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THE PROOF OF THE EXISTENCE OF THE THIRD SOLUTION OF A NONLINEAR BIHARMONIC EQUATION BY DEGREE THEORY

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Abstract. We investigate the multiplicity of solutions of the nonlinear biharmonic equation with Dirichlet boundary condition, $\Delta^2 u + c\Delta u = bu^+ + s$, in Ω, where $c \in R$ and Δ^2 denotes the biharmonic operator. We show by degree theory that there exist at least three solutions of the problem.

1. Introduction

Let Ω be a bounded set in R^n with smooth boundary $\partial\Omega$. In this paper we study the multiplicity of the solutions of the nonlinear biharmonic equation with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where $u^+ = \max\{u, 0\}$, $c \in R$, $s \in R$ and Δ^2 denotes the biharmonic operator. Equations with nonlinearities of this type have been extensively studied for the second order elliptic operators (cf. [7]). Tarantello [12] also studied this type equation. She showed that if $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and $c < \lambda_1$, then the problem

$$\Delta^2 u + c\Delta u = b[(u+1)^+ - 1] \qquad \text{in } \Omega,$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

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has at least one solution, which is negative if and only if $b \geq \lambda_1(\lambda_1 - c)$. Choi and Jung [4] proved that if $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, or if $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$, then problem (1.1) has at least two solutions by use of the variational reduction method.

In this paper, we prove that if $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, then problem (1.1) has at least three solutions by use of the degree theory.

In section 2 we state the main result and in section 3 we prove the main theorem.

2. Statement of main result

Let Ω be a bounded set in R^n with smooth boundary $\partial\Omega$. Let λ_k, $k = 1, 2, \ldots$, denote the eigenvalues and ϕ_k, $k = 1, 2, \ldots$, the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem

$$\Delta^2 u + c\Delta u = \nu u \qquad \text{in } \Omega,$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has infinitely many eigenvalues

$$\nu_k = \lambda_k(\lambda_k - c), \qquad k = 1, 2, \ldots,$$

and corresponding eigenfunctions $\phi_k(x)$. The set of functions $\{\phi_k\}$ is an orthonormal base for $L^2(\Omega)$. Let us denote an element u, in $L^2(\Omega)$, as

$$u = \sum h_k \phi_k, \qquad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \sum |\lambda_k(\lambda_k - c)| h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$|||u||| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have
(1) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(2) $|||u||| \geq C\|u\|$ for some $C > 0$.
(3) $\|u\|_{L^2(\Omega)} = 0$ if and only if $|||u||| = 0$.
For the proof, refer to Choi and Jung [4].
In this paper we consider weak solutions of the boundary value problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{2.1}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega.$$

A weak solution of (2.1), which is called a solution in H, is of the form

$$u = \sum h_k \phi_k, \qquad \Delta^2 u + c\Delta u = \sum \lambda_k(\lambda_k - c) h_k \phi_k \in L^2(\Omega).$$

For simplicity of notation, a weak solution of (2.1) is characterized by

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } H. \tag{2.2}$$

Now, we state the main result of this paper, which is a sharp result for the multiplicity of solutions of a nonlinear biharmonic equation.

THEOREM 2.1. Assume that $c < \lambda_1, \lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$. Then problem (2.2) has at least three solutions.

3. Proof of theorem 2.1

For the proof of Theorem 2.1 we need some lemmas.

LEMMA 3.1. Let $c < \lambda_1$, $b \geq 0$ and $b \neq \lambda_k(\lambda_k - c)$, $k \geq 1$, Then the problem

$$\Delta^2 u + c\Delta u = bu^+ \qquad \text{in } H \tag{3.1}$$

has only the trivial solution.

For the proof, refer to Theorem 1.3 (ii) and Lemma 2.9 in [4].

LEMMA 3.2. Let $c < \lambda_1$, $s < 0$ and $\alpha > 0$ be given. Then there exists an $R_0 > 0$ (depending on s and α) such that for all b with $\lambda_1(\lambda_1 - c) + \alpha \leq b \leq \lambda_2(\lambda_2 - c) - \alpha$, the solutions u of (2.2) satisfy $|||u||| < R_0$.

Proof. If not, then there exists a sequence (b_n, u_n) with $\lambda_1(\lambda_1 - c) + \alpha \leq b_n \leq \lambda_2(\lambda_2 - c) - \alpha$, $|||u||| \to \infty$ such that

$$u_n = (\Delta^2 + c\Delta)^{-1}(bu_n^+ + s). \tag{3.3}$$

The functions $w_n = \frac{u_n}{|||u_n|||}$ satisfy the equation

$$w_n = (\Delta^2 + c\Delta)^{-1}(bw_n^+ + \frac{s}{|||u_n|||}).$$

Now $(\Delta^2 + c\Delta)^{-1}$ is a compact operator. Therefore we may assume that $w_n \to w_0$, $b_n \to b_0$ and $0 < \lambda_1(\lambda_1 - c) < b_0 < \lambda_2(\lambda_2 - c)$. Since $|||w_n||| = 1$, it follows that $|||w_0||| = 1$ and

$$w_0 = (\Delta^2 + c\Delta)^{-1}(bw_0^+) \qquad \text{in } H. \tag{3.4}$$

This contradicts Lemma 3.1 and proved the lemma.

LEMMA 3.3. Under the assumptions and the notations of Lemma 3.2

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_R, 0) = 1$$

for all $R \geq R_0$, where d_{LS} denotes the Leray-Schauder degree.

Proof. Let R_0 be such that solutions of

$$u - (\Delta^2 + c\Delta)^{-1}(bu^+ + \lambda s) = 0, \qquad 0 \leq \lambda \leq 1,$$

satisfy $|||u||| \leq R_0$. Since the degree is invariant under a homotopy, we get

$$\begin{aligned} & d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + \lambda s), B_R(0), 0) \\ = \;& d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+), B_R(0), 0) \end{aligned}$$

for $R \geq R_0$. The equation

$$u - (\Delta^2 + c\Delta)^{-1}(bu^+) = 0$$

has only the trivial solution $u = 0$ in $B_R(0)$. Thus we have

$$\begin{aligned} & d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+), B_R(0), 0) \\ = \;& d_{LS}(u, B_R(0), 0) = 1, \end{aligned}$$

since the map is simply identity. Now, we will show the existence of the negative solution of (2.2).

LEMMA 3.4. Assume that $c < \lambda_1$ and $s < 0$. Then problem (2.2) has a negative solution $u_0(x)$.

For the proof, refer to [4].

Now, we consider the local Leray-Schauder degree of $u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s)$ at the negative solution u_0 with respect to zero.

LEMMA 3.5. *Assume that* $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ *and* $s < 0$. *Then there exists* $d > 0$ *such that*

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_d(u_0), 0) = 1, \tag{3.5}$$

where u_0 *is the negative solution of (2.2).*
Proof. Since every solution of problem (2.2) is discrete and u_0 is a solution of (2.2), there exists $d > 0$ such that there is no the other solution of (2.2) in the neighborhood $B_d(u_0)$ of u_0 with radius d. Then we have

$$\begin{aligned} & d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_d(u_0), 0) \\ = \; & d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(s), B_d(u_0), 0) = 1, \end{aligned}$$

since the map is simply a translation of the identity and since $\|(\Delta^2 + c\Delta)^{-1}s\| < d$ by Lemma 3.4.

Next, we will consider the local Leray-Schauder degree of $u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s)$ at the changing sign solution of (2.2) with respect to zero.

Now, we denote that for given u, $\chi(u)$ is the characteristic function of the positive set of u, i.e.,

$$[\chi(u)](x) = \begin{cases} 1, & \text{if } u(x) > 0, \\ 0, & \text{if } u(x) \leq 0. \end{cases}$$

We consider the following eigenvalue problem

$$(\Delta^2 + c\Delta)u = \nu b\chi(u)u \qquad \text{in } H, \tag{3.6}$$

when $\mu(\{x|u(x) = 0\}) = 0$, where μ is the Lebesgue measure.

We assume that $c < \lambda_1$ and $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$. Let

$$v = \sum h_m\phi_m, \qquad Lv = \sum \lambda_m(\lambda_m - c)h_m\phi_m.$$

For eigenvalues $\lambda_m(\lambda_m - c)$, $m \geq 1$, the corresponding eigenvalues $\nu_m(b\chi(u))$ are nontrivial solutions of (3.6) and

$$\nu_1(b\chi(u)) < \nu_2(b\chi(u)) < \ldots \to +\infty. \tag{3.7}$$

Then we have the following lemma.

Lemma 3.6. *Assume that $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$. Then if u is a solution of (2.2) which changes sign, then*

$$0 < \nu_1(b\chi(u)) < 1.$$

Proof. We know that (2.2) has the negative solution u_0. Writing (2.2) for u and u_0 and subtracting we get

$$(\Delta^2 + c\Delta)(u - u_0) = bu^+.$$

If we use the notation $\frac{bu^+}{u-u_0}$, then we have

$$0 \le \frac{bu^+}{u - u_0} < b\chi(u) \le b. \tag{3.8}$$

By (3.6), $\nu_m(\frac{bu^+}{u-u_0}) = 1$ for some $m \ge 1$ and by (3.8), $m = 1$, i.e., $\nu_1(\frac{bu^+}{u-u_0}) = 1$. Since

$$0 < \nu_1(b\chi(u)) < \nu_1(\frac{bu^+}{u - u_0}) = 1,$$

we obtain the desired result.

The final step in the proof of our theorem is described in

Lemma 3.7. *Assume that $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$. Then if u_* is a solution of (2.2) which changes sign, then there exists $\epsilon > 0$ such that*

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_\epsilon(u_*), 0) = +1 \qquad \text{or } -1.$$

Proof. Let u_* be a solution of (2.2) which changes sign. Since the solutions of (2.2) are discrete, we can choose small $\epsilon' > 0$ such that $B_{\epsilon'}(u_*)$ does not contain the other solutions of (2.2). Let us choose $u \in B_{\epsilon'}(u_*)$ and set $v = u - u_*$. Then there exists $\epsilon_* < \epsilon'$ such that u_* and $u_* + v$ have same sign, so the following holds:

$$\begin{aligned} & u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s) = (u_* + v) - (\Delta^2 + c\Delta)^{-1}(b(u_* + v)^+ + s) \\ = \ & v - (\Delta^2 + c\Delta)^{-1}(b(u_* + v)^+ - bu_*^+) \\ = \ & v - (\Delta^2 + c\Delta)^{-1}(b\chi(u_*)v), \end{aligned}$$

where $u \in B_{\epsilon_*}(u_*)$. Thus we have

$$\begin{aligned} & d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_{\epsilon_*}(u_*), 0) \\ = \ & d_{LS}(v - (\Delta^2 + c\Delta)^{-1}(b\chi(u_*)v), B_{\epsilon_*}(0), 0). \end{aligned}$$

The eigenvalues of the operator $v - (\Delta^2 + c\Delta)^{-1}(b\chi(u_*)v)$ *are connected with the eigenvalues* ν *of the eigenvalue problem* $(\Delta^2 + c\Delta)v = \nu b\chi(u_*)v$ *by*

$$v - (\Delta^2 + c\Delta)^{-1}(b\chi(u_*)v) = \rho v \Longleftrightarrow (\rho - 1)(\Delta^2 + c\Delta)v = -b\chi(u_*)v$$

or $\rho = \frac{\nu-1}{\nu}$. *It follows from Lemma 3.6 and (3.7) that*

$$0 < \nu_1(b\chi(u)) \ldots < \nu_n(b\chi(u)) < 1 < \nu_{n+1}(b\chi(u)) \ldots$$

and thus there are $(-1)^n$ *negative eigenvalues* ρ. *Thus the desired degree is* $+1$ *or* -1. *So the lemma is proved.*

Proof of Theorem 2.1.

The equation (2.2) can be written in the form

$$u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s) = 0.$$

The degree of $u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s)$ *on a large ball of radius* $R > R_0$ *is* $+1$ *by Lemma 3.3. From Lemma 3.4 and 3.5, the constant sign solution of (2.2) is only the negative solution* u_0 *and the degree on the small neighborhood* $B_d(u_0)$ *is* $+1$. *Choi and Jung [4] proved that under the same assumptions of Theorem 2.1, there exists another solution of problem (2.2) which changes sign. If* u_* *is a solution of (2.2) which changes sign, then from Lemma 3.7 , the degree on the ball* $B_{\epsilon_*}(u_*)$ *is* $+1$ *or* -1. *Choosing* $R > R_0$ *so that* B_R *contains all solutions of (2.2), we can conclude that*

$$d_{LS}(u-(\Delta^2+c\Delta)^{-1}(bu^+ +s), B_R\backslash\{B_d(u_0)\cup B_{\epsilon_*}(u_*)\}, 0) = -1 \qquad \text{or } +1.$$

Thus there exist at least three solutions in B_R.

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