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# A BOUNDARY CONTROL PROBLEM FOR THE TIME-DEPENDENT 2D NAVIER–STOKES EQUATIONS

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Abstract. In this paper, a boundary control problem for a flow governed by the time-dependent two dimensional Navier-Stokes equations is considered. We derive a mathematical formulation and a relevant process for an appropriate control along the part of the boundary to minimize the drag due to the flow.

After showing the existence of an optimal solution, the first order optimality conditions are derived. The strict differentiability of the state solution in regard to the control parameter shall be exposed rigorously, and the necessary conditions along with the system for the optimal solution shall be deduced in conjunction with the evaluation of the first order Gateaux derivative to the performance functional.

## 1. Introduction

In this paper, we are concerned with a boundary control problem for a flow which is governed by the time-dependent two dimensional Navier-Stokes equations. Let us describe the boundary control problem for a time-dependent Navier-Stokes system that models the drag minimization in a flow domain. For practical purposes, we assume that the boundary $\Gamma$ of the flow domain $\Omega$ is composed of two disjoint parts ; the homogeneous part $\Gamma_0$ and the non-homogeneous control part $\Gamma_c$ such that $\Gamma = \Gamma_0 \cup \Gamma_c$ .

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We consider a two dimensional flow over the time interval $[0, T]$ in the physical flow domain $\Omega$ with the control effected over the part $\Gamma_c$:

$$\frac{d\vec{u}}{dt} - \nu\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \vec{f} \quad \text{in } (0,T)\times\Omega\,, \tag{1.1}$$

$$\nabla\cdot\vec{u} = 0 \quad \text{in } (0,T)\times\Omega \tag{1.2}$$

along with the Dirichlet boundary condition

$$\vec{u} = \begin{cases} \vec{g} & \text{on } (0,T)\times\Gamma_c \\ \vec{0} & \text{on } (0,T)\times\Gamma_0\,, \end{cases} \tag{1.3}$$

and an initial condition

$$\vec{u}(0,x) = \vec{u}_0(x) \quad \text{for } x\in\Omega\,. \tag{1.4}$$

Here, the vector field $\vec{u}(t,x) = (u_1(t,x), u_2(t,x))$ denotes the velocity of the two dimensional flow, $p$ the pressure, and $\nu > 0$ the inverse of the Reynolds number whenever the variables are appropriately non-dimensionalized. We will use the time variable by $t$, the state variable by $x$ in the flow domain $\Omega$, and the boundary variable by $s$ for consistency. In our problem, the control parameter is the boundary velocity $\vec{g}$ along $\Gamma_c$. For the compatibility in the whole system (1.1)–(1.4), the control parameter $\vec{g}$ should satisfy

$$\int_{\Gamma_c} \vec{g}\cdot\vec{n}\,ds = 0\,, \tag{1.5}$$

where $\vec{n}$ is the unit outward normal vector along the boundary $\Gamma_c$, and

$$\vec{u_0}(x) = \vec{g}(0,x) \quad \text{for } x\in\Gamma_c\,. \tag{1.6}$$

In order to keep the balance between the initial and boundary conditions in (1.3)–(1.4) together, it is natural to assume that $\vec{u}_0$ satisfies

$$\nabla\cdot\vec{u}_0 = 0 \ \text{ in } \Omega,\ \vec{u}_0 = \vec{0} \text{ on } \Gamma_0, \text{ and } \int_{\Gamma_c} \vec{u}_0\cdot\vec{n}\,ds = 0\,. \tag{1.7}$$

One could examine several physically meaningful objective functionals for the boundary control in practices, e.g., seeking the desired velocity tracking over the special region of the flow body $\Omega$ as in [9], or pursuing an optimal drag reduction profile.

The modeling boundary control problem we are concerned with is stated as follows:

Find the boundary control $\vec{g}$ along $\Gamma_c$ and a velocity field $\vec{u}$ such that the performance functional

$$\mathcal{J}(\vec{u},\vec{g}) = 2\nu \int_0^T \int_\Omega D(\vec{u}) : D(\vec{u})\, dxdt + \frac{\alpha}{2} \int_0^T \int_{\Gamma_c} |\vec{g}|^2\, dsdt \tag{1.8}$$

is minimized subject to $(\vec{u},\vec{g})$ satisfying the time-dependent two dimensional Navier-Stokes equations (1.1)–(1.7) with its compatibility between the initial and boundary conditions.

In (1.8), we denote the deformation tensor due to the flow by $D(\vec{u}) = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$ and $\alpha$ a nonnegative constant which is regarded as a valuable parameter in times. This choice of nonnegative parameter may be needed in general purposes. For this reason, it would be helpful to refer some special occasions remarked in [13]. Especially, for $\alpha = 0$, the functional (1.8) represents the rate of energy dissipation associated with the deformation followed by the flow. Physically, except for an additive constant, this can be identified with the viscous drag of the flow.

The motivation for our choice of the functional can be described by the following consideration. Under the constitutive laws, the stress tensor $\mathcal{S}$ due to the Newtonian fluid is given by

$$\mathcal{S} = -pI + 2\nu D(\vec{u})\,.$$

Suppose the flow body $\Omega$ is immersed in the flow. Then the total force acting on the fluid body during the time interval $(0,T)$ is given by

$$\vec{\mathcal{F}} = \int_0^T \int_{\partial\Omega} \mathcal{S}\cdot\vec{n}\, dsdt\,,$$

where $\vec{n}$ denotes the unit normal vector along the boundary, which points into the body([6]). Since the component $\mathcal{F}_{\vec{u}}$, of the force acting in the direction of the velocity vector field $\vec{u}$, is given by $\vec{u} \cdot \vec{\mathcal{F}}$, using the integration by parts and the solenoidal condition (1.2), we have

$$\begin{aligned}
\mathcal{F}_{\vec{u}} &= \vec{u} \cdot \vec{\mathcal{F}} \\
&= \int_0^T \int_{\partial\Omega} \vec{u} \cdot \mathcal{S} \cdot \vec{n}\, ds \\
&= 2\nu \int_0^T \int_\Omega D(\vec{u}) : D(\vec{u})\, dxdt - \int_0^T \int_\Omega p(\nabla \cdot \vec{v})\, dxdt \\
&\qquad\qquad + \nu \int_0^T \int_\Omega \vec{v} \cdot \nabla(\nabla \cdot \vec{v})\, dxdt \\
&= 2\nu \int_0^T \int_\Omega D(\vec{u}) : D(\vec{u})\, dxdt\,.
\end{aligned}$$

Thus, the first term in the performance functional (1.8) stands for the viscous drag forced by the flow, or kinematic dissipation energy inside of the domain as in our case.

The plan of the study is as follows. In the rest of this section, we introduce some notations and preliminary results that will be useful in what follows. In section 2, we give a precise description of the model boundary control problem, and then state and prove some results concerning the existence of an optimal solution. In section 3, we will examine the differentiability with respect to the control parameter for the concerned velocity as well as the functional. As a result, the first order necessary conditions can be found through a direct sensitivity analysis. In section 4, we organize the first order necessary conditions, which is identified in section 3, into the optimality system. For completeness of discussion, some suggestions for an algorithm to achieve an optimal control shall be proposed with some closing remarks.

**1.1. Notations and Preliminaries.** Throughout this paper, $C$ denotes generic constants whose values depend on the context. Let $\Omega$ be a bounded open connected set in $\mathbb{R}^2$ with $\mathcal{C}^2$ boundary. We denote by $H^s(\mathcal{O})$, $s \in \mathbb{R}$, the standard Sobolev space of order $s$ in regards the set $\mathcal{O}$, which is either the flow domain $\Omega$, or its boundary $\Gamma$, or part of its boundary. Whenever $m$ is a nonnegative integer, the inner product over $H^m(\mathcal{O})$ is denoted by $(f,g)_m$, and $(f,g)$ denotes the inner product over $H^0(\mathcal{O}) = L^2(\mathcal{O})$. Hence, we associate $H^m(\mathcal{O})$ with its natural norm $\|f\|_m = \sqrt{(f,f)_m}$. In fact, $H^m(\mathcal{O})$ is defined as the closure of $C^\infty(\mathcal{O})$

in the norm

$$\|f\|_m^2 = \sum_{|\alpha|\le m} \int_{\mathcal{O}} |\Big(\frac{\partial}{\partial x}\Big)^{\alpha} f(x)|^2\, dx\,.$$

The closure of $C_0^\infty(\mathcal{O})$ under the norm $\|\cdot\|_m$ will be denoted by $H_0^m(\mathcal{O})$. For details about these spaces, see, e.g., [2], [5], [7] and [14].

We will use boldface notations for spaces of vector-valued functions. For example, $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^2$ denotes the space of $\mathbb{R}^2$-valued functions such that each component belongs to $H^s(\Omega)$. Of special interest is the space

$$\mathbf{H}^1(\Omega) = \left\{ v_j \in L^2(\Omega) \;\Big|\; \frac{\partial v_j}{\partial x_k} \in L^2(\Omega) \quad \text{for } j,k = 1,2 \right\}$$

equipped with the norm $\|\vec{v}\|_1 = (\sum_{k=1}^{2} \|v_k\|_1^2)^{1/2}$. Whenever $\Gamma_0 \subset \Gamma$ has positive measure, we shall denote the space with the homogeneous boundary condition along $\Gamma_0$ by $\mathbf{H}^1_{\Gamma_0}(\Omega) = \{\vec{v} \in \mathbf{H}^1(\Omega) \,|\, \vec{v} = \vec{0} \;\text{ on } \Gamma_0\}$, and we let $\mathbf{H}^1_0(\Omega) = \mathbf{H}^1_\Gamma(\Omega)$.

We define the space of infinitely differentiable solenoidal vector fields by

$$\mathcal{V}(\Omega) = \{\, \vec{u} \in \mathbf{C}^\infty(\bar{\Omega}) \mid \nabla\cdot\vec{u} = 0 \text{ in } \Omega\,, \quad \vec{u} = \vec{0} \text{ on } \Gamma_0 \}\,,$$

and its completion in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$ by

$$\mathbf{H} = \{\, \vec{u} \in \mathbf{L}^2(\Omega) \mid \nabla\cdot\vec{u} = 0 \;\text{ in } \Omega\,,\; \vec{u} = \vec{0} \;\text{ on } \Gamma_0 \}\,,$$

and

$$\mathbf{V} = \{\, \vec{u} \in \mathbf{H}^1_{\Gamma_0}(\Omega) \mid \nabla\cdot\vec{u} = 0 \;\text{ in } \Omega \,\}\,,$$

respectively. Also, we will denote the space of solenoidal vector fields equipped with the homogeneous boundary condition along the whole boundary $\Gamma$ by

$$\mathcal{V}_0(\Omega) = \{\vec{u} \in \mathbf{C}_0^\infty(\Omega) \mid \nabla\cdot\vec{u} = 0 \;\text{ in } \Omega \,\}\,,$$

and its completion by

$$\mathbf{H}_0 = \{\vec{u} \in \mathbf{L}^2(\Omega) \mid \nabla\cdot\vec{u} = 0 \text{ in } \Omega,\, \vec{u} = \vec{0} \text{ on } \Gamma \}$$

and

$$\mathbf{V}_0 = \{\vec{u} \in \mathbf{H}^1_0(\Omega) \,|\, \nabla\cdot\vec{u} = 0 \;\text{ in } \Omega \,\}\,.$$

It is well-known that $\mathbf{H}$ is the closure of $\mathcal{V}(\Omega)$ in the space $\mathbf{L}^2(\Omega)$, $\mathbf{V}$ the closure of $\mathcal{V}(\Omega)$ in the space $\mathbf{H}^1(\Omega)$, and $\mathbf{H}_0$ and $\mathbf{V}_0$ the closure of $\mathcal{V}_0(\Omega)$. The norm on $\mathbf{H}$ shall be defined by $|\vec{u}|$. We also define the norm on $\mathbf{V}$ by the seminorm $||\vec{u}|| = |\nabla\vec{u}|$. According to the Poincare's inequality([5]), this seminorm is equivalent to $\mathbf{H}^1$-norm. We also note that $\mathbf{H}^1$-norm of the vector $\vec{v}$ is given by

$$\|\vec{v}\|_1^2 = |\vec{v}|^2 + \|\vec{v}\|^2 . \tag{1.9}$$

Let us denote the dual space of $\mathbf{V}$ by $\mathbf{V}^*$ and the duality between $\mathbf{V}^*$ and $\mathbf{V}$ by $< \cdot, \cdot >_{\mathbf{V}^*}$. Since $\mathbf{V}$ is densely embedded in $\mathbf{H}$, we have the canonical framework for the weak formulation in the sense that the following inclusions imply dense embedding:

$$\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}^* .$$

For the traces to the control boundary $\Gamma_c$, we will define $\gamma_c^0 : \mathbf{H}^1_{\Gamma_0}(\Omega) \to \mathbf{H}^{1/2}(\Gamma_c)$ by $\gamma_c^0(\vec{u}) = \vec{u}\Big|_{\Gamma_c}$ and $\gamma_c^1 : \mathbf{H}^1_{\Gamma_0}(\Omega) \to \mathbf{H}^{-1/2}(\Gamma_c)$ by $\gamma_c^1(\vec{u}) = \dfrac{\partial\vec{u}}{\partial\vec{n}}\Big|_{\Gamma_c}$.

In order to define a weak form for the Navier-Stokes equations, we introduce the continuous bilinear form

$$a(\vec{u},\vec{v}) = 2\nu \int_\Omega D(\vec{u}) : D(\vec{v})\, dx \quad \forall\, \vec{u},\vec{v} \in \mathbf{H}^1(\Omega)\,, \tag{1.10}$$

and the trilinear form on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$

$$b(\vec{w};\vec{u},\vec{v}) = \int_\Omega (\vec{w}\cdot\nabla)\vec{u}\cdot\vec{v}\, dx = \sum_{i,j=1}^{2} \int_\Omega w_j \left(\frac{\partial u_i}{\partial x_j}\right) v_i\, dx\,. \tag{1.11}$$

Here $D(\vec{u}) : D(\vec{v})$ denotes the tensor product $\displaystyle\sum_{i,j=1}^{2} D_{ij}(\vec{u})D_{ij}(\vec{v})$, where $D_{ij}(\vec{u}) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$. Obviously, $a(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ and $b(\cdot;\cdot,\cdot)$ is a continuous trilinear form on $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$, which can be verified by the Sobolev embedding of $\mathbf{H}^1(\Omega) \subset \mathbf{L}^4(\Omega)$ and Hölder's inequality. For details, one may refer [2], [5] and [14]. We also have the coercivity property

$$a(\vec{v},\vec{v}) \geq C\|\vec{v}\|_1^2 \quad \forall\, \vec{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega)\,. \tag{1.12}$$

It is worthwhile to notice that

$$2\nabla\cdot D(\vec{u}) = \Delta\vec{u} + \nabla(\nabla\cdot\vec{u})\,.$$

If we take a dot product with $\vec{v}$ and then integration, we obtain by Green's formula

$$2\int_\Gamma \vec{v}\cdot D(\vec{u})\,\vec{n}\,ds = \int_\Omega \Delta\vec{u}\cdot\vec{v}\,dx + 2\int_\Omega D(\vec{u})\,:\,\nabla\vec{v}\,dx\,.$$

Since $D(\vec{u})$ is a symmetric tensor, we have

$$\int_\Omega D(\vec{u})\,:\,\nabla\vec{v}\,dx = \int_\Omega D(\vec{u}) : D(\vec{v})\,dx\,.$$

Hence, for $\vec{v}\in\mathbf{V}\cap\mathbf{H}_0^1(\Omega)$, it follows that

$$(1.13)\qquad 2\int_\Omega D(\vec{u})\,:\,D(\vec{v})\,dx = -\int_\Omega \Delta\vec{u}\cdot\Delta\vec{v}\,dx = \int_\Omega \nabla\vec{u}\,:\,\nabla\vec{v}\,dx\,.$$

Related to the duality pairing $<\cdot,\cdot>_{\mathbf{V}^*}$, we will make use of the following operators:

$$\mathcal{A}\,:\,\mathbf{V}\longrightarrow\mathbf{V}^*\,,$$

which is defined by

$$(1.14)\qquad <\mathcal{A}\vec{u},\vec{v}>_{\mathbf{V}^*} = a(\vec{u},\vec{v})\quad \forall\,\vec{u},\,\vec{v}\in\mathbf{V}\,,$$

and

$$\mathcal{B}\,:\,\mathbf{V}\times\mathbf{V}\longrightarrow\mathbf{V}^*$$

defined by

$$(1.15)\qquad <\mathcal{B}(\vec{u},\vec{v}),\vec{w}>_{\mathbf{V}^*} = b(\vec{u};\vec{v},\vec{w})\quad \forall\,\vec{u},\,\vec{v},\,\vec{w}\in\mathbf{V}\,.$$

If we encounter no other confusion, we will denote $\mathcal{B}(\vec{u},\vec{u})$ by $\mathcal{B}(\vec{u})$, and $\mathbf{V}^*$ will be dropped out in the duality between $\mathbf{V}^*$ and $\mathbf{V}$ so that $<\cdot,\cdot>_{\mathbf{V}^*} = <\cdot,\cdot>$.

For the operator $\mathcal{B}$ and its associated trilinear form $b(\cdot;\cdot,\cdot)$, the following results will be widely used in the sequel.

LEMMA 1.1

*(i) The map* $\mathcal{B}\,:\,\mathbf{V}\to\mathbf{V}^*\,;\,(\vec{u}\mapsto\mathcal{B}(\vec{u}))$ *is differentiable, and we have*

$$(1.16)\qquad \mathcal{B}'(\vec{u};\vec{v}) = \frac{d}{d\tau}\mathcal{B}(\vec{u}+\tau\vec{v},\vec{u}+\tau\vec{v})\Big|_{\tau=0} = \mathcal{B}(\vec{u},\vec{v})+\mathcal{B}(\vec{v},\vec{u})\,.$$

*Furthermore, if we represent the corresponding adjoint of* $\mathcal{B}'(\cdot;\cdot)$ *by* $\mathcal{B}'(\cdot;\cdot)^*$ *so that*

$$<\mathcal{B}'(\vec{u};\vec{v})^*,\vec{w}> = <\mathcal{B}'(\vec{u};\vec{w}),\vec{v}>\,,$$

*then it follows*

$$(1.17)\quad <\mathcal{B}'(\vec{u};\vec{v})^*,\vec{w}>=\int_\Omega \sum_{i,j=1}^{2} w_j\left(\frac{\partial u_i}{\partial x_j}v_i-u_i\frac{\partial v_j}{\partial x_i}\right)dx \quad \forall\,\vec{w}\in\mathbf{V}_0\,.$$

*(ii) The trilinear form $b(\cdot;\cdot,\cdot)$ has the following properties:*

$$(1.18)\qquad \textit{(orthogonality)}\quad b(\vec{u};\vec{v},\vec{v})=0 \quad \forall\,\vec{u}\in\mathbf{V}_0\,,\ \forall\,\vec{v}\in\mathbf{H}^1(\Omega)\,,$$

$$(1.19)\quad \begin{cases} |\,b(\vec{u};\vec{v},\vec{w})|\le C\|\vec{u}\|\|\vec{v}\|\|\vec{w}\| & \forall\,\vec{u},\ \vec{v},\ \vec{w}\in\mathbf{V}\,,\\ |\,b(\vec{u};\vec{v},\vec{w})|\le C\|\vec{u}\|\|\vec{v}\||\vec{w}|^{1/2}\|\vec{w}\|^{1/2} & \forall\,\vec{u},\ \vec{v},\ \vec{w}\in\mathbf{V}\,.\end{cases}$$

*Moreover, for $\vec{u}\in\mathbf{V}$, $\vec{v}\in\mathbf{V}\cap\mathbf{H}^2(\Omega)$, and $\vec{w}\in\mathbf{H}$, we have*

$$(1.20)\quad \begin{aligned} |\,b(\vec{u};\vec{v},\vec{w})| &\le C\,|\vec{u}|^{1/2}\|\vec{u}\|^{1/2}\|\vec{v}\|^{1/2}|\mathcal{A}\vec{v}|^{1/2}|\vec{w}|\\ &\le C\|\vec{u}\|\|\vec{v}\|^{1/2}|A\vec{v}|^{1/2}|\vec{w}|\,.\end{aligned}$$

*Proof* : The estimates (1.19) and (1.20) can be obtained by applying the Poincare's inequality to the results listed in [4] and [14]. For (1.17), by taking the test function $\vec{w}$ in $\mathbf{V}_0$, we have

$$\begin{aligned} <\mathcal{B}'(\vec{u};\vec{w}),\vec{v}> &= <\mathcal{B}(\vec{w},\vec{u}),\vec{v}>+<\mathcal{B}(\vec{u},\vec{w}),\vec{v}>\\ &= \int_\Omega\sum_{i,j=1}^{2}\Big(w_j\frac{\partial u_i}{\partial x_j}v_i+u_i\frac{\partial w_j}{\partial x_i}v_j\Big)\,dx\\ &= \int_\Omega\sum_{i,j=1}^{2}w_j\Big(\frac{\partial u_i}{\partial x_j}v_i-u_i\frac{\partial v_j}{\partial x_i}\Big)\,dx\\ &\qquad -\int_\Omega(\nabla\cdot\vec{u})\vec{w}\cdot\vec{v}\,dx-\int_\Gamma(\vec{w}\cdot\vec{v})\vec{u}\cdot\vec{n}\,ds\\ &= \int_\Omega\sum_{i,j=1}^{2}w_j\Big(\frac{\partial u_i}{\partial x_j}v_i-u_i\frac{\partial v_j}{\partial x_i}\Big)\,dx\,.\end{aligned}$$

Hence (1.17) follows. □

From (1.16), it follows that $\mathcal{B}'(\vec{u};\vec{v})$ corresponds to the linearized form for the nonlinear convective term $\mathcal{B}(\vec{u})=(\vec{u}\cdot\nabla)\vec{u}$ in the $\vec{v}$-direction, so that

$$(1.21)\qquad \mathcal{B}'(\vec{u};\vec{v})=(\vec{u}\cdot\nabla)\vec{v}+(\vec{v}\cdot\nabla)\vec{u}\,.$$

On the while, one can see from (1.17) that its adjoint $\mathcal{B}'(\vec{u};\vec{v})^*$ is represented by

$$\mathcal{B}'(\vec{u};\vec{v})^* = (\nabla\vec{u})^t\vec{v} - (\vec{u}\cdot\nabla)\vec{v}\,, \tag{1.22}$$

where $(\nabla\vec{u})^t$ denotes the transpose of the tensor. The following skew-symmetric condition ensues from the orthogonality property (1.18):

$$b(\vec{u};\vec{v},\vec{w}) = -\,b(\vec{u};\vec{w},\vec{v}) \quad \forall\,\vec{u}\in\mathbf{V}\cap\mathbf{H}_0^1(\Omega)\,,\ \forall\,\vec{v}\,,\vec{w}\in\mathbf{H}^1(\Omega)\,. \tag{1.23}$$

Final remarks is in the order. Let $X$ be a Banach space defined on a domain $\Omega$. For the time-dependent system, we shall denote the function space $\mathcal{C}(0,T;X)$ to be the class of functions $y(t,x)$ such that for a fixed $x\in\Omega$, $t\mapsto y(t,x)$ is a continuous function over $[0,T]$, while $y(t)\equiv y(t,\cdot)$ belongs to $X$. For $1\le p\le\infty$, $L^p(0,T;X)$ denotes the completion of $\mathcal{C}(0,T;X)$ with respect to the norm

$$\|y\|_{L^p(0,T;X)} = \Big(\int_0^T \|y(t)\|_X^p\,dt\Big)^{1/p}.$$

For a reflexive Banach space $X$, we will use the notion $L^\infty(0,T;X)$ to denote the dual space of $L^1(0,T;X)$. In most cases, $X$ will be Hilbert spaces.

## 2. Existence results

We shall project the system (1.1)–(1.7) into the dual space of the solenoidal vector fields as in [1] and [4]. For the boundary control problem (1.8), we will show the existence of an optimal solution.

**2.1. The mathematical setting of the problem.** We first note that

$$\mathbf{L}^2(\Omega) = \mathbf{H}_0\oplus\mathbf{H}_0^\perp\,,$$

where

$$\mathbf{H}_0^\perp = \{\vec{u}\in\mathbf{L}^2(\Omega)\,|\,\vec{u}=\nabla\phi \text{ for some } \phi\in H^1(\Omega)\,\}\,.$$

If $\vec{u}\in\mathbf{H}_0$, from $\displaystyle\int_\Omega \vec{u}\cdot\nabla\phi\,dx = \int_\Gamma \phi\vec{u}\cdot\vec{n}\,ds - \int_\Omega(\nabla\cdot\vec{u})\phi\,dx$, it follows that $\displaystyle\int_\Omega \vec{u}\cdot\nabla\phi\,dx = 0$ for every $\phi\in H^1(\Omega)$. On the other hand, if $\vec{u}\in\mathbf{L}^2(\Omega)$ satisfies $\vec{u}=\vec{0}$ on $\Gamma$ and $\displaystyle\int_\Omega \vec{u}\cdot\nabla\phi\,dx = 0$ for all $\phi\in H^1(\Omega)$, then by the

Green's formula we have $\nabla \cdot \vec{u} = 0$ in $\Omega$, so that $\vec{u} \in \mathbf{H}_0$. This justifies the orthogonal decomposition of $\mathbf{L}^2(\Omega)$.

By $\mathcal{P}$, we define the orthogonal projector $\mathcal{P} : \mathbf{L}^2(\Omega) \to \mathbf{H}_0$. It is obvious that the operator in (1.14) corresponds to $\mathcal{A} = \mathcal{P}(-\Delta)$, and the operator in (1.15) to $\mathcal{B}(\vec{u}, \vec{v}) = \mathcal{P}((\vec{u} \cdot \nabla)\vec{v})$. In perspective points of view, the major advantage we can get by applying the projector $\mathcal{P}$ to the Navier-Stokes system is that the pressure term can be excluded, so that it is reduced to the system only the velocity concerned. After finding the velocity, the pressure then can be retrieved by applying de Rham's lemma. For details, see, e.g., [4], [6] or [10].

According to this formulation, the Navier-Stokes system (1.1)–(1.4) can be written by

$$\left\{ \begin{array}{l} \dfrac{d\vec{u}}{dt} + \nu \mathcal{A}\vec{u} + \mathcal{B}(\vec{u}) = \mathcal{P}\vec{f} \quad \text{in } (0,T) \times \Omega\,, \\ \nabla \cdot \vec{u} = 0 \quad \text{in } (0,T) \times \Omega\,, \\ \vec{u} = \vec{g} \quad \text{on } (0,T) \times \Gamma_c\,, \\ \vec{u} = \vec{0} \quad \text{on } (0,T) \times \Gamma_0\,, \\ \vec{u}(0,x) = \vec{u}_0(x) \quad \forall\, x \in \Omega\,. \end{array} \right. \tag{2.1}$$

For consistency in the mathematical formulation, we assume the body force satisfies $\nabla \cdot \vec{f} = 0$ in $(0,T) \times \Omega$, so that $\mathcal{P}\vec{f} = \vec{f}$ in (2.1).

Ahead of presenting the well-posedness of the system (1.1)–(1.7) as well as the relevant regularity results, we need to pay some careful attention to the compatibility conditions between the boundary and initial conditions. From (1.6) and (1.7), the initial data $\vec{u}_0$ must satisfy the conditions

$$\vec{u}_0 \in \mathbf{V}\,, \quad \vec{u}_0(s) = \vec{g}(0,s)\ \forall\, s \in \Gamma_c, \quad \text{and} \quad \int_{\Gamma_c} \vec{u}_0 \cdot \vec{n}\, ds = 0\,. \tag{2.2}$$

For the boundary vector fields in our need, we use the space

$$\mathbf{W} = \{\vec{g} \in \mathbf{H}_0^{1/2}(\Gamma_c) \mid \int_{\Gamma_c} \vec{g} \cdot \vec{n}\, ds = 0\}\,.$$

Then, $\mathbf{W}$ is a closed subspace of $\mathbf{H}^{1/2}(\Gamma_c)$, and the boundary condition $\vec{g}$, which is comprised the control parameter in our case, belongs to the space $\mathcal{W} \equiv L^2(0,T;\mathbf{W})$.

For the norm of $\vec{g}$ in $\mathcal{W}$, one may take

$$\|\vec{g}\|_{\Gamma_c} = \left( \int_0^T |\vec{g}|^2_{\Gamma_c}\, dt \right)^{1/2},$$

where $|\cdot|_{\Gamma_c}$ denotes the $\mathbf{H}^{1/2}$-norm on $\Gamma_c$. We let $\mathbf{H}^{-1/2}(\Gamma_c)$ denote the dual space of $\mathbf{H}^{1/2}_0(\Gamma_c)$, and $< \cdot, \cdot >_{\Gamma_c}$ denote the duality between $\mathbf{H}^{-1/2}(\Gamma_c)$ and $\mathbf{H}^{1/2}_0(\Gamma_c)$. When $\vec{s}^* \in \mathbf{H}^{-1/2}_0(\Gamma_c)$, from the definition for the dual norm, the norm of $\vec{s}^*$ in $\mathbf{H}^{-1/2}(\Gamma_c)$ is given by

$$|\vec{s}^*|_{-1/2,\Gamma_c} = \sup_{\vec{h}\in \mathbf{H}^{1/2}_0(\Gamma_c)} \frac{< \vec{s}^*, \vec{h} >_{\Gamma_c}}{|\vec{h}|_{\Gamma_c}}.$$

In [8], it is shown that

$$|\vec{s}^*|_{-1/2,\Gamma_c} = \sup_{\vec{v}\in \mathbf{H}^1_{\Gamma_0}(\Omega)} \frac{< \vec{s}^*, \gamma^0_c(\vec{v}) >_{\Gamma_c}}{\|\vec{v}\|_1}$$

can be used as an alternative equivalent norm for the space $\mathbf{H}^{-1/2}(\Gamma_c)$. Since the dual space of $\mathcal{W} = L^2(0,T;\mathbf{W})$ is $\mathcal{W}^* = L^2(0,T;\mathbf{W}^*)$, the duality between $\mathcal{W}^*$ and $\mathcal{W}$ can be given by

$$< \vec{s}^*, \vec{h} >_{\mathcal{W}^*} = \int_0^T < \vec{s}^*, \gamma^0_c(\vec{v}) >_{\Gamma_c} dt$$

for $\vec{v} \in L^2(0,T;\mathbf{H}^1_{\Gamma_0}(\Omega))$ with $\gamma^0_c(\vec{v}) = \vec{h}$.

In the following theorem, we present some classical results concerning the well-posedness and the regularity result for the time-dependent two dimensional Navier-Stokes system.

THEOREM 2.1 *Let $\Omega$ be a bounded domain with the $\mathcal{C}^2$ boundary. Let $\vec{u}_0$ and $\vec{g}$ satisfy the compatibility conditions (1.5)–(1.7).*

*(i) Let $\vec{f} \in L^2(0,T;\mathbf{H})$. Then, the system (2.1) has a unique solution $\vec{u}$ which belongs to $X \equiv L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$, and the system is well posed in a sense*

$$\|\vec{u}\|^2_X \le C \left( \|\vec{g}\|^2_{\Gamma_c} + \|\vec{u}_0\|^2 + \|\vec{f}\|^2_{L^2(0,T;\mathbf{H})} \right). \quad (2.3)$$

*Moreover, $\dfrac{d\vec{u}}{dt}$ belongs to $L^2(0,T;\mathbf{V}^*)$.*

*(ii) Let $\vec{f}$ and $\dfrac{d\vec{f}}{dt} \in L^2(0,T;\mathbf{H})$ with $\vec{f}(0,\cdot) \in \mathbf{H}$, $\vec{u}_0 \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$, and for each $0 \le t \le T$, $\vec{g} \in L^2(0,T;\mathbf{W} \cap \mathbf{H}^{3/2}(\Gamma_c))$. Then the*

*solution* $\vec{u}$ *of the system (2.1) belongs to* $L^2(0,T;\mathbf{H}^2(\Omega))$, *and* $\dfrac{d\vec{u}}{dt}$ *to* $L^2(0,T;\mathbf{V})\cap L^\infty(0,T;\mathbf{H})$.

For proof, one may follow the compactness argument by employing the Galerkin approximation method as in [11], [12] and [14]. The condition and the result on regularity in (ii) are standard.

Based on this preliminary facts, we show the existence result for the boundary control problem in the next section.

**2.2. Existence of an optimal solution.** We provide a precise formulation for the control problem (1.8) and prove the existence of an optimal solution. To comply with our discussion in the previous section, we set the admissible family of sets by

$$\begin{aligned}\mathcal{U}_{\rm ad} = \ & \{(\vec{u},\vec{g})\in L^2(0,T;\mathbf{V})\times L^2(0,T;\mathbf{W})\mid \ \mathcal{J}(\vec{u},\vec{g})<\infty, \\ & \qquad (\vec{u},\vec{g}) \text{ satisfies (2.1) with } \vec{u}_0 \text{ satisfies (2.2)}\}\,.\end{aligned}$$

Then the boundary control problem we are concerned with can be stated in the following way:

> Given $\vec{u}_0\in\mathbf{V}$ along with the compatibility condition (2.2), find $(\vec{u},\vec{g})\in\mathcal{U}_{\rm ad}$ such that the boundary control $\vec{g}$ minimizes the performance functional

$$\mathcal{J}(\vec{u},\vec{g}) = 2\nu\int_0^T\int_\Omega D(\vec{u}):D(\vec{u})\,dxdt + \frac{\alpha}{2}\int_0^T\int_{\Gamma_c}|\vec{g}|^2\,dsdt\,. \tag{2.4}$$

The first term in (2.4) expresses the energy dissipation forced by the deformation due to the flow and the second part represents the driven control along the control boundary $\Gamma_c$.

We now turn to the question of the existence of an optimal solution.

THEOREM 2.2 *Let* $\vec{f}\in L^2(0,T;\mathbf{H})$ *and* $\vec{u}_0\in\mathbf{V}$ *be given. Suppose* $\vec{u}_0$ *satisfies the compatibility condition* $\int_{\Gamma_c}\vec{u}_0\cdot\vec{n}\,ds=0$. *Then, there exists at least one optimal solution* $(\vec{u},\vec{g})\in\mathcal{U}_{ad}$ *which minimizes the functional (2.4), and* $\vec{u}=\vec{u}(\vec{g})$ *satisfies* $\gamma_c^0(\vec{u})=\vec{g}$ *and* $\vec{u}(0,x)=\vec{u}_0(x)$, $\forall\, x\in\Omega$.

*Proof*: An admissible solution can be found by first setting $\widehat{g}(t,\cdot) = \gamma_c^0(\vec{u}_0)$ for $0 \le t \le T$, and then by solving the system

$$\begin{cases} \dfrac{d\widehat{u}}{dt} + \nu\mathcal{A}\widehat{u} + \mathcal{B}(\widehat{u}) = \vec{f} \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\widehat{u} = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \widehat{u} = \widehat{g} \quad \text{on } (0,T)\times\Gamma_c\,, \\ \widehat{u} = \vec{0} \quad \text{on } (0,T)\times\Gamma_0\,, \\ \widehat{u}(0,x) = \vec{u}_0(x) \quad \forall\, x\in\Omega\,. \end{cases}$$

According to Theorem 2.1, the solution $(\widehat{u},\widehat{g})$ exists and belongs to $\mathcal{U}_{\text{ad}}$.

Since the set of admissible solutions $\mathcal{U}_{\text{ad}}$ is not empty and the set of the values assumed by the functional is bounded from below, there exists a minimizing sequence $\vec{g}_m \in \mathcal{W}$, and the corresponding sequence for the velocity $\vec{u}_m = \vec{u}(\vec{g}_m)$, where $\vec{u} = \vec{u}_m$ is a solution of the system (2.1) with $\vec{g} = \vec{g}_m$. Then since the sequence $\vec{g}_m$ is uniformly bounded in $L^2(0,T;\mathbf{W})$, by (2.3) the corresponding sequence $\vec{u}_m$ is also uniformly bounded in $L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$. Thus one can extract from the sequence $\{(\vec{u}_m,\vec{g}_m)\}$ a subsequence (denoted again by the same notation) in $L^2(0,T;\mathbf{V})\times L^2(0,T;\mathbf{W})$ which converges weakly to $(\vec{u},\vec{g})$.

Hence one can write

$$\begin{array}{llll} \vec{g}_m \to \vec{g} & \textit{weakly} & \text{in} & L^2(0,T;\mathbf{H}^{1/2}(\Gamma_c))\,, \\ \vec{u}_m \to \vec{u} & \textit{weakly} & \text{in} & L^2(0,T;\mathbf{V})\,, \\ \vec{u}_m \to \vec{u} & \textit{strongly} & \text{in} & L^2(0,T;\mathbf{H})\,, \\ \vec{u}_m \to \vec{u} & \textit{weakstarly} & \text{in} & L^\infty(0,T;\mathbf{H})\,. \end{array}$$

Since $\mathcal{J}(\cdot,\cdot)$ is coercive and strongly continuous by (1.12) and (2.3), the performance functional $\mathcal{J}$ is lower semicontinuous. Hence passing to the limit in $\mathcal{U}_{\text{ad}}$, we have

$$\mathcal{J}(\vec{u},\vec{g}) \le \liminf_{n\to\infty} \mathcal{J}(\vec{u}_m,\vec{g}_m)\,,$$

so that the functional is minimized at $(\vec{u},\vec{g})$.

To complete the proof, it remains to show that $(\vec{u},\vec{g})$ belongs to the admissible set $\mathcal{U}_{\text{ad}}$. First of all, we need to note: a priori estimate by using the compactness argument in the fractional time order Sobolev space yields the strong convergence of $\vec{u}_m$ to $\vec{u}$ in $L^2(0,T;\mathbf{H})$ as stated above, see [4] and [14] for details.

Note that for $\vec{w} \in L^2(0,T;\mathbf{V})$, we have

$$\begin{aligned} &b(\vec{u}_m;\vec{w},\vec{u}_m) - b(\vec{u};\vec{w},\vec{u}) \\ &\quad = b(\vec{u}_m-\vec{u};\vec{w},\vec{u}_m-\vec{u}) + b(\vec{u}_m-\vec{u};\vec{w},\vec{u}) + b(\vec{u};\vec{w},\vec{u}_m-\vec{u})\,. \end{aligned}$$

Using (1.19), the strong convergence of $\vec{u}_m$ to $\vec{u}$ in $L^2(0,T;\mathbf{H})$, and the Sobolev embedding $\mathbf{H}^1(\Omega) \subset \mathbf{L}^4(\Omega)$, we have for every $\vec{w} \in L^2(0,T;\mathbf{V})$

$$\begin{aligned} < \mathcal{B}(\vec{u}_m,\vec{u}_m),\vec{w} > &= - < \mathcal{B}(\vec{u}_m,\vec{w}),\vec{u}_m > \\ &\longrightarrow\ - < \mathcal{B}(\vec{u},\vec{w}),\vec{u} > = < \mathcal{B}(\vec{u},\vec{u}),\vec{w} > \ . \end{aligned}$$

Hence $\vec{u}$ satisfies the equation

$$\frac{d\vec{u}}{dt} + \nu\,\mathcal{A}\vec{u} + \mathcal{B}(\vec{u}) = \vec{f}\,.$$

Since $\vec{g}_m = \gamma_c^0(\vec{u}_m)$ weakly converges to $\vec{g}$ and the lifting $\vec{u}_m$ of $\vec{g}_m$ strongly converges to $\vec{u}$ in $L^2(0,T;\mathbf{V})$, by the continuity of the trace([5]) it follows that $\gamma_c^0(\vec{u}) = \vec{g}$, which implies that $\vec{u} = \vec{u}(\vec{g})$.

Finally, we need to show that $\vec{u}(0,\cdot) = \vec{u}_0$. Note that $\vec{u}_m \in L^2(0,T;\mathbf{V})$ is a solution of an initial problem for the parabolic system

$$\text{(2.5)} \qquad \begin{cases} \dfrac{d}{dt}\vec{u}_m &= \vec{f} - \nu\,\mathcal{A}\vec{u}_m - \mathcal{B}(\vec{u}_m) \quad \text{on } L^2(0,T;\mathbf{V}^*)\,, \\ \vec{u}_m(0) &= \vec{u}_0 \quad \text{for every } m\,. \end{cases}$$

Since $\mathcal{C}^1(0,T) \times \mathcal{V}_0(\Omega)$ is dense in $\mathrm{L}^2(0,T;\mathbf{V}_0)$, multiplying (2.5) by a trial function $\phi(t)\vec{\zeta}$ such that $\phi \in \mathcal{C}^1(0,T)$ with $\phi(T) = 0$ and $\vec{\zeta} \in \mathcal{V}_0$, and taking integration by parts, we have

$$\begin{aligned} -\int_0^T < \vec{u}_m,\ \ \phi'(\tau)\vec{\zeta} > \, d\tau &= < \vec{u}_0, \phi(0)\vec{\zeta} > + \int_0^T < \vec{f}, \phi(\tau)\vec{\zeta} > \, d\tau \\ &\quad - \nu \int_0^T a(\vec{u}_m, \phi(\tau)\vec{\zeta})\, d\tau - \int_0^T b(\vec{u}_m;\vec{u}_m,\phi(\tau)\vec{\zeta})\, d\tau\,. \end{aligned}$$

After passing to the limit and integrating of the first term by parts, this yields

$$< (\vec{u}(0) - \vec{u}_0), \vec{\zeta} > \phi(0) = 0 \quad \forall\,\vec{\zeta} \in \mathbf{V}_0\,.$$

Hence, if we choose $\phi$ with $\phi(0) \neq 0$, it follows $\vec{u}(0) = \vec{u}_0$. □

## 3. Sensitivity Analysis

We investigate the question of what relations characterizes an optimal solution. For this purpose, we examine the differentiability for the performance functional as well as the corresponding velocity vector field with respect to the control parameter. Our ultimate goal is to derive the necessary conditions for an optimal solution. By Theorem 2.1, the solution $\vec{u}$ for the system (2.1) can be described as a function of the control parameter. For this reason, the functional $\mathcal{J}(\cdot,\cdot)$ can be casted equivalently into the functional

$$\mathcal{J}(\vec{g}) = \mathcal{J}(\vec{u}(\vec{g}), \vec{g}) \quad \text{for } \vec{g} \in \mathcal{W}\,. \tag{3.1}$$

Let us investigate the rate of variation of $\mathcal{J}(\vec{g})$ with respect to the control parameter $\vec{g}$. The rate of variation at $\vec{g}$ in the direction of $\vec{h}$ can be measured as a directional semi-derivative

$$d\mathcal{J}(\vec{g};\vec{h}) = \frac{d}{d\tau}\mathcal{J}(\vec{g}+\tau\vec{h})\Big|_{\tau=0}\,. \tag{3.2}$$

This derivation for $\mathcal{J}(\vec{g})$ is said to be Gateaux-differentiable if

- ∘ $d\mathcal{J}(\vec{g};\vec{h})$ exists for every $\vec{h}$,
- ∘ $\vec{h} \mapsto d\mathcal{J}(\vec{g};\vec{h})$ is linear and continuous.

When the functional $\mathcal{J}$ is Gateaux-differentiable, the rate of variation $d\mathcal{J}(\vec{g};\vec{h})$ is called the Gateaux-derivative at $\vec{g}$ in the $\vec{h}$-direction. Before proving differentiability, we recall the following inequalities which are now classical.

Lemma 3.1

*(i)* (Gronwall's Lemma) *Let $\phi$ and $\psi$ be real continuous functions over the interval $[t_0, t_1]$ with $\psi(\tau) \geq 0$ for all $\tau \in [t_0, t_1]$. Suppose for some constant $C$*

$$\phi(t) \leq C + \int_{t_0}^{t} \phi(\tau)\psi(\tau)\,d\tau \quad \forall\, t \in [t_0, t_1]\,,$$

*then it follows that*

$$\phi(t) \leq C\,exp\Big(\int_{t_0}^{t} \psi(\tau)\,d\tau\Big) \quad \forall\, t \in [t_0, t_1]. \tag{3.3}$$

*(ii)* (Young's inequality) *Let $\phi$ be a continuous function such that $\phi(0) = 0$ and $\phi$ is strictly increasing over $[0,\infty)$, and let $\psi$ be an*

*inverse function of $\phi$, then for every positive constants $a$ and $b$ we have*

$$ab \le \int_0^a \phi(\tau)\, d\tau + \int_0^b \psi(\tau)\, d\tau \,. \tag{3.4}$$

Of special interest to us, we need the following specific variations of (3.3) and (3.4).

LEMMA 3.2

*(i)* (Gronwall's inequality) *Let $\phi$ and $\psi$ be continuous functions and let $r(t)$ be continuously differentiable function over the interval $[t_0, t_1]$. Suppose that $\psi(t) \ge 0$ and $r'(t) \ge 0$ for all $t \in [t_0, t_1]$, and that*

$$\phi(t) \le r(t) + \int_{t_0}^t \phi(\tau)\psi(\tau)\, d\tau \quad \forall\, t \in [t_0, t_1]\,, \tag{3.5}$$

*then it follows that*

$$\phi(t) \le\, r(t)\, exp\Big(\int_{t_0}^t \psi(\tau)\, d\tau\Big) \quad \forall\, t \in [t_0, t_1]. \tag{3.6}$$

*(ii)* (Young's inequality) *Let $a$ and $b$ be positive constants. Let $\epsilon > 0$ be given. For $1 < p < \infty$, we have*

$$ab \le \frac{1}{p}(\epsilon^{1/p}\, a)^p + \frac{1}{q}\Big(\frac{b}{\epsilon^{1/p}}\Big)^q, \quad (\frac{1}{p} + \frac{1}{q} = 1)\,. \tag{3.7}$$

*Proof*: (ii) is directly obtained by taking $\phi(\tau) = \epsilon\tau^{p-1}$ in (3.4).

Let us show (i). If we put $R(t) = r(t) + \int_{t_0}^t \phi(\tau)\psi(\tau)\, d\tau$, then from $\phi(t) \le R(t)$, it follows that $R'(t) \le r'(t) + \psi(t)R(t)$, and the following differential inequality is derived as a result:

$$\frac{d}{dt}(R(t)\, exp(-\int_{t_0}^t \psi(\tau)\, d\tau)) \le r'(t)\, exp(-\int_{t_0}^t \psi(\tau)\, d\tau)\,. \tag{3.8}$$

Let us put $\chi(t) = exp(-\int_{t_0}^t \psi(\tau)\, d\tau)$. Then, $\chi(t_0) = 1$ and $\chi'(t) = -\psi(t)\chi(t) \le 0$, for $\chi(t) \ge 0$ and $\psi(t) \ge 0$ for all $t$. Hence, $\chi(t) \le \chi(t_0) = 1$ and (3.8) leads to

$$\frac{d}{dt}(R(t)\chi(t)) \le r'(t)\,. \tag{3.9}$$

Since $R(t_0)\chi(t_0) = R(t_0) = r(t_0)$, taking integration in (3.9), we have

$$R(t)\chi(t) \le r(t) \quad \forall\, t \in [t_0, t_1]\,,$$

so that

$$\phi(t) \le R(t) \le r(t)\, exp\Big(\int_{t_0}^{t} \psi(\tau)\, d\tau\Big)\,. \qquad \Box$$

Young's inequality (3.7) is mainly used to detach one positive constant from the multiplied constants and to modulate the size of the constant by imposing appropriate weights.

**3.1. Sensitivity for the state solution.** We show that the state solution $\vec{u}$ is strictly differentiable with respect to the control parameter.

THEOREM 3.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^2$ boundary. Let $\vec{f} \in L^2(0,T;\mathbf{H})$ and $\vec{u}_0 \in \mathbf{V}$ be given. Suppose $\vec{u}_0$ satisfies the compatibility conditions (2.2). Then, the mapping*

$$\vec{u} \,:\, L^2(0,T;\mathbf{W}) \to L^2(0,T;\mathbf{V})\,; \quad \Big(\vec{g} \mapsto \vec{u}(\vec{g})\Big)$$

*is differentiable. Furthermore, if we represent the Gateaux-derivative of $\vec{u}$ at $\vec{g}$ in the $\vec{h}$-direction by $\widehat{w}(\vec{h}) \equiv d\vec{u}(\vec{g};\vec{h})$, then $\widehat{w}(\vec{h})$ is the solution of the linearized equations*

$$\text{(3.10)} \qquad \begin{cases} \dfrac{d\widehat{w}}{dt} + \nu\mathcal{A}\widehat{w} + \mathcal{B}'(\vec{u}(\vec{g});\widehat{w}) = \vec{0} \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\widehat{w} = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \widehat{w} = \vec{h} \quad \text{on } (0,T)\times\Gamma_c\,, \\ \widehat{w} = \vec{0} \quad \text{on } (0,T)\times\Gamma_0\,, \\ \widehat{w}(0,x) = \vec{0} \quad \forall\, x \in \Omega\,, \end{cases}$$

*and $\widehat{w} = \widehat{w}(\vec{h})$ belongs to $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$.*

*Proof* : If the data are good enough, the time-dependent linearized two dimensional Navier-Stokes system is well posed(see [11] and [14] for details), and the solution $\widehat{w}$ belongs to $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$.

To show that $\vec{u}(\vec{g})$ is strictly differentiable, we need to prove

$$\text{(3.11)} \quad \|\vec{u}(\vec{g}+\tau\vec{h}) - \vec{u}(\vec{g}) - \tau\widehat{w}(\vec{h})\|_{L^2(0,T;\mathbf{V})} \le C\,|\tau|^k \quad \text{for some } k > 1\,.$$

If we set $\vec{r} = \vec{u}(\vec{g} + \tau\vec{h}) - \vec{u}(\vec{g}) - \tau\widehat{w}(\vec{h})$, where $\widehat{w}$ is a solution of the system (3.10), it is obvious that $\vec{r}$ is a solution of the system

$$(3.12)\quad \begin{cases} \dfrac{d\vec{r}}{dt} + \nu\mathcal{A}\vec{r} + \mathcal{B}(\vec{u}(\vec{g} + \tau\vec{h})) - \mathcal{B}(\vec{u}(\vec{g})) \\ \qquad\qquad -\tau\mathcal{B}'(\vec{u}(\vec{g}); \widehat{w}(\vec{h})) = \vec{0} \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\vec{r} = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \vec{r} = \vec{0} \quad \text{on } (0,T)\times\Gamma\,, \\ \vec{r}(0,x) = \vec{0} \quad \forall\, x\in\Omega\,. \end{cases}$$

Let us denote $\vec{u}(\vec{g}+\tau\vec{h}) - \vec{u}(\vec{g}) \equiv \widetilde{u}$. Obviously, $\widetilde{u}$ satisfies the following nonlinear system

$$(3.13)\quad \begin{cases} \dfrac{d\widetilde{u}}{dt} + \nu\mathcal{A}\widetilde{u} + \mathcal{B}'(\vec{u}(\vec{g}); \widetilde{u}) + \mathcal{B}(\widetilde{u}) = \vec{0} \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\widetilde{u} = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \widetilde{u} = \tau\vec{h} \quad \text{on } (0,T)\times\Gamma_c\,, \\ \widetilde{u} = \vec{0} \quad \text{on } (0,T)\times\Gamma_0\,, \\ \widetilde{u}(0,x) = \vec{0} \quad \forall\, x\in\Omega\,. \end{cases}$$

If we set

$$(3.14)\qquad \mathcal{B}(\vec{u}(\vec{g} + \tau\vec{h})) - \mathcal{B}(\vec{u}(\vec{g})) - \mathcal{B}'(\vec{u}(\vec{g}); \widetilde{u}) = \vec{\xi}\,,$$

then the system (3.12) is converted into the homogeneous system

$$(3.15)\quad \begin{cases} \dfrac{d\vec{r}}{dt} + \nu\mathcal{A}\vec{r} + \mathcal{B}'(\vec{u}(\vec{g}); \vec{r}) = -\vec{\xi} \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\vec{r} = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \vec{r} = \vec{0} \quad \text{on } (0,T)\times\Gamma\,, \\ \vec{r}(0,x) = \vec{0} \quad \forall\, x\in\Omega\,. \end{cases}$$

At this point, one cannot say the well-posedness of this system, since $\vec{\xi}$ contains the term $\widetilde{u}$. Assuming momentarily $\widetilde{u} \in L^2(0,T;\mathbf{V})$, we observe that for $\vec{v} \in L^2(0,T;\mathbf{V}_0)$,

$$\begin{aligned} |<\vec{\xi},\vec{v}>| &= |<(\mathcal{B}(\vec{u}+\tau\vec{h}),\vec{v}> - <\mathcal{B}(\vec{u}(\vec{g}),\vec{v}> \\ &\qquad\qquad - <\mathcal{B}'(\vec{u}(\vec{g});\widetilde{u}),\vec{v}>| \\ &= |\,b(\vec{u}(\vec{g}+\tau\vec{h});\vec{u}(\vec{g}+\tau\vec{h}),\vec{v}) - b(\vec{u}(\vec{g});\vec{u}(\vec{g}),\vec{v}) \\ &\qquad\qquad - b(\vec{u}(\vec{g});\widetilde{u},\vec{v}) - b(\widetilde{u};\vec{u}(\vec{g}),\vec{v})| \\ &= |\,b(\widetilde{u};\widetilde{u},\vec{v})| \qquad \text{from the orthogonality (1.18)} \\ &\le C\|\widetilde{u}\|^2\|\vec{v}\| \qquad \text{from (1.19)}\,. \end{aligned}$$

Hence, we have

$$\|\vec{\xi}\|_{\mathbf{V}^*} \le C\|\widetilde{u}\|^2\,, \quad \text{whenever } \widetilde{u} \text{ belongs to } L^2(0,T;\mathbf{V})\,. \tag{3.16}$$

In order to estimate $\|\vec{r}\|$, we take a weak product with $\mathcal{A}\vec{r}$ in (3.15). Then, using the facts $r(\vec{t},\cdot) \in \mathbf{V}_0$ and $\mathcal{B}'(\vec{u};\vec{r}) = \mathcal{B}(\vec{u},\vec{r}) + \mathcal{B}(\vec{r},\vec{u})$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\vec{r}\|^2 + \nu\,|\mathcal{A}\vec{r}|^2 + b(\vec{u};\vec{r},\mathcal{A}\vec{r}) + b(\vec{r};\vec{u},\mathcal{A}\vec{r}) = <\vec{\xi},\mathcal{A}\vec{r}>\ . \tag{3.17}$$

Note that by applying the Young's inequality, we obtain

$$|<\vec{\xi},\mathcal{A}\vec{r}>| \le \|\vec{\xi}\|_{\mathbf{V}^*}|\mathcal{A}\vec{r}| \le \alpha_1\|\vec{\xi}\|^2_{\mathbf{V}^*} + \beta_1|\mathcal{A}\vec{r}|^2$$

for some appropriate chosen positive constants $\alpha_1$ and $\beta_1$. Similarly, using the first inequality in (1.20) and (3.7), we have

$$\begin{aligned} |\,b(\vec{u};\vec{r},\mathcal{A}\vec{r})| &\le C\|\vec{u}\|\|\vec{r}\|^{1/2}|\mathcal{A}\vec{r}|^{3/2} \le \alpha_2\|\vec{u}\|^4\|\vec{r}\|^2 + \beta_2|\mathcal{A}\vec{r}|^2\,, \\ |\,b(\vec{r};\vec{u},\mathcal{A}\vec{r})| &\le C\|\vec{r}\|\|\vec{u}\|^{1/2}|\mathcal{A}\vec{u}|^{1/2}|\mathcal{A}\vec{r}|^{1/2} \\ &\le \alpha_3\|\vec{u}\||\mathcal{A}\vec{u}|\|\vec{r}\|^2 + \beta_3|\mathcal{A}\vec{r}|^2 \end{aligned}$$

for some constants $\alpha_i$ and $\beta_i$. One can take $\beta_i\,,(i=1,2,3)$, sufficiently small, so that $\delta = 2\nu - (\beta_1+\beta_2+\beta_3) > 0$. Combining all this results in (3.17), the following inequality is obtained:

$$\frac{d}{dt}\|\vec{r}\|^2 + \delta|\mathcal{A}\vec{r}|^2 \le \alpha\|\vec{\xi}\|^2_{\mathbf{V}^*} + \beta(\|\vec{u}\|^4 + \|\vec{u}\||\mathcal{A}\vec{r}|)\|\vec{r}\|^2\,, \tag{3.18}$$

where $\alpha$, $\beta$ and $\delta$ are all positive. Hence, from the Gronwall's inequality in Lemma 3.2, we obtain

$$\|\vec{r}\|^2 \le \alpha(t)\|\vec{\xi}\|^2_{\mathbf{V}^*}\,, \tag{3.19}$$

where $\alpha(t)$ is a bounded function of $t$. Consequently, combined with (3.16), we can derive

$$\|\vec{r}\|^2_{L^2(0,T;\mathbf{V})} = \int_0^T \|\vec{r}\|^2\,dt \le C\int_0^T \|\widetilde{u}\|^4\,dt \tag{3.20}$$

for some constant $C > 0$.

It remains to estimate $\|\widetilde{u}\|$, where $\widetilde{u}$ is a solution for the system (3.13) with the non-homogeneous boundary condition. To show that $\widetilde{u}$ belongs to $\mathrm{L}^2(0,T;\mathbf{V})$, let us decompose $\widetilde{u}$ as $\widetilde{u}_1 + \widetilde{u}_2$, where $\widetilde{u}_1$ is a solution of the linearized system with the non-homogeneous boundary condition along $\Gamma_c$

$$\begin{cases} \dfrac{d\widetilde{u}_1}{dt} + \nu\mathcal{A}\widetilde{u}_1 + \mathcal{B}'(\vec{u}(\vec{g});\widetilde{u}_1) = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\widetilde{u}_1 = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \widetilde{u}_1 = \tau\vec{h} \quad \text{on } (0,T)\times\Gamma_c\,, \\ \widetilde{u}_1 = \vec{0} \quad \text{on } (0,T)\times\Gamma_0\,, \\ \widetilde{u}_1(0,x) = \vec{0} \quad \forall\, x\in\Omega\,. \end{cases} \tag{3.21}$$

Then, $\widetilde{u}_2$ corresponds to the solution of the following nonlinear system with the homogeneous boundary condition

$$\begin{cases} \dfrac{d\widetilde{u}_2}{dt} + \nu\mathcal{A}\widetilde{u}_2 + \mathcal{B}'(\widetilde{u}_1 + \vec{u}(\vec{g});\widetilde{u}_2) \\ \qquad\qquad +\mathcal{B}(\widetilde{u}_2) = -\mathcal{B}(\widetilde{u}_1) \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\widetilde{u}_2 = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \widetilde{u}_2 = \vec{0} \quad \text{on } (0,T)\times\Gamma\,, \\ \widetilde{u}_2(0,x) = \vec{0} \quad \forall\, x\in\Omega\,. \end{cases} \tag{3.22}$$

Since the linearized system (3.21) is well posed, we have

$$\|\widetilde{u}_1\|_{L^2(0,T;\mathbf{V})} \le C\,|\tau|\|\vec{h}\|_{\Gamma_c} \tag{3.23}$$

for some constant $C > 0$.

To evaluate $\|\widetilde{u}_2\|$, taking a weak product with $\mathcal{A}\widetilde{u}_2$ in (3.22), then we obtain

$$\begin{aligned}(3.24)\quad &\frac{1}{2}\frac{d}{dt}\|\widetilde{u}_2\|^2 + \nu\,|\mathcal{A}\widetilde{u}_2| + b(\widetilde{u}_2, \widetilde{u}_1 + \vec{u}(\vec{g}), \mathcal{A}\widetilde{u}_2)\\ &+b(\widetilde{u}_1 + \vec{u}(\vec{g}), \widetilde{u}_2, \mathcal{A}\widetilde{u}_2) + b(\widetilde{u}_2, \widetilde{u}_2, \mathcal{A}\widetilde{u}_2) = -b(\widetilde{u}_1, \widetilde{u}_1, \mathcal{A}\widetilde{u}_2)\,.\end{aligned}$$

It is worthwhile to notice the followings:

$$\begin{aligned}|b(\widetilde{u}_1, \widetilde{u}_1, \mathcal{A}\widetilde{u}_2)| &\le C\,|\widetilde{u}_1|\|\widetilde{u}_1\||\mathcal{A}\widetilde{u}_1|^{1/2}|\mathcal{A}\widetilde{u}_2|\\ &\le \alpha_1|\widetilde{u}_1|^2\|\widetilde{u}_1\|^2|\mathcal{A}\widetilde{u}_1| + \beta_1|\mathcal{A}\widetilde{u}_2|^2\\ &\le \alpha_1|\tau|^2\|\vec{h}\|^2_{\Gamma_c}|\widetilde{u}_1|^2|\mathcal{A}\widetilde{u}_1|^2 + \beta_1|\widetilde{u}_2|^2\,,\end{aligned}$$

and

$$\begin{aligned}|b(\widetilde{u}_2, \widetilde{u}_2, \mathcal{A}\widetilde{u}_2)| &\le C\,|\widetilde{u}_2|^{1/2}\|\widetilde{u}_2\||\mathcal{A}\widetilde{u}_2|^{3/2}\\ &\le \alpha_2|\widetilde{u}_2|^2\|\widetilde{u}_2\|^2 + \beta_2|\mathcal{A}\widetilde{u}_2|^2\,.\end{aligned}$$

The constants listed above are all positive, which the values depend on the context. Above estimates are all obtained by using (3.6) and the second inequality in (1.20). Analogously from the other two terms for the trilinear forms in (3.24), one can separate the term $|\mathcal{A}\widetilde{u}_2|^2$ from the others. Hence, from (3.24) one can introduce the following differential inequality for $\|\widetilde{u}_2\|^2$:

$$(3.25)\qquad \frac{d}{dt}\|\widetilde{u}_2\|^2 \le \alpha|\tau|^2\|\vec{h}\|^2_{\Gamma_c} + \beta\psi(t)\|\widetilde{u}_2\|^2$$

for some appropriate constant $\alpha$ and $\beta$. Here, $\psi(t)$ is a nonnegative bounded function. Again by the Gronwall's inequality, it follows that

$$(3.26)\qquad \|\widetilde{u}_2\|^2 \le \alpha(t)|\tau|^2\|\vec{h}\|^2_{\Gamma_c}\,,$$

where $\alpha(t) > 0$ is bounded. Thus, combining this result with (3.23), we have

$$\begin{aligned}\|\widetilde{u}\|^2_{L^2(0,T;\mathbf{V})} &= \int_0^T \|\widetilde{u}_1 + \widetilde{u}_2\|^2\,dt\\ (3.27)\qquad &\le C\int_0^T (\|\widetilde{u}_1\|^2 + \|\widetilde{u}_2\|^2)\,dt\\ &\le C\,|\tau|^2\|\vec{h}\|^2_{\Gamma_c}\,.\end{aligned}$$

Consequently, joining (3.27) to (3.20), the following result is followed:

$$(3.28)\qquad \|\vec{r}\|^2_{L^2(0,T;\mathbf{V})} \le \int_0^T \|\vec{r}\|^2\,dt \le C\int_0^T \|\widetilde{u}\|^4\,dt \le C\,|\tau|^4$$

for some constant $C > 0$. Therefore, $k = 2$ in (3.11) is derived, and it completes the proof. □

REMARK : By taking a weak product with $\widetilde{u}$ in (3.13) and using the second inequality of (1.19), one can also show that $\widetilde{u}$ belongs to $L^\infty(0,T;\mathbf{H})$ and its essential bound is given by $|\widetilde{u}|_{L^\infty(0,T;\mathbf{H})} \leq C\,|\tau|\|\vec{h}\|_{\Gamma_c}$. Hence, along with (3.27), one can say that the nonlinear system (3.13) is well posed and the solution $\widetilde{u}$ belongs to $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$.

**3.2. Sensitivity for the performance functional.** In this section, we will examine the differentiability of the performance functional. As a result, we will get a first order necessary conditions for the optimal solution to the problem. For this purpose, we need the following preliminary result.

LEMMA 3.4 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^2$-boundary. Let $\vec{u}_0 \in \mathbf{V}$ and $\vec{h} \in \mathcal{W} = L^2(0,T;\mathbf{W})$ be given. Suppose $\widehat{w} \equiv \widehat{w}(\vec{h})$ is a solution of the system (3.10), then for every $\vec{e} \in L^2(0,T;\mathbf{H})$, we have*

$$\int_0^T \int_\Omega \vec{e} \cdot \widehat{w}(\vec{h})\, dxdt = \int_0^T \int_{\Gamma_c} -\nu \frac{\partial \vec{w}}{\partial \vec{n}} \cdot \vec{h}\, dsdt\,, \tag{3.29}$$

*where $\vec{w} = \vec{w}(\vec{e})$ is the solution of the adjoint linearized homogeneous problem*

$$\begin{cases} -\dfrac{d\vec{w}}{dt} + \nu\,\mathcal{A}\vec{w} + \mathcal{B}'(\vec{u}(\vec{g});\vec{w})^* = \vec{e} \quad \text{in } (0,T)\times\Omega\,, \\ \nabla\cdot\vec{w} = 0 \quad \text{in } (0,T)\times\Omega\,, \\ \vec{w} = \vec{0} \quad \text{on } (0,T)\times\Gamma\,, \\ \vec{w}(T,x) = \vec{0} \quad \forall\, x\in\Omega\,. \end{cases} \tag{3.30}$$

*Proof* : We note that since $\vec{e} \in L(0,T;\mathbf{H})$, the linearized system (3.30) has a solution in $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$(c.f. [11], [14]). To begin with, we provide some noticeable facts. Since $\widehat{w}(0,x) = 0 = \vec{w}(T,x)$ for all $x \in \Omega$, we have $\displaystyle\int_\Omega \vec{w}(t,x)\cdot\widehat{w}(t,x)\Big|_{t=0}^{T} dx = 0$.

Also, since the traces $\widehat{w}\Big|_{\Gamma_c} = \vec{h}$ and $\vec{w}\Big|_{\Gamma_c} = \vec{0}$, the Green's 2nd identity yields

$$
\begin{aligned}
&< \mathcal{A}\vec{w}, \widehat{w} > - < \mathcal{A}\widehat{w}, \vec{w} > \\
&\qquad = - < \gamma_c^1(\vec{w}), \gamma_c^0(\widehat{w}) >_{\Gamma_c} + < \gamma_c^1(\widehat{w}), \gamma_c^0(\vec{w}) >_{\Gamma_c} \\
&\qquad = - < \gamma_c^1(\vec{w}), \vec{h} >_{\Gamma_c} \ .
\end{aligned}
$$

Now, we replace $\vec{e}$ by the left hand equation for $\vec{w}$ in (3.30) and take integration by parts with respect to the time variable. Considering all of the above respects, we obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega \vec{e} \cdot \widehat{w}(\vec{h}) \, dxdt \\
&= \int_0^T < \Big( - \frac{d\vec{w}}{dt} + \nu \mathcal{A}\vec{w} + \mathcal{B}'(\vec{u};\vec{w})^* \Big), \widehat{w} > \, dt \\
&= \int_0^T < -\frac{d\vec{w}}{dt}, \widehat{w} > \, dt + \int_0^T < \nu\mathcal{A}\vec{w}, \widehat{w} > \, dt + \int_0^T < \mathcal{B}'(\vec{u},\vec{w})^*, \widehat{w} > \, dt \\
&= \int_0^T < \frac{d\widehat{w}}{dt}, \vec{w} > \, dt + \int_0^T < \nu\mathcal{A}\widehat{w}, \vec{w} > \, dt \\
&\qquad\qquad - \int_0^T < \nu\gamma_c^1(\vec{w}), \vec{h} >_{\Gamma_c} dt + \int_0^T < \mathcal{B}'(\vec{u};\widehat{w}), \vec{w} > \, dt \\
&= \int_0^T < \Big( \frac{d\widehat{w}}{dt} + \nu\mathcal{A}\widehat{w} + \mathcal{B}'(\vec{u};\widehat{w}) \Big), \vec{w} > \, dt - \int_0^T < \nu\gamma_c^1(\vec{w}), \vec{h} >_{\Gamma_c} dt \\
&= \int_0^T - < \nu\gamma_c^1(\vec{w}), \vec{h} >_{\Gamma_c} \, dt = \int_0^T \int_{\Gamma_c} -\nu \frac{\partial \vec{w}}{\partial \vec{n}} \cdot \vec{h} \, dsdt \, . \qquad \Box
\end{aligned}
$$

We are now ready to examine the differential structure for the performance functional.

THEOREM 3.5 *Let $(\vec{u}, \vec{g}) \in \mathcal{U}_{ad}$ be an optimal pair for the boundary control problem (2.4) with $\vec{u} = \vec{u}(\vec{g})$. Then the Gateaux-derivative of the functional $\mathcal{J}$ at $\vec{g}$ in the $\vec{h}$-direction is given by*

$$
d\mathcal{J}(\vec{g};\vec{h}) = \int_0^T < \vec{\sigma}(\widetilde{w}) + \alpha\,\vec{g}, \vec{h} >_{\Gamma_c} \, dt \, , \tag{3.31}
$$

*where $\widetilde{w}$ is the solution of the adjoint linearized system*

$$
(3.32)\quad \begin{cases} -\dfrac{d\widetilde{w}}{dt} + \nu\,\mathcal{A}\widetilde{w} + \mathcal{B}'(\vec{u};\widetilde{w})^* = 2\nu\,\mathcal{A}\vec{u} \quad in\ (0,T)\times\Omega\,, \\ \nabla\cdot\widetilde{w} = 0 \quad in\ (0,T)\times\Omega\,, \\ \widetilde{w} = \vec{0} \quad on\ (0,T)\times\Gamma\,, \\ \widetilde{w}(T,x) = \vec{0} \quad \forall\, x\in\Omega\,, \end{cases}
$$

*and the vector field $\vec{\sigma} = -\nu\gamma_c^1(\widetilde{w})$ acting along the control part $\Gamma_c$ is defined by*

$$
(3.33)\qquad \int_{\Gamma_c} \vec{\sigma}\cdot\vec{v}\,ds = < \left(-\frac{d\widetilde{w}}{dt} + \nu\mathcal{A}\widetilde{w} + \mathcal{B}'(\vec{u};\widetilde{w})^* + 2\nu\mathcal{A}\vec{u}\right), \vec{v} >
$$

for every $\vec{v}\in\mathbf{H}_0^1(\Omega)$.

*Proof*: We need to recall Theorem 3.3 that the state solution $\vec{u} = \vec{u}(\vec{g})$ is differentiable and its differential is given by

$$
d\vec{u}(\vec{g};\vec{h}) = \widehat{w}(\vec{h})\,,
$$

where $\widehat{w} = \widehat{w}(\vec{h})$ is the solution of the linearized system (3.10). Also, from (1.13), it is followed

$$
\int_\Omega D(\vec{u}) : D(\widehat{w})\,dx = -\frac{1}{2}\int_\Omega \Delta\vec{u}\cdot\widehat{w}\,dx\,.
$$

If we evaluate the Gateaux-derivative of $\mathcal{J}$ at $\vec{g}$ in the $\vec{h}$-direction, then by Lemma 3.4 we have

$$
\begin{aligned}
d\mathcal{J}(\vec{g};\vec{h}) &= 4\nu\int_0^T\int_\Omega D(\vec{u}(\vec{g})) : D(d\vec{u}(\vec{g};\vec{h}))\,dxdt + \alpha\int_0^T\int_{\Gamma_c}\vec{g}\cdot\vec{h}\,dsdt \\
&= 4\nu\int_0^T\int_\Omega D(\vec{u}(\vec{g})) : D(\widehat{w}(\vec{h}))\,dxdt + \alpha\int_0^T\int_{\Gamma_c}\vec{g}\cdot\vec{h}\,dsdt \\
&= \int_0^T\int_\Omega(-2\nu\Delta\vec{u})\cdot\widehat{w}(\vec{h})\,dxdt + \alpha\int_0^T\int_{\Gamma_c}\vec{g}\cdot\vec{h}\,dsdt \\
&= \int_0^T\int_{\Gamma_c} -\nu\frac{\partial\widetilde{w}}{\partial\vec{n}}\cdot\vec{h}\,dsdt + \alpha\int_0^T\int_{\Gamma_c}\vec{g}\cdot\vec{h}\,dsdt\,,
\end{aligned}
$$

where $\widetilde{w}$ is the solution of the adjoint linearized system (3.32). Hence, the Gateaux-derivative to the functional $\mathcal{J}$ can be given by

$$
(3.34)\qquad d\mathcal{J}(\vec{g};\vec{h}) = \int_0^T\int_{\Gamma_c}\left(-\nu\frac{\partial\widetilde{w}}{\partial\vec{n}} + \alpha\vec{g}\right)\cdot\vec{h}\,dsdt\,.
$$

The formulation for the control action $\vec{\sigma}$ expressed in (3.33) can be identified by taking analogous procedure as shown in the proof of Lemma 3.4. It is implicitly used that the $\gamma_c^0$ has a continuous lifting in $\mathbf{H}^1(\Omega)$. □

As a result of Theorem 3.5, we can say that the functional $\mathcal{J}$ is Gateaux-differentiable in the space $\mathcal{W} = L^2(0,T;\mathbf{W})$, and the Gateaux-differential corresponds to the gradient of the functional $\mathcal{J}$, so that in the duality pairing between $\mathcal{W}^*$ and $\mathcal{W}$ the differential framework for $\mathcal{J}$ can be understood by

$$d\mathcal{J}(\vec{g};\vec{h}) = < \nabla\mathcal{J}(\vec{g}), \vec{h} >_{\mathcal{W}^*} = \int_0^T < \vec{\sigma}(\widetilde{w}) + \alpha\vec{g}, \vec{h} >_{\Gamma_c} dt\,. \tag{3.35}$$

Hence, compared with (3.34), the gradient can be written by

$$\nabla\mathcal{J}(\vec{g}) = \alpha\vec{g} - \nu\frac{\partial\widetilde{w}}{\partial\vec{n}} \quad \text{along } (0,T)\times\Gamma_c\,. \tag{3.36}$$

## 4. The optimality system and some remarks on algorithm

In the previous section, with the aid of the strict differentiability of the velocity field we have derived the differential structure of the first order for the performance functional. In this procedure, the differential for $\vec{u} = \vec{u}(\vec{g})$ has basically contributed to determine the form of the adjoint system (3.32) through the relation (3.29). The relation (3.29) practically plays the role of transferring the derivation terms of $\mathcal{J}$ in $\Omega$ to the boundary control actuator.

Suppose $\vec{g} \in \mathcal{W}$ is a minimizer for $\mathcal{J}$. Since we do not have any constraint on the control space, it is necessary to have

$$d\mathcal{J}(\vec{g};\vec{h}) = \frac{d}{d\tau}\mathcal{J}(\vec{g}+\tau\vec{h})\Big|_{\tau=0} = 0 \quad \forall\,\vec{h}\in\mathcal{W}\,.$$

Hence, the minimizer $\vec{g}$ should satisfy the equation

$$\alpha\vec{g} - \nu\,\frac{\partial\widetilde{w}}{\partial\vec{n}} = \vec{0} \quad \text{along } (0,T)\times\Gamma_c\,. \tag{4.1}$$

This can be regarded as a first order necessary conditions for the minimizer. Thus, we have close at hand the optimality system that is needed to obtain the solution of the optimal control problem.

We have to solve

◦ the Navier-Stokes system: seek $\vec{u} = \vec{u}(\vec{g}) \in L^2(0,T;\mathbf{V})$ satisfying

$$(4.2)\qquad \begin{cases} \dfrac{d\vec{u}}{dt} + \nu\mathcal{A}\vec{u} + \mathcal{B}(\vec{u}) = \vec{f} \quad \text{in } (0,T)\times\Omega\,, \\ \vec{u} = \vec{g} \quad \text{on } (0,T)\times\Gamma_c\,, \\ \vec{u} = \vec{0} \quad \text{on } (0,T)\times\Gamma_0\,, \\ \vec{u}(0,x) = \vec{u}_0(x) \quad \forall\, x\in\Omega\,; \end{cases}$$

◦ the adjoint system: seek $\widetilde{w} = \widetilde{w}(\vec{u}) \in L^2(0,T;\mathbf{V}_0)$ satisfying

$$(4.3)\qquad \begin{cases} -\dfrac{d\widetilde{w}}{dt} + \nu\,\mathcal{A}\widetilde{w} + \mathcal{B}'(\vec{u};\widetilde{w})^* = 2\nu\,\mathcal{A}\vec{u} \ \text{ in } (0,T)\times\Omega\,, \\ \widetilde{w} = \vec{0} \ \text{ on } (0,T)\times\Gamma\,, \\ \widetilde{w}(T,x) = \vec{0} \quad \forall\, x\in\Omega\,; \end{cases}$$

◦ the boundary control equation: seek $\vec{g} \in L^2(0,T;\mathbf{W})$ satisfying

$$(4.4)\qquad \alpha\vec{g} - \nu\,\frac{\partial\widetilde{w}}{\partial\vec{n}} = \vec{0} \quad \text{along } (0,T)\times\Gamma_c\,.$$

Remark : According to (1.22), the first equation in the adjoint system (4.3) constitutes a weak formulation of the equation

$$-\frac{d\widetilde{w}}{dt} - \nu\,\Delta\widetilde{w} + (\nabla\vec{u})^t\widetilde{w} - (\vec{u}\cdot\nabla)\widetilde{w} + \nabla\widetilde{q} = -2\nu\,\Delta\vec{u}\,.$$

In this expression, $\widetilde{q}$ corresponds to the adjoint variable for the pressure $p$ in the state equation (1.1). This is also identified by de Rham's lemma. We also remark that the adjoint system has to be solved by backward time steps.

Since $\widetilde{w}$ is a function of $\vec{u}$ which is the solution of nonlinear systems (1.1)–(1.7), the equation (4.1) is inevitably nonlinear. Hence, one can solve it with the aid of deliberately designed numerical algorithm. As suggested in [3], we recommend to examine Fletcher-Powell method, when we are available the Gateaux-derivative of the first order of the functional to be minimized. Of course, it is not so simple finding an efficient algorithm to implement this kind of problem. Even though one may propose an effective algorithm depended on purely theoretical basis, the ongoing practices seem to be another problem.

For the sake of completeness, we provide a basic algorithm based on the gradient method:

- **(step I: initialization)** given $\vec{g}^{(0)} \in \mathcal{W}$, solve $\vec{u}^{(0)} = \vec{u}(\vec{g}^{(0)})$, and then find $\widetilde{w}^{(0)} = \widetilde{w}(\vec{u}^{(0)})$.
- **(step II: update)** given $\rho > 0$, set $\vec{g}^{(1)} = \vec{g}^{(0)} - \rho\,(\alpha\vec{g}^{(0)} - \nu\gamma_c^1(\widetilde{w}^{(0)}))$.
- **(step III: iteration)** recursively, evaluate the following until satisfactory

$$\vec{g}^{(n+1)} = \vec{g}^{(n)} - \rho\,(\alpha\vec{g}^{(n)} - \nu\gamma_c^1(\widetilde{w}^{(n)}))\,,$$

where $\vec{u}^{(n)} = \vec{u}(\vec{g}^{(n)})$ and $\widetilde{w}^{(n)} = \widetilde{w}(\vec{u}^{(n)})$.

It is evident that we have to solve the state system (4.2) and the adjoint system (4.3) at each iterations, so that the numerical tasks required are quite massive.

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