditions (1.3), (1.4) and (1.5) hold and f, $g \in H$ are π-periodic, even in x and t and bounded functions with $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\phi_{00} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\phi_{00} = 0$. Then the system*

$$\tag{2.3} \begin{cases} u_{tt} - u_{xx} = au + bv + f, \\ v_{tt} - v_{xx} = cu + dv + g, \\ u(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) = u(x, t) = u(-x, t) = u(x, -t), \\ v(x, t+\pi) = v(x, t) = v(-x, t) = v(x, -t) \end{cases}$$

has a unique solution $(\check{u}, \check{v}) \in E$.

Proof. Let $\delta > 0$ and $\delta > \max\{b, c\}$. Let us consider the modified system

$$\tag{2.4} \begin{cases} u_{tt} - u_{xx} - au - bv + \lambda_{00}u + \delta u = f, \\ v_{tt} - v_{xx} - cu - dv + \lambda_{00}v + \delta v = g, \\ u(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) = u(x, t) = u(-x, t) = u(x, -t), \\ v(x, t+\pi) = v(x, t) = v(-x, t) = v(x, -t). \end{cases}$$

Let us set

$$L_\delta U = U_{tt} - AU + \lambda_{00}U + \delta U, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then system (2.4) is invertible. Thus there exists a inverse operator $L_\delta^{-1} : L^2(Q) \times L^2(Q) \to E$ which is a linear and compact operator such that $(u, v) = L_\delta^{-1}(f, g)$. Thus we have that if (u, v) is a solution of (2.3) if and only if

$$\tag{2.5} (u, v) = L_\delta^{-1}((f, g) + \lambda_{00}(u, v) + \delta(u, v))$$

Thus we have

$$(I - (\lambda_{00} + \delta)L_\delta^{-1})((f, g) + \lambda_{00}(u, v) + \delta(u, v)) = (f, g).$$

By the conditions (1.3) and (1.4), $\frac{1}{\lambda_{00}+\delta} \notin \sigma(L_\delta^{-1})$. Since L_δ^{-1} is a compact operator, system (2.5) has a unique solution, thus system (2.4) has a unique solution. □

Proof of Theorem 1.1 By Lemma 2.3 and Lemma 2.6, $(u_*+\check{u}, v_*+\check{v})$ is a solution of the system

$$
(2.6) \qquad \begin{cases} u_{tt} - u_{xx} = au + bv + s\phi_{00} + f, \\ v_{tt} - v_{xx} = cu + dv + t\phi_{00} + g, \\ u(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) = v(x,t) = v(-x,t) = v(x,-t), \end{cases}
$$

where $u_* = [\frac{b}{\lambda_{00}-a}(\frac{cs+t(\lambda_{00}-a)}{(\lambda_{00}-a)(\lambda_{00}-d)-bc}) + \frac{s}{\lambda_{00}-a}]\phi_{00}$ and $v_* = [\frac{cs+t(\lambda_{00}-a)}{(\lambda_{00}-a)(\lambda_{00}-d)-bc}]\phi_{00}$. Here $u_* > 0, v_* > 0$ and $u_*^+ = v_*^+ = 0$. Hence $U(x,t) = \binom{u_*}{v_*}$ is a negative solution of (1.1).

Therefore there exists (s_1, t_1) with $s_1 < 0$ and $t_1 < 0$ such that $u_*+\check{u} < 0$ and $v_* + \check{v} < 0$ is a negative solution of (1.1) for $s < s_1$ and $t < t_1$.

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