ORBITAL SHADOWING PROPERTY

BAHMAN HONARY AND ARIEZA ZAMANI BAHABADI

ABSTRACT. Let $M$ be a generalized homogeneous compact space, and let $Z(M)$ denote the space of homeomorphisms of $M$ with the $C^0$ topology. In this paper, we show that if the interior of the set of weak stable homeomorphisms on $M$ is not empty then for any open subset $W$ of $Z(M)$ containing only weak stable homeomorphisms the orbital shadowing property is generic in $W$.

1. Introduction

The concept of shadowing is investigated by many authors (see e.g. [2, 4, 8]). In [3] Corless and Pilyugin proved that weak shadowing is a $C^0$ generic property for discrete dynamical systems of a compact smooth manifold $M$. Subsequently, Pilyugin and Plamenevskaya [9] improved this result by showing $C^0$ genericity of the shadowing property. Both proofs given in [3] and [9] required that $M$ be a $C^\infty$ smooth manifold. Mazur in [7] showed that for $C^0$ genericity of weak shadowing neither the differential structure on $M$, nor even being a manifold is a crucial assumption, but what matters is a generalized version of a topological property called homogeneity. Kosicielnik and Mazur in [5] have given a proof for $C^0$ genericity of periodic orbital shadowing on a compact topological manifold of dimension at least 2. In this note we show that if the space $M$ is generalized homogeneous and has no isolated points, then in an open subset of $Z(M)$ the orbital shadowing property is generic.

2. Notations

Let $(M, d)$ be a compact metric space and let $f : M \to M$ be a homeomorphism (a discrete dynamical system on $M$). A sequence $\{x_n\}_{n \in \mathbb{Z}}$ is called an orbit of $f$, denote by $o(x, f)$, if for each $n \in \mathbb{Z}$, $x_{n+1} = f(x_n)$ and we call it a $\delta$-pseudo-orbit of $f$ if,

$$d(f(x_n), x_{n+1}) \leq \delta, \ \forall n \in \mathbb{Z}.$$ 

The homeomorphism $f$ is said to have the weak shadowing property if for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $\delta$-pseudo-orbit $\{x_n\}_{n \in \mathbb{Z}}$ of $f$ we...
can find a point $y \in M$ such that $\{x_n\}_{n \in \mathbb{Z}} \subset N_{\epsilon}(o(y, f))$, where $N_{\epsilon}(S)$ is the $\epsilon$-neighborhood of the set $S \subset M$.

A system $f$ is said to have the orbital shadowing property if for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $\delta$-pseudo-orbit $\{x_n\}_{n \in \mathbb{Z}}$ of $f$ we can find a point $y \in M$ with the property that $\{x_n\}_{n \in \mathbb{Z}} \subset N_{\epsilon}(o(y, f))$ and $o(y, f) \subset N_{\epsilon}(\{x_n\}_{n \in \mathbb{Z}})$. We denoted the set of all homeomorphisms of $M$ by $Z(M)$.

Introduce in $Z(M)$ the complete metric

$$d_0(f, g) = \max_{x \in M} \{\max_{\epsilon > 0} d(f(x), g(x)), \max_{x \in M} d(f^{-1}(x), g^{-1}(x))\},$$

which generates the $C^0$ topology.

The space $M$ is said to be generalized homogeneous if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\} \subset M$ is a pair of sets of mutually disjoint elements satisfying $d(x_i, y_i) \leq \delta$, $i \in \{1, \ldots, n\}$, then there exists $h \in Z(M)$ satisfying $d_0(h, id_M) \leq \epsilon$ and $h(x_i) = y_i, i \in \{1, \ldots, n\}$. Such a $\delta$ is called an $\epsilon$-modulus of homogeneity of $M$. We say that $x \in M$ is a weak stable point for $f$ if for any $\epsilon > 0$ there is $\delta > 0$ and positive integer $N$ such that $o(z, f) \subset N_{\epsilon}(\{f^n(z) : i = -N, \ldots, N\})$ for every $z \in M$ with $d(x, z) < \delta$. We say that $f$ is weak stable if every point of $M$ is a weak stable point for $f$.

A property $P$ is said to be generic for elements of a topological space $M$ if the set of all $x \in M$ satisfying $P$ is residual, i.e., it includes a countable intersection of open and dense subsets of $M$.

### 3. Results

**Proposition 1.** Let $f$ be a homeomorphism on a compact metric space $M$. Then the set of weak stable points is residual in $M$.

**Proof.** Fix an $\epsilon > 0$. Let $U = \{U_i : i = 1, \ldots, k\}$ be a finite covering of $M$ by open sets with diameter less than $\frac{\epsilon}{2}$. Put $K = \{1, 2, \ldots, k\}$. For each $x \in M$, choose a subset $L_x$ of $K$ satisfying the following conditions:

$$o(x, f) \subset \cup\{U_i : i \in L_x\}$$

$$o(x, f) \cap U_i \neq \emptyset$$

for all $i \in L_x$.

Let $A_x$ be the set of all $x \in M$ such that there is $\delta_x$ and positive integer $N_x$ such that $o(z, f) \subset N_{\epsilon}(\{f^n(z) : i = -N_x, \ldots, N_x\})$ for every $z \in M$ with $d(x, z) < \delta_x$. We claim that $A_x$ is open and dense in $M$. Clearly $A_x$ is open. To see that $A_x$ is dense, let $x \in M$ be arbitrary. We can find a positive integer $T$ such that

$$\{f^i(x) : i = -T, \ldots, T\} \cap U_j \neq \emptyset$$

for all $j \in L_x$. Hence $o(x, f) \subset N_{\epsilon}(\{f^i(x) : i = -T, \ldots, T\})$. Choose $\delta > 0$ such that

$$d(f^i(x), f^j(z)) < \frac{\epsilon}{2}$$

for all $i = -T, -T + 1, \ldots, T$ for every $z \in M$ with $d(x, z) < \delta$. Suppose that $x \notin A_x$. Given any $\zeta$ with $0 < \zeta < \delta$ there is $x_1 \in N_\zeta(x)$ such that $f^{T_i}(x_1) \notin N_\zeta(\{f^i(x) : i = -T, \ldots, T\})$
for some $T_1$ with $|T_1| > T$. If $f^{T_1}(x_1) \in N_{\frac{\epsilon}{2}}(\{f^{i}(x) : i = -T, \ldots, T\})$, then we have

$$d(f^{T_1}(x_1), f^i(x_1)) \leq d(f^{T_1}(x_1), f^i(x)) + d(f^i(x), f^{i}(x_1)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for some $i \in \{-T, \ldots, T\}$. This is a contradiction. Thus

$$f^{T_1}(x_1) \not\in N_{\frac{\epsilon}{2}}(\{f^{i}(x) : i = -T, \ldots, T\}).$$

Since $f^j(x_1) \notin U_j$ for all $j \in L_x$, there is $j \in K - L_x$ such that $f^{T_1}(x_1) \in U_j$. Thus $L_x$ is a proper subset of $L_x$. We can find $T_2 \geq |T_1|$ such that

$$o(x_1, f) \subset N_r(\{f^{i}(x) : i = -T_2, \ldots, T_2\}).$$

Choose $\delta_1 > 0$ such that $d(f^i(x_1), f^i(z)) < \frac{\delta}{2}$ for all $i \in \{-T_2, -T_2 + 1, \ldots, T_2\}$ for every $z \in M$ with $d(x_1, z) < \delta_1$. If $x_1 \in A_e$ we are done, otherwise there is $x_2 \in N_\xi(X_1) \subset N_\xi(x)$ such that

$$f^{T_3}(x_2) \not\in N_r(\{f^{i}(x) : i = -T_3, \ldots, T_3\})$$

for some $T_3$ with $|T_3| > T_2$, where $\xi = \min(\zeta, \delta_1)$. Since

$$f^{T_3}(x_2) \not\in N_{\frac{\delta}{2}}(\{f^{i}(x) : i = -T_3, \ldots, T_3\}),$$

$f^j(x_2) \notin U_j$ for all $j \in L_x$. There is $j \in K - L_x$ such that $f^{T_3}(x_2) \in U_j$. Thus $L_{x_1}$ is a proper subset of $L_{x_2}$. By continuing this process, since $K$ is finite, we can find $x' \in N_\xi(x)$ such that $L_{x'} = K$. Then $x' \in A_e$. Thus $A_e$ is dense in $M$. Now let $R = \cap_{n=1}^{\infty} A_{\frac{\xi}{n}}$ then $R$ is a residual subset of $M$ consisting of weak stable points.

A homeomorphism $f : M \to M$ is called minimal if $f(A) = A$, $A$ closed, implies either $A = M$ or $A = \emptyset$. It is easy to see that $f$ is minimal if and only if $o(x, f) = M$ for each $x \in M$.

**Proposition 2.** Let $f$ be a homeomorphism on a compact metric space $M$. If $f$ is minimal, then $f$ is weak stable.

**Proof.** Let $x \in M$ and $\epsilon > 0$ be arbitrary. Let $U = \{U_i : i = 1, \ldots, k\}$ be a finite covering of $M$ by open sets with diameter less than $\frac{\epsilon}{2}$. Since $o(x, f) = M$, there are $n_1, n_2, \ldots, n_k \in \mathbb{Z}$, such that $f^{n_i}(x) \in U_i$ for $i = 1, 2, \ldots, k$. Let

$$N = \max\{|n_i| : 1 \leq i \leq k\}.$$

There exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f^i(x), f^i(y)) < \frac{\delta}{2}$ for all $-N \leq i \leq N$. Let $y \in N_\epsilon(x)$. Given any $n \in \mathbb{Z}$, since $f^n(y) \in M = \bigcup_{i=1}^{k} U_i$, $f^n(y) \in U_i$ for some $i = 1, \ldots, k$. But $f^n(x) \in U_i$, so $d(f^n(y), f^n(x)) \leq \text{diam}(U_i) < \frac{\epsilon}{2}$. Since $d(x, y) < \delta$ and $-N \leq n_i \leq N$ we have $d(f^n(x), f^n(y)) < \frac{\epsilon}{2}$. Thus we have

$$d(f^n(y), f^n(x)) \leq d(f^n(y), f^{n_i}(x)) + d(f^{n_i}(x), f^n(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $o(y, f) \subset N_\epsilon(\{f^i(x) : i = -N, \ldots, N\})$. This shows that $x$ is an weak stable point and $f$ is an weak stable homeomorphism. \qed
We denote the set of all weak stable homeomorphisms by WSH.

**Theorem 1.** Let \( \text{int}(WSH) \neq \emptyset \) and let \( W \) be an open subset of \( Z(M) \) containing only weak stable homeomorphisms. Then there is a residual subset \( R_1 \) of \( W \) such that for each \( f \in R_1 \) and \( \epsilon > 0 \), there is a neighborhood \( U_f \) of \( f \) and positive integer \( N_f \) such that \( o(x, g) \subset \bigcup_{i=-N_f}^{N_f} \overline{N}_\epsilon(g^i(x)) \) for every \( g \in U_f \) and \( x \in M \).

To prove this theorem we need the following two lemmas.

**Lemma 1.** Let \( \epsilon > 0 \) be arbitrary. Then the function \( \psi_\epsilon : W \rightarrow \mathbb{N} \) defined by
\[
\psi_\epsilon(f) = N_f,
\]
where
\[
N_f = \min \{N \in \mathbb{N} : o(x, f) \subset \bigcup_{i=-N}^{N} \overline{N}_\epsilon(f^i(x)) \forall x \in M \},
\]
is lower semi-continuous.

**Proof.** For \( f \in W \) there is \( x_0 \in M \) such that \( o(x_0, f) \notin \bigcup_{i=-N_f}^{N_f} \overline{N}_\epsilon(f^i(x_0)) \). So \( f^k(x_0) \notin \bigcup_{i=-N_f}^{N_f} \overline{N}_\epsilon(f^i(x_0)) \) for some \( k \in \mathbb{Z} \) with \( |k| \geq N_f \). Choose \( \epsilon' > 0 \) such that
\[
d(f^{N_f}(x_0), f^1(x_0)) \geq \epsilon + \epsilon', -N_f + 1 \leq l \leq N_f - 1.
\]
Choose a neighborhood \( U_f \) of \( f \) such that \( d(f^l(x), g^l(x)) < \frac{\epsilon'}{2}, |l| \leq k + 1 \) for each \( x \in M \) and \( g \in U_f \). If \( N_g < N_f \), then \( d(g^k(x_0), g^l(x_0)) < \epsilon \) for some \(-N_f + 1 \leq l \leq N_f - 1 \). So
\[
d(f^k(x_0), f^l(x_0)) \leq d(f^k(x_0), g^k(x_0)) + d(g^k(x_0), g^l(x_0)) + d(g^l(x_0), f^l(x_0))
\]
\[
\leq \frac{\epsilon'}{2} + \epsilon + \frac{\epsilon'}{2} = \epsilon + \epsilon',
\]
which contradicts \((*)\). Hence \( N_g \geq N_f \). This complete the proof of the lemma. \( \square \)

Now, we recall a topology lemma, for the proof see [6].

**Lemma 2.** Let \( X \) be a Bair topological space and \( \Gamma : X \rightarrow \mathbb{N} \) be a lower semi-continuous map. Then there exists a residual subset \( R \) of \( X \) such that \( \Gamma \mid R \) is locally constant on each point of \( R \).

**Proof of Theorem 1.** Using lemmas 1 and 2, for any \( \epsilon > 0 \) let \( R_\epsilon \) be a residual subset of \( W \) such that \( \psi_\epsilon \) is locally constant on \( R_\epsilon \). Then \( R_1 = \cap \{R_\epsilon : n = 1, 2, \ldots \} \) is the required residual set. \( \square \)

**Theorem 2.** Let \( M \) be a generalized homogeneous space with no isolated point. Then either \( \text{int}(WSH) = \emptyset \), or for every \( f \in \text{int}(WSH) \) and every open neighborhood \( W \) of \( f \) in \( \text{int}(WSH) \) the orbital shadowing property is generic in \( W \).
For the proof we need the following lemma from [10].

**Lemma 3.** If $h$ is upper semi-continuous, then for any $\epsilon > 0$ the set of all $x \in X$ such that there exists a neighborhood $U$ of $x$ with the property that $d_H(h(x), h(y)) \leq \epsilon$ for all $y \in U$, is open and dense in $X$. Here $d_H$ is the Hausdorff metric.

**Proof of Theorem 2.** Assuming $\text{int}(WSH) \neq \emptyset$. Let $f \in \text{int}(WSH)$ and $W$ be an open neighborhood of $f$ in $\text{int}(WSH)$. Let $\epsilon > 0$ be arbitrary, and $A = \{U_1, \ldots, U_k\}$ be a finite covering of $M$ by closed sets with diameter less than $\frac{\epsilon}{2}$. Consider the set $K = \{1, 2, \ldots, k\}$ as a compact metric space with discrete metric. Let $C(K)$ be the set of all subset of $K$, then define the map $\varphi : Z(M) \to C(C(K))$ by

$$
\varphi_x(f) = \{L \subset K : \exists x \in M \text{ such that } o(x, f) \subset \bigcup \{U_i : i \in L\}, o(x, f) \cap U_i \neq \varnothing \forall i \in L\}.
$$

The map $\varphi_x$ is upper semi-continuous [1]. Let $R_{\epsilon}$ be the set of all $f \in Z(M)$ such that there exists a neighborhood $U_f$ of $f$ with the property that $d_H(\varphi_x(f), \varphi_x(g)) \leq \epsilon$ for every $g \in U_f$. The set $R_{\epsilon}$ is open and dense in $Z(M)$ by Lemma 3. Moreover, it is easy to see that the map $\varphi_x$ is locally constant on the set $R_{\epsilon}$ if $\epsilon < 1$ that is for any $f \in R_{\epsilon}$ there is a neighborhood $U_f$ of $f$ satisfying $\varphi_x(f) = \varphi_x(g)$ for all $g \in U_f$. Choose $R_2 = \bigcap_{n=1}^{\infty} R_{\frac{\epsilon}{2}}$ and $R = R_2 \cap R_1 \cap W$, where $R_1$ is as in Theorem 1. To complete the proof, it remains to show that the set $R$ has orbital shadowing property. Let $0 < \epsilon < 1$ be arbitrary and $f \in R$. There is a neighborhood $U_f$ of $f$ satisfying $\varphi_x(f) = \varphi_x(g)$ and $\psi_\epsilon(f) = \psi_\epsilon(g)$ for all $g \in U_f$. Choose $\beta > 0$ such that $N_\beta(f) \subset U_f$. Let $\gamma > 0$ be a $\beta$-modulus of homogeneity of $M$, and put $0 < \delta < \min\{\frac{\epsilon}{2}, \frac{\gamma}{2}\}$. Fix any $\delta$-pseudo-orbit $y = \{y_n\}_{n \in \mathbb{Z}}$. There exists a positive integer $l \geq N_f$ such that $y \in N_f(y_i)$ where $y_i = \{y_n\}_{n=-l}^l$. Since $M$ has no isolated point we can easily find a finite $2\delta$-pseudo-orbit $y'_i = \{y'_n\}_{n=-l}^l$ such that $y'_i \subset N_f(y_i)$ and $y'_i \subset N_f(y_i)$ for $i \neq j$. Since $d(f(y'_i), f(y'_j)) < 2\delta < \gamma$ there exists $h \in Z(M)$ such that $d(h, \text{id}_M) \leq \beta$ and $h(f(y'_i)) = f(y'_i)$ for all $i = -l, \ldots, l$. Set $g = ho f$. Then the sequence

$$
o(y_0, g) = \{\ldots, g^{-2}(y_{-1}), g^{-1}(y_0), g(y'_0), g(y'_1), \ldots, g^{-2}(y_{-1}), g^{-1}(y_0), g(y'_0), g(y'_1), \ldots\}
$$

is an orbit of $g$. Since $g \in N_\beta(f)$ we have $\varphi_x(f) = \varphi_x(g)$ and $\psi_\epsilon(f) = \psi_\epsilon(g)$. Choose $L \in \varphi_x(f)$ such that $o(y_0, g) \cap L \subset U_i$ and $o(y_0, g) \cap U_i \neq \emptyset$ for all $i \in L$. But $L \in \varphi_x(f)$, thus there exists $x \in M$ satisfying $o(x, f) \cap \bigcup_{i \in L} U_i$ and $o(x, f) \cap U_i \neq \emptyset$ for all $i \in L$. This implies that $y \subset N_{3\delta}(o(x, f))$, $o(x, f) \subset N_f(o(y_0, g))$. Since $\psi_\epsilon(f) = \psi_\epsilon(g)$ we have $o(y_0, g) \subset \bigcup_{n=-N_f}^{N_f} N_f(y_n) \subset \bigcup_{n=-N_f}^{N_f} N_f(y_n)$ and $g(y'_0) \subset N_f(y_i) \subset N_f(y_i)$. Hence we get $o(x, f) \subset N_{3\delta}(y')$. This complete the proof of Theorem 2.

As Mazur has shown in [7] the spaces (i), (ii) and (iii) in the following corollary are homogeneous. Thus using Theorem 2 we have:
Corollary. If the space $M$ is one of the followings:

(i) a topological manifold with boundary ($\dim M \geq 2$ if $\partial M \neq \emptyset$).
(ii) a cartesian product of a countably infinite number of manifolds with nonempty boundary.
(iii) a cantor set.

Then orbital shadowing is a generic property in $W$.

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References