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OPTION PRICING IN VOLATILITY ASSET MODEL

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Abstract. We deal with the closed forms of European option pricing for the general class of volatility asset model and the jump-type volatility asset model by several methods.

1. Introduction

There are many asset models which are modified by stochastic volatilities. As we know, Black-Scholes volatility asset model is

$$dS_t = S_t(\mu dt + \sigma_t dW_t), \tag{1}$$

$$d\sigma_t = b(\sigma_t)dt + a(\sigma_t)d\bar{W}_t, \tag{2}$$

where μ is a constant, W_t is a Brownian motion, σ_t is called the volatility and $\bar{W}_t$ is a Browinan motion which is independent with W_t.

A stochastic volatility asset model assumes that σ_t is a stochastic process dependent on a risk factor. We can say that one of another risk factors (which may be correlated with W_t) occurs in incomplete market models in general. Therefore the premium for volatility risk has to be exogenously given to determine an arbitrage-free price for derivative securities written on S_t.

Many authors defined volatility models as several types of the solutions of stochastic differential equations, and use various objects. Several authors set the premium equal to zero to recover an analog of the Black-Scholes formula for European options giving the economic motivation of a diversifiable volatility risk.

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If we refer some volatility models, we can cite so-called the H/W model (3)([4]), the S/S model (4)([7]), and the Heston model (5)([3]):

$$\frac{d\sigma_t^2}{\sigma_t^2} = pdt + qd\bar{W}_t, \tag{3}$$

$$d\sigma_t = -\delta(\sigma_t - \theta)dt + kd\bar{W}_t, \tag{4}$$

$$d\sigma_t^2 = \delta(\theta - \sigma_t^2)dt + k\sigma_t d\bar{W}_t. \tag{5}$$

In paper [5], we can see a general class of stochastic volatility models of the form

$$dS_t/S_t = \mu dt + S_t^\gamma f(\sigma_t)[\sqrt{1-\rho^2}dW_t^1 + \rho dW_t^2], \tag{6}$$

$$d\sigma_t/\sigma_t = \beta(\sigma_t)dt + g(\sigma_t)dW_t^2, \tag{7}$$

with independent standard Brownian motions W_t^1 and W_t^2 on a same probability space $(\Omega, \mathbf{F}, P)$. $S_t > 0$ denotes the price of the (traded) asset and $\sigma_t > 0$ is the (non-traded) stochastic local return variance. In this cited paper, we can see various volatility models by assuming functions f, β, and g, and numbers ρ and γ([5]). Thus, we can calculate option prices by using same method for almost all of volatility models which are derived by (6) and (7).

On the other hand, in papers [1] and [2], we can see some volatility asset models of jump-type. If we write one, it is a model in [2];

$$dS_t/S_{t-} = \mu_t dt + \sigma_t dW_t + \gamma_t dM_t, \tag{8}$$

where

$$S_{t-} := \lim_{h \downarrow 0} S_{t-h},$$

and γ_t is a predictable process. Some conditions for μ_t, σ_t, and γ_t are given in [2].

In this paper, first, we calculate (European) option price of the volatility asset model (6) and (7) for the case $\gamma = 0$. We will use the Heston volatility model derived by (7). For the equations (6) and (7), if we put as $f(\sigma_t) = \sqrt{\sigma_t}$, $\beta(\sigma_t) = \delta(\theta - \sigma_t)/\sigma_t$ and $g(\sigma_t) = k/\sqrt{\sigma_t}$, we can get the Heston volatility model of the form

$$\frac{d\sigma_t}{\sigma_t} = \frac{\delta(\theta - \sigma_t)}{\sigma_t}dt + \frac{k}{\sqrt{\sigma_t}}d\bar{W}_t. \tag{9}$$

For this volatility model (9), we can get a closed form of solution;

$$\sigma_t = \sigma_0 \exp\{\frac{\delta(\theta - \sigma_t)}{\sigma_t}t - \frac{1}{2}(\frac{k}{\sqrt{\sigma_t}})^2 t + (\frac{k}{\sqrt{\sigma_t}})\bar{W}_T\}. \tag{10}$$

If we denote the density function f of random variable W_T and $\bar{f}$ of random variable $\bar{W}_T$, for the Heston model, we can get the European option price $u(x,0)$ with maturity T at time $t = 0$ as a closed form. If the option is call and $S_T \geq K$,

$$\begin{aligned} u(x,0) &= e^{-rT} x \exp\{rT\} \\ &\cdot \int_{R^+} \int_{-1}^{\infty} [\exp\{-\frac{1}{2}[\sigma_0 \exp\{\frac{\delta(\theta - \sigma_T)}{\sigma_T}T - \frac{1}{2}(\frac{k}{\sqrt{\sigma_T}})^2 T + (\frac{k}{\sqrt{\sigma_T}})z\}]T \\ &+ [\sigma_0 \exp\{\frac{\delta(\theta - \sigma_T)}{\sigma_T}T - \frac{1}{2}(\frac{k}{\sqrt{\sigma_T}})^2 T + (\frac{k}{\sqrt{\sigma_T}})z\}]^{1/2} y\}] f(y) dy \bar{f}(z) dz \\ &- e^{-rT} K. \end{aligned}$$

If we use the functional $E[h(\sigma_t)]$, $h(v) = \exp\{-\frac{1}{2}vT + \sqrt{v}y\}$, of diffusion process σ_t, we get if $S_T \geq K$,

$$\begin{aligned} u(x,0) &= e^{-rT} x \exp\{rT\} E[\exp\{-\frac{1}{2}\sigma_T + \sqrt{\sigma_T} W_T\}] - e^{-rT} K \\ &= e^{-rT} x \exp\{rT\} \int_{-1}^{\infty} \int_R h(v) p(T, z, v) dv f(y) dy - e^{-rT} K. \end{aligned}$$

Second, for the volatility asset model (8), if we use the H/W volatility model (3), we can get

$$\begin{aligned} u(x,0) &= e^{-rT} x \exp\{rT\} \\ &\cdot E[\Pi_{0 \leq s \leq T}(1 + \gamma_s)\Delta N_s \exp\{rT - \frac{1}{2}\int_0^T [\sigma_0^2 \exp\{ps - \frac{1}{2}(q^2 s) + q\bar{W}_s\}] ds \\ &- \int_0^T \gamma_s^2 \lambda [1 + \Psi_s] ds + \int_0^T [\sigma_0^2 \exp\{ps - \frac{1}{2}(q^2 s) + q\bar{W}_s\}]^{1/2} dW_s^Q\}] \\ &- e^{-rT} K. \end{aligned}$$

In Section 2, we get the solutions of two types of volatility asset models and volatility models by closed form. In Section 3, we get the closed forms of option prices of European call options for two types of volatility asset models: the Heston model and the H/W model.

2. Stochastic volatility asset models

A volatility model, which is so-called as the Heston model, is defined by (5), or from (6) and (7), is represented by

$$d\sigma_t = \delta(\theta - \sigma_t)dt + k\sqrt{\sigma_t}d\bar{W}_t. \tag{11}$$

But, if we put in (6) and (7), $f(\sigma_t) = \sqrt{\sigma}_t$, $\beta(\sigma_t) = \delta(\theta - \sigma_t)/\sigma_t$, $g(\sigma_t) = k/\sqrt{\sigma_t}$, $\rho \in [-1, 1]$, and $\gamma = 0$, we get a modified Heston volatility asset model

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\sigma_t}[\sqrt{1-\rho^2}dW_t^1 + \rho dW_t^2], \tag{12}$$

$$\frac{d\sigma_t}{\sigma_t} = \frac{\delta(\theta - \sigma_t)}{\sigma_t}dt + \frac{k}{\sqrt{\sigma_t}}dW_t^2. \tag{13}$$

For this volatility model (13), we get a closed form of solution;

$$\sigma_t = \sigma_0 \exp\{\frac{\delta(\theta - \sigma_t)}{\sigma_t}t - \frac{1}{2}(\frac{k}{\sqrt{\sigma_t}})^2 t + (\frac{k}{\sqrt{\sigma_t}})W_T^2\}. \tag{14}$$

For the equation (12), if we assume for simplicity $\rho = 0 \in [-1, 1]$, we get a stochastic differential equation

$$dS_t/S_t = \mu dt + \sqrt{\sigma_t}dW_t. \tag{15}$$

The solution of this stochastic differential equation is a closed form as following;

$$S_t = S_0 \exp\{\mu t - \frac{1}{2}\sigma_t t + \sqrt{\sigma_t}W_t\}. \tag{16}$$

On the other hand, For the H/W volatility model of the form (3), if we write the solution, we get a closed form

$$\sigma_t^2 = \sigma_0^2 \exp\{pt - \frac{1}{2}(q^2 t) + q\bar{W}_t\}. \tag{17}$$

For the asset model (8), we also can get the solutions of closed form. When $M_t = \bar{W}_t$, the asset price S_t at time t is represented by

(18)

$$S_t = S_0 \exp\{\mu t - \frac{1}{2}\int_0^t \sigma_s^2 ds - \frac{1}{2}\int_0^t \gamma_s^2 ds + \int_0^t \sigma_s dW_s^Q + \int_0^t \gamma_s^2 d\bar{W}_s^Q\}.$$

When $M_t = N_t - \lambda t$, we get the asset price S_t at time t as

$$
\begin{aligned}
S_t = S_0 \Pi_{0\le s\le t}(1+\gamma_s)\Delta N_s \exp\{rt - \frac{1}{2}\int_0^t \sigma_s^2 ds \\
- \int_0^t \gamma_s^2 \lambda[1+\Psi_s]ds + \int_0^t \sigma_s dW_s^Q\},
\end{aligned}
\tag{19}
$$

where Q is an equivalent martingale measure (c.f. [2]).

3. The calculation of the price of European options

Let S_t be the price of given asset at time t. The price of European option with maturity T at time t is defined by

$$u(x,t) = E[e^{-r(T-t)}g(S_T)|S_t = x],$$

where the function g is a pay-off function. The option is call if $g(x) = (x-K)^+$, and is put if $g(x) = (K-x)^+$, where $(x)^+ = max\{0,x\}$ and K is the strike price.

3.1. Heston Volatility Asset Model.

To get the option price for maturity T at time $t=0$, from the solution (16), we think the random variable S_T at time T as following;

$$S_T = S_0 \exp\{\mu T - \frac{1}{2}\sigma_T T + \sqrt{\sigma_T} W_T\}. \tag{20}$$

From the closed form of solution of Heston volatility model (14), we get

(21)

$$
\begin{aligned}
S_T &= S_0 \exp\{\mu T - \frac{1}{2}\sigma_T T + \sqrt{\sigma_T} W_T\}, \\
&= S_0 \exp\{\mu T - \frac{1}{2}[\sigma_0 \exp\{\frac{\delta(\theta-\sigma_T)}{\sigma_T}T - frac12(\frac{k}{\sqrt{\sigma_T}})^2 T + (\frac{k}{\sqrt{\sigma_T}})\bar{W}_T\}]T \\
&+ [\sigma_0 \exp\{\frac{\delta(\theta-\sigma_T)}{\sigma_T}T - \frac{1}{2}(\frac{k}{\sqrt{\sigma_T}})^2 T + (\frac{k}{\sqrt{\sigma_T}})\bar{W}_T\}]^{1/2} W_T\},
\end{aligned}
$$

because of

$$\sigma_T = \sigma_0 \exp\{\frac{\delta(\theta-\sigma_T)}{\sigma_T}T - \frac{1}{2}(\frac{k}{\sqrt{\sigma_T}})^2 T + (\frac{k}{\sqrt{\sigma_T}})\bar{W}_T\}. \tag{22}$$

From the definition of option price, the price of European call option at time $t=0$ is

$$\begin{aligned} u(x,0) &= E[e^{-rT} g(S_T)|S_0 = x] \\ &= E[e^{-rT}(S_T - K)^{+}|S_0 = x] \\ &= e^{-rT} E[S_T|S_0 = x] - e^{-rT} K, \quad if \quad S_T \geq K \\ &= e^{-rT} x E[\exp\{\mu T - \frac{1}{2}\sigma_T T + \sqrt{\sigma_T} W_T\}] - e^{-rT} K. \end{aligned} \tag{23}$$

If we assume the Brownian motion W_t has the mean and the variance,we get the density function f of W_t. For the solutions of the stochastic differential equations (11)and (15), if we assume(know) the Brownian motion $\bar{W}_t$, which is independent with Brownian motion W_t, has the mean and the variance. Then, we get the density function $\bar{f}$ of random variable $\bar{W}_T$. From the equation (23), we get if $S_T \geq K$,

$$\begin{aligned} & u(x,0) \\ &= e^{-rT} x \exp \mu T E[\exp\{-\frac{1}{2}\sigma_T T + \sqrt{\sigma_T} W_T\}] - e^{-rT} K \\ &= e^{-rT} x \exp\{\mu T\} \\ & \cdot E[\exp\{-\frac{1}{2}[\sigma_0 \exp\{\frac{\delta(\theta - \sigma_T)}{\sigma_T} T - \frac{1}{2}(\frac{k}{\sqrt{\sigma_T}})^2 T + (\frac{k}{\sqrt{\sigma_T}})\bar{W}_T\}]T \\ & + [\sigma_0 \exp\{\frac{\delta(\theta - \sigma_T)}{\sigma_T} T - \frac{1}{2}(\frac{k}{\sqrt{\sigma_T}})^2 T + (\frac{k}{\sqrt{\sigma_T}})\bar{W}_T\}]^{1/2} W_T\}] \\ & - e^{-rT} K. \end{aligned} \tag{24}$$

Thus, we get the closed form of European call option price at $t=0$;

(25)

$$\begin{aligned} & u(x,0) \\ &= e^{-rT} x \exp\{\mu T\} \int_{R^+} \int_{-1}^{\infty} [\exp\{-\frac{1}{2}[\sigma_0 \exp\{\frac{\delta(\theta - \sigma_T)}{\sigma_T} T \\ & - \frac{1}{2}(\frac{k}{\sqrt{\sigma_T}})^2 T + (\frac{k}{\sqrt{\sigma_T}}) z\}]T + [\sigma_0 \exp\{\frac{\delta(\theta - \sigma_T)}{\sigma_T} T - \frac{1}{2}(\frac{k}{\sqrt{\sigma_T}})^2 T \\ & + (\frac{k}{\sqrt{\sigma_T}}) z\}]^{1/2} y\}] f(y) dy \bar{f}(z) dz - e^{-rT} K, \quad if \quad S_T \geq K. \end{aligned}$$

By using an equivalent martingale measure Q, we can change μ to the risk-free interest rate r. From this, we can also calculate option price if we get the value of σ_T. The value of σ_T, sometimes, can be obtained by a simulation method by using following definition (c.f. [2]);

$$\sigma_T := [\frac{1}{T}\int_0^T \sigma_s^2 ds]^{\frac{1}{2}}.$$

From the form of (24) and (25), we can link to the works of leverage by using $dW_t d\bar{W}_t = \rho dt$ in the Heston model.

Further, from the first line of (24), to get $E[\exp\{-\frac{1}{2}\sigma_T T + \sqrt{\sigma_T} W_T\}]$, we put $h(v) = \exp\{-\frac{1}{2}vT + \sqrt{v}y\}$, and use

$$E[h(\sigma_t)|\sigma_0 = z] = \int_0^\infty h(v)p(t,z,v)dv,$$

where $p(T,z,v)$ is the fundamental solution. Then we can get, if $S_T \geq K$,

$$\begin{aligned} u(x,0) &= e^{-rT}x\exp\{rT\}E[\exp\{-\frac{1}{2}\sigma_T + \sqrt{\sigma_T}W_T\}] - e^{-rT}K \\ &= e^{-rT}x\exp\{rT\}\int_{-1}^{\infty}\int_R h(v)p(T,z,v)dvf(y)dy - e^{-rT}K. \end{aligned} \tag{26}$$

3.2. Jump-diffusion H/W Volatility Asset Model.

To get the option price for maturity T at time $t=0$ from the process (18) and (19), we think two types random variables S_T at time T:
(27)

$$S_T = S_0\exp\{rT - \frac{1}{2}\int_0^T \sigma_s^2 ds - \frac{1}{2}\int_0^T \gamma_s^2 ds + \int_0^T \sigma_s dW_s^Q + \int_0^T \gamma_s^2 d\bar{W}_s^Q\},$$

and, when $M_t = N_t - \lambda t$,

$$\begin{aligned} S_T =& S_0\Pi_{0\leq s\leq T}(1+\gamma_s)\Delta N_s\exp\{rT - \frac{1}{2}\int_0^T \sigma_s^2 ds \\ &- \int_0^T \gamma_s^2\lambda[1+\Psi_s]ds + \int_0^T \sigma_s dW_s^Q\} \end{aligned} \tag{28}$$

for an equivalent martingale measure Q, (c.f. [2]). For the H/W volatility model of the form (3), we can get a closed form (17) of solution of (3), and get the option price for asset model given by (27) with maturity T;

(29)
$$
\begin{aligned}
&u(x,0)\\
&= E[e^{-rT} g(S_T)|S_0 = x]\\
&= E[e^{-rT}(S_T - K)^+|S_0 = x]\\
&= e^{-rT}E[S_T|S_0 = x] - e^{-rT}K, \quad if \quad S_T \geq K\\
&= e^{-rT} x \exp\{rT\} \cdot E[\exp\{-\frac{1}{2}\int_0^T \sigma_s^2 ds - \frac{1}{2}\int_0^T \gamma_s^2 ds\\
&\quad + \int_0^T \sigma_s dW_s^Q + \int_0^T \gamma_s^2 d\bar{W}_s^Q\}] - e^{-rT}K.
\end{aligned}
$$

Thus, if we use σ_t^2 of (17), we get

(30)
$$
\begin{aligned}
&u(x,0)\\
&= e^{-rT}E[S_T|S_0 = x] - e^{-rT}K, \quad if \quad S_T \geq K\\
&= e^{-rT} x \exp\{rT\} \cdot E[\exp\{-\frac{1}{2}\int_0^T [\sigma_0^2 \exp\{ps - \frac{1}{2}(q^2 s) + q\bar{W}_s\}]ds\\
&- \frac{1}{2}\int_0^T \gamma_s^2 ds + \int_0^T [\sigma_0^2 \exp\{ps - \frac{1}{2}(q^2 s) + q\bar{W}_s\}]^{1/2} dW_s^Q + \int_0^T \gamma_s^2 d\bar{W}_s^Q\}]\\
&- e^{-rT}K.
\end{aligned}
$$

For asset model given by (30), we get a closed form of European option price;

(31)
$$
\begin{aligned}
&u(x,0)\\
&= e^{-rT}E[S_T|S_0 = x] - e^{-rT}K, \quad if \quad S_T \geq K\\
&= e^{-rT} x \exp\{rT\}\\
&\cdot E[\Pi_{0 \leq s \leq T}(1+\gamma_s)\Delta N_s \exp\{rT - \frac{1}{2}\int_0^T \sigma_s^2 ds - \int_0^T \gamma_s^2 \lambda[1+\Psi_s]ds\\
&+ \int_0^T \sigma_s dW_s^Q\}] - e^{-rT}K.
\end{aligned}
$$

Thus, if we use σ_t^2 of (17), we get

(32)
$$
u(x,0) = e^{-rT} s_0 \exp\{rT\} \cdot E[\Pi_{0 \leq s \leq T}(1+\gamma_s)\Delta N_s
$$

$$\exp\{rT - \frac{1}{2}\int_0^T [\sigma_0^2 \exp\{ps - \frac{1}{2}(q^2 s) + q\bar{W}_s\}]ds$$
$$- \int_0^T \gamma_s^2 \lambda[1 + \Psi_s]ds + \int_0^T [\sigma_0^2 \exp\{ps - \frac{1}{2}(q^2 s) + q\bar{W}_s\}]^{1/2} dW_s^Q\}]$$
$$- e^{-rT} K.$$

4. Conclusions

In (24) and (25), if we get the value of σ_t by using the simulation method, or know the distribution of random variable σ_T, we can calculate option price $u(x, 0)$. When the correlation is not zero, our work for Heston model can be linked to leverage works of volatility models. For the analytical calculation of expectation for a functional of σ_t in (26), we can use the diffusion equation theory because the solution of (2) is a diffusion process.

Variables p and q in (3) and (17) may depend on time t and volatility σ_t, but they do not depend on S_t in (1) and (8). From (17), if we know the coefficients of (27) and (28), we can calculate option prices (30) and (32) by using the density functions of Brownian motions and/or the covariance of W_t^Q and $\bar{W}_t$.

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