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UNIQUE POSITIVE SOLUTION FOR A CLASS OF THE SYSTEM OF THE NONLINEAR SUSPENSION BRIDGE EQUATIONS

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Abstract. We prove the existence of a unique positive solution for a class of systems of the following nonlinear suspension bridge equations with Dirichlet boundary conditions and periodic conditions

$$\begin{cases} u_{tt} + u_{xxxx} + \frac{1}{4}u_{ttxx} & +av^+ = \phi_{00} + \epsilon_1 h_1(x,t) \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \\ v_{tt} + v_{xxxx} + \frac{1}{4}u_{ttxx} & +bu^+ = \phi_{00} + \epsilon_2 h_2(x,t) \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \end{cases}$$

where $u^+ = \max\{u, 0\}$, ϵ_1, ϵ_2 are small numbers and $h_1(x,t)$, $h_2(x,t)$ are bounded, π-periodic in t and even in x and t and $\|h_1\| = \|h_2\| = 1$. We first show that the system has a positive solution, and then prove the uniqueness by the contraction mapping principle on a Banach space.

1. Introduction and main result

In this paper we investigate the uniqueness of the solution of the following system of the nonlinear suspension bridge equations with Dirichlet boundary conditions and periodic conditions

$$\begin{cases} u_{tt} + u_{xxxx} & +\frac{1}{4}u_{ttxx} + av^+ = \phi_{00} + \epsilon_1 h_1(x,t) \\ & \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \\ v_{tt} + v_{xxxx} & +\frac{1}{4}u_{ttxx} + bu^+ = \phi_{00} + \epsilon_2 h_2(x,t) \\ & \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \\ u(\pm\frac{\pi}{2}, t) & = u_{xx}(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = v_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{1.1}$$

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where $u^+ = \max\{u, 0\}$, ϵ_1, ϵ_2 are small numbers and $h_1(x,t)$, $h_2(x,t)$ are bounded, π-periodic in t, even in x and t and $\|h_1\| = \|h_2\| = 1$. Here the effect of the inertia term u_{ttxx} is weak in the oscillation of the beam. So we take the small coefficient $\frac{1}{4}$. McKenna and Walter([6]) found the physical model of jumping problem from a bridge suspended by cables under a load. System (1.1) of the nonlinear suspension bridge equations with Dirichlet boundary condition is considered as a model of the cross of the two nonlinear oscillations in differential equation. For the case of the single suspension bridge equation McKenna and Walter ([6]), Choi and Jung ([3], [4] and [5]) etc., investigate the multiplicity of the solutions via the degree theory or the critical point theory or the variational reduction method. In this paper we improve the multiplicity results of the single suspension bridge equation to the case of the system of the nonlinear suspension bridge equations. The system (1.1) can be rewritten by

$$\begin{cases} U_{tt} + U_{xxxx} & + \frac{1}{4}U_{ttxx} + AU^+ = \binom{\phi_{00}+\epsilon_1 h_1(x,t)}{\phi_{00}+\epsilon_2 h_2(x,t)}, \\ U(\pm\frac{\pi}{2}, t) & = U_{xx}(\pm\frac{\pi}{2}, t) = \binom{0}{0}, \\ U(x, t+\pi) & = U(x,t) = U(-x,t) = U(x,-t), \end{cases} \tag{1.2}$$

where $U = \binom{u}{v}$, $U^+ = \binom{u^+}{v^+}$, $U_{tt} = \binom{u_{tt}}{v_{tt}}$, $U_{xxxx} = \binom{u_{xxxx}}{v_{xxxx}}$, $U_{ttxx} = \binom{u_{ttxx}}{v_{ttxx}}$, $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in M_{2\times 2}(R)$. The eigenvalue problem for $u(x,t)$,

$$u_{tt} + u_{xxxx} + \frac{1}{4}u_{ttxx} = \lambda u \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R,$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0,$$

$$u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t)$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 + (2n+1)^2 m^2 - 4m^2 \qquad (m, n = 0, 1, 2, \ldots)$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m, n \geq 0)$ given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi}\cos(2n+1)x \qquad \text{for } n \geq 0,$$

$$\phi_{mn} = \frac{2}{\pi}\cos 2mt \cdot \cos(2n+1)x \qquad \text{for } m > 0, n \geq 0.$$

We can check easily that the eigenvalues in the interval (-26,81) are given by

$$\lambda_{20} = -11 < \lambda_{10} = -1 < \lambda_{00} = 1.$$

The main result of this paper is the following:

THEOREM 1.1. *(Existence of the unique positive solution) Assume that*

$$\lambda_{mn}^2 + ab \neq 0, \qquad \textit{for all} \;\; m, n \tag{1.3}$$

$$a < 0, \qquad b < 0 \qquad \textit{and} \qquad 1 - ab > 0. \tag{1.4}$$

Then, for each $h_1(x,t)$, $h_2(x,t) \in H$ *with* $\|h_1(x,t)\| = 1$, $\|h_2(x,t)\| = 1$, *there exist small numbers* ϵ_1 *and* ϵ_2 *such that system (1.1) has a unique positive solution.*

In section 2 we show that system (1.1) has a positive solution by direct computation. In section 3 we prove the uniqueness by the contraction mapping principle on the Banach space.

2. Existence of the positive solution

Let Q be the square $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) |\ u \text{ is even in } x \text{ and } t \text{ and } \int_Q u = 0\}.$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u, in H_0, by

$$u = \sum h_{mn}\phi_{mn}.$$

We define a subspace H of H_0 as follows

$$H = \{u \in H_0 : \sum_{mn} |\lambda_{mn}| h_{mn}^2 < \infty\}.$$

Then this space is a Banach space with norm

$$\|u\|^2 = [\sum |\lambda_{mn}| h_{mn}^2]^{\frac{1}{2}}.$$

Let us set $E = H \times H$. We endow the Hilbert space E the norm

$$\|(u, v)\|_E^2 = \|u\|^2 + \|v\|^2 \qquad \forall (u, v) \in E.$$

We are looking for the weak solutions of (1.1) in E, that is, (u, v) such that $u \in H$, $v \in H$, $u_{tt}+u_{xxxx}+av^+ = \cos x+\epsilon_1 h_1(x,t)$, $v_{tt}+v_{xxxx}+bu^+ = \cos x + \epsilon_2 h_2(x,t)$.

Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

LEMMA 2.1. *(i)* $\|u\| \geq \|u\|_{L^2(Q)}$, *where* $\|u\|_{L^2(Q)}$ *denotes the* L^2 *norm of* u.
(ii) $\|u\| = 0$ *if and only if* $\|u\|_{L^2(Q)} = 0$.
(iii) $u_{tt} + u_{xxxx} \in H$ *implies* $u \in H$.

LEMMA 2.2. *Suppose that* c *is not an eigenvalue of* L, $Lu = u_{tt}+u_{xxxx}$, *and let* $f \in H_0$. *Then we have* $(L-c)^{-1}f \in H$.

Proof. When n is fixed, we define

$$\lambda_n^+ = \inf_m \{\lambda_{mn} : \ \lambda_{mn} > 0\} = 8n^2 + 8n + 1,$$

$$\lambda_n^- = \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -8n^2 - 8n - 3.$$

We see that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$ as $n \to \infty$. Hence the number of elements in the set $\{\lambda_{mn} : \ |\lambda_{mn}| < |c|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$f = \sum h_{mn}\phi_{mn}.$$

Then

$$(L-c)^{-1}f = \sum \frac{1}{\lambda_{mn} - c} h_{mn}\phi_{mn}.$$

Hence we have the inequality

$$\|(L-c)^{-1}f\| = \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn}-c)^2} h_{mn}^2 \leq C \sum h_{mn}^2$$

for some C, which means that

$$\|(L-c)^{-1}f\| \leq C_1 \|f\|_{L^2(Q)}, \qquad C_1 = \sqrt{C}.$$

□

LEMMA 2.3. *Assume that conditions (1.3) and (1.4) hold. Then the system*

$$\begin{cases} u_{tt} + u_{xxxx} & +\frac{1}{4}u_{ttxx} + av = \phi_{00} \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\ v_{tt} + v_{xxxx} & +\frac{1}{4}u_{ttxx} + bu = \phi_{00} \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\ u(\pm\frac{\pi}{2}, t) & = u_{xx}(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = v_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{2.1}$$

has a positive solution $(u_*, v_*) \in E$*, which is of the form*

$$u_* = [\frac{-a}{\lambda_{00}}\frac{-b+\lambda_{00}}{\lambda_{00}^2 - ab} + \frac{1}{\lambda_{00}}]\phi_{00}, \tag{2.2}$$

$$v_* = [\frac{-b+\lambda_{00}}{\lambda_{00}^2 - ab}]\phi_{00}.$$

Proof. We note that (u^*, v^*) is a solution of the system (2.1) with $u_* > 0$ and $v_* > 0$. □

Define $\mathcal{L}U = (Lu, Lv)$, $Lu = u_{tt} + u_{xxxx} + \frac{1}{4}u_{ttxx}$. We need to find a spectral analysis for the linear operator $\mathcal{L}U + AU$. The following lemma need a simple 'Fourier Series' argument.

Lemma 2.4. *Let* a, $b \in R$ *and let* $\mathcal{L}_{ab} : H \times H \to H_0 \times H_0$ *be defined by* $\mathcal{L}_{ab}(u, v) = (Lu + av, Lv + bu)$*. For* $\mu \in R$ *we have*
(a) if $(\lambda_{mn} - \mu)^2 \neq ab$ *for every* m, n, *then*

$$(\mathcal{L}_{ab} - \mu I)^{-1} : H_0 \times H_0 \to H_0 \times H_0$$

is well defined and continuous;
(b) if $(\lambda_{mn} - \mu)^2 = ab$ *for some* m, n, *then*

$$Ker(\mathcal{L}_{ab} - \mu I) = span\{\phi_{mn} : \ (\lambda_{mn} - \mu)^2 = ab\};$$

moreover if $X_\mu = \overline{span\{\phi_{mn} : \ (\lambda_{mn} - \mu)^2 \neq ab\}}$, *then*

$$(\mathcal{L}_{ab} - \mu I)^{-1} : X_\mu \times X_\mu \to X_\mu \times X_\mu$$

is well defined and continuous.
Notice that if $ab < 0$*, the second case can never occur.*

Proof. To prove (a) we take (f, g) in $H_0 \times H_0$. We can write $f = \sum_{mn} f_{mn}\phi_{mn}$ with $\sum_{mn} f_{mn}^2 < +\infty$ and $g = \sum_{mn} g_{mn}\phi_{mn}$ with $\sum_{mn} g_{mn}^2 < +\infty$. We define, for m, n integers,

$$u_{mn} = \frac{(\lambda_{mn} - \mu)f_{mn} - ag_{mn}}{(\lambda_{mn} - \mu)^2 - ab}, \qquad v_{mn} = \frac{(\lambda_{mn} - \mu)g_{mn} - bf_{mn}}{(\lambda_{mn} - \mu)^2 - ab},$$

which make sense since $(\lambda_{mn} - \mu)^2 \neq ab$ for every m, n. We have

$$|u_{mn}| \leq \frac{C}{|\lambda_{mn}|}(|f_{mn}| + |g_{mn}|) \Longrightarrow \lambda_{mn}^2 u_{mn}^2 \leq C_1(f_{mn}^2 + g_{mn}^2)$$

for suitable constants C, C_1 not depending on mn. The same inequality applies for v_{mn}. So if $u = \sum_{mn} u_{mn}\phi_{mn}$, $v = \sum_{mn} v_{mn}\phi_{mn}$, then $(u, v) \in H \times H$. Arguing componentwise it is simple to check that $\mathcal{L}_{ab}(u, v) -$

$\mu I(u,v) = (f,g)$. So $(\mathcal{L}_{ab} - \mu I)^{-1} : H_0 \times H_0 \to H_0 \times H_0$ is well defined. To prove (b) we first observe that if $(\lambda_{mn} - \mu)^2 = ab$, then $(\mathcal{L}_{ab} - \mu I)\phi_{mn} = 0$, as one can easily check. Secondly given (f,g) in X_μ we can argue as in the first case since $f_{mn} = g_{mn} = 0$ for all mn such that $(\lambda_{mn} - \mu)^2 = ab$. This allows to define u_{mn} and v_{mn} as before for all mn such that $(\lambda_{mn} - \mu)^2 \neq ab$ and $u_{mn} = v_{mn} = 0$ for all mn such that $(\lambda_{mn} - \mu)^2 = ab$. □

Using Lemma 2.4 with the case $\mu = 0$ we can easily derive Lemma 2.5

LEMMA 2.5. *Assume that the conditions (1.3) and (1.4) hold. Then for each* $h_1(x,t)$, $h_2(x,t) \in H_0$ *with* $\|h_1\| = 1$ *and* $\|h_2\| = 1$, *there exist small numbers* ϵ_1 *and* ϵ_2 *such that system*

the system

$$\begin{cases} u_{tt} + u_{xxxx} & +av = \epsilon_1 h_1(x,t), \\ v_{tt} + v_{xxxx} & +bu = \epsilon_2 h_2(x,t), \\ u(\pm\frac{\pi}{2},t) & = u_{xx}(\pm\frac{\pi}{2},t) = v(\pm\frac{\pi}{2},t) = v_{xx}(\pm\frac{\pi}{2},t) = 0, \\ u(x,t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x,t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{2.3}$$

has a unique solution $(u_{\epsilon_1\epsilon_2}, v_{\epsilon_1\epsilon_2}) \in E = H \times H$.

PROOF OF THE EXISTENCE OF A POSITIVE SOLUTION By Lemma 2.3 and Lemma 2.5, $(u_* + u_{\epsilon_1\epsilon_2}, v_* + v_{\epsilon_1\epsilon_2})$ is a solution of the system

$$\begin{cases} u_{tt} + u_{xxxx} & +av = \phi_{00} + \epsilon_1 h_1(x,t), \\ v_{tt} + v_{xxxx} & +bv = \phi_{00} + \epsilon_2 h_2(x,t), \\ u(\pm\frac{\pi}{2},t) & = u_{xx}(\pm\frac{\pi}{2},t) = v(\pm\frac{\pi}{2},t) = v_{xx}(\pm\frac{\pi}{2},t) = 0, \\ u(x,t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x,t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{2.4}$$

where $u_* = [\frac{-a}{\lambda_{00}}\frac{-b+\lambda_{00}}{\lambda_{00}^2-ab} + \frac{1}{\lambda_{00}}]\phi_{00} > 0$, $v_* = [\frac{-b+\lambda_{00}}{\lambda_{00}^2-ab}]\phi_{00} > 0$. By Lemma 2.4, $u_{\epsilon_1\epsilon_2} \in H$ and $v_{\epsilon_1\epsilon_2} \in H$. Since the elements of H lies in C^1, the elements $u_{\epsilon_1\epsilon_2}$, $v_{\epsilon_1\epsilon_2} \in C^1$. Thus we can find small numbers ϵ_1 and ϵ_2 such that $u_* + u_{\epsilon_1\epsilon_2} > 0$ and $v_* + v_{\epsilon_1\epsilon_2} > 0$, which is also a positive solution of system (1.1).

3. Uniqueness

Assume that the conditions (1.3) and (1.4) hold. To prove the uniqueness of Theorem 1.1 we will use the contraction mapping principle. By the assumption (1.4), $-1 < -\sqrt{ab} < 0 < \sqrt{ab} < 1 < 3 = -\lambda_{10}$. Let us set $\delta = -\frac{1}{2}(-3+1) = 1$. The system (1.2) is equivalent to

$$U = (\mathcal{L} + \delta I)^{-1}[(\delta I - A)U^+ - \delta I U^- + \begin{pmatrix} \phi_{00} + \epsilon_1 h_1(x,t) \\ \phi_{00} + \epsilon_2 h_2(x,t) \end{pmatrix}], \tag{3.1}$$

where $U^+ = \binom{u^+}{v^+}$, $U^- = \binom{u^-}{v^-}$ and $(L+\delta)^{-1}$ is a compact, self-adjoint, linear map from $H_0 \times H_0$ into $H_0 \times H_0$ with norm $\frac{1}{2}$. We note that

$$\|(\delta I - A)(U_2^+ - U_1^+) - \delta I(U_2^- - U_1^-)\| \le \max det(\delta I - A), det(\delta I)\|U_2 - U_1\|$$
$$< \|U_2 - U_1\|.$$

It follows that the right hand side of (3.1) defines a Lipschitz mapping of $H_0 \times H_0$ into $H_0 \times H_0$ with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $U = \binom{u}{v} \in H_0 \times H_0$ of (1.2). By Lemma 2.2, $U = \binom{u}{v} \in H \times H$. Thus the uniqueness of the solution of System (1.1) is proved. Thus the positive solution $(u_* + u_{\epsilon_1\epsilon_2}, v_* + v_{\epsilon_1\epsilon_2})$ is the unique solution of system (1.1). Thus we prove Theorem 1.1.

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