ON THE WEAK FORMS OF CHOICE IN TOPOI

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ABSTRACT. In topoi, there are various forms of the axiom of choice such as (ES), (AC) and (WO). And also there are various weak forms of the axiom of choice such as (DES), (IAC) and (ASC). First we investigate the relation between (IAC) and (ASC), and then we study the relation between (AC) and (WO). We get equivalent forms of the axiom of choice in a well-pointed topos.

1. INTRODUCTION

There are various forms of the axiom of choice in topoi:

(ES) Every epimorphism is a section.

(AC) For any noninitial object $A$ and $f : A \to B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.

(WO) For any $P$ and $q : U \to \Omega^P$, if there exist $\alpha : V \to U$ and $p : V \to P$ such that $(q\alpha, p)$ factors through $\epsilon_P \to \Omega^P \times P$, then there exists $\alpha_0 : V_0 \to U$ and $p_0 : V_0 \to P$ such that $(q\alpha_0, p_0)$ factors through $\epsilon_P$, and such that for all $\beta : W \to V_0$ and all $p_1 : W \to P$, if $(q\alpha_0\beta, p_1)$ factors through $\epsilon_P$, then $(p_0\beta, p_1)$ factors through a monomorphism $P_1 \to P \times P$.

Also there are various weak forms of the axiom of choice in a topos:

(DES) Every decidable epimorphism is a section.

(ASC) Every separated epimorphism of $E$ is a section.

(IAC) Every object of $E$ is internally projective.

Mawanda [7] showed that (ES), (DES) and (ASC) are equivalent in a well-pointed topos. Goldblatt [4] showed that (ES) and (AC) are equivalent in a well-pointed topos. In this paper we show that (IAC) is not equivalent to (ASC) in some topoi, but (IAC) is equivalent to (ASC) in a well-pointed topos. Also we show that (AC)
is not equivalent to (WO) in some topoi, but (AC) is equivalent to (WO) in a well-pointed topos. Therefore we know that (ES), (AC), (WO), (DES), (ASC) and (IAC) are equivalent in a well-pointed topos.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

Definition 2.1. An elementary topos is a category \( \mathcal{E} \) that satisfies the following:

(T1) \( \mathcal{E} \) is finitely complete,
(T2) \( \mathcal{E} \) has exponentiation,
(T3) \( \mathcal{E} \) has a subobject classifier.

(T2) means that for every object \( A \) in \( \mathcal{E} \), the endofunctor \((-) \times A\) has its right adjoint \((-)^A\). Hence for every object \( A \) in \( \mathcal{E} \), there exists an object \( B^A \), and a morphism \( ev_A : B^A \times A \to B \), called the evaluation map of \( A \), such that for any \( Y \) and \( f : Y \times A \to B \) in \( \mathcal{E} \), there exists a unique morphism \( g \) such that \( ev_A \circ (g \times i_A) = f \);

\[
\begin{array}{ccc}
Y \times A & \xrightarrow{f} & B \\
g \times i_A \downarrow & & \downarrow i_B \\
B^A \times A & \xrightarrow{ev_A} & B
\end{array}
\]

And subobject classifier in (T3) is an \( \mathcal{E} \)-object \( \Omega \), together with a morphism \( \top : 1 \to \Omega \) such that for any monomorphism \( h : D \to C \), there is a unique morphism \( \chi_h : C \to \Omega \), called the character of \( h : D \to C \) which makes the following diagram a pull-back;

\[
\begin{array}{ccc}
D & \xrightarrow{!} & 1 \\
\downarrow h & & \downarrow \top \\
C & \xrightarrow{\chi_h} & \Omega
\end{array}
\]

Example 2.2. If \( M_2 \) is a monoid with two elements, then the category \( M_2 - \text{Set} \) is a topos.

Consider \( (M_2, \circ, e) \) where \( M_2 = \{ e, a \} \) and \( \circ \) is defined by \( e \circ e = e, \ e \circ a = a \circ e = a \circ a = a \). Then \( M_2 \) is a monoid with identity \( e \), in which \( a \) has no inverse. The set \( L_2 \) of left ideals of \( M_2 \) has three elements, that is, \( M_2, \emptyset \), and \( \{ a \} \). Thus in \( M_2 - \text{Set}, \Omega = (L_2, \omega) \), where the action \( \omega : M_2 \times L_2 \to L_2 \) is defined by
\[ \omega(m, B) = \{ n | n \circ m \in B \} . \]

In fact, \( M - \text{Set} \) is a topos for any monoid \( M \) (see [4]).

**Definition 2.3.** We say that **supports split** (SS) for every \( \mathcal{E} \)-object \( A \) in the topos \( \mathcal{E} \) if the epic part of the epi-monic factorization of \( ! : A \to 1 \) has a right inverse.

**Lemma 2.4.** The topos \( M_2 - \text{Set} \) satisfies (SS).

**Proof.** Since supports split if and only if every subobject of the terminal object \( 1 = \{ \ast \} \) is a projective object, we only need to show that the terminal object is a projective object. Let \( f : (X, \psi) \to (Y, \phi) \) be an epimorphism in \( M_2 - \text{Set} \) where \( \psi : M_2 \times X \to X \) and \( \phi : M_2 \times Y \to Y \) are actions of \( M_2 \) on \( X \) and \( Y \), respectively. By the property of \( M_2 \), we have \( \psi(e, x) = x \) for all \( x \in X \). Let \( h : \{ \ast \} \to Y \) be a morphism in \( M_2 - \text{Set} \) where \( \nu : M_2 \times \{ \ast \} \to \{ \ast \} \) is a trivial action. Since \( f \) is an epimorphism, for \( h(\ast) \in Y \) there is a \( x' \in X \) such that \( f(x') = h(\ast) \). We define a morphism \( k : \{ \ast \} \to X \) by \( k(\ast) = \psi(a, x') \). Then \( k(\nu(n, \ast)) = \psi(n, k(\ast)) = k(\ast) \) and \( f \circ k(\ast) = f(\psi(a, x')) = \phi(a, f(x')) = \phi(a, h(\ast)) = h(\ast) \).

**Definition 2.5.** We say that \( \mathcal{E} \) satisfies the axiom (SG) if the subobjects of \( 1 = \{ \ast \} \) in \( \mathcal{E} \) form a class of generators.

**Definition 2.6.** A topos \( \mathcal{E} \) is called **Boolean** if for every object \( D \) in \( \mathcal{E} \), \( (\text{Sub}(D), \in) \) is a Boolean algebra where \( \text{Sub}(D) \) is the class of monomorphism with common codomain \( D \), and \( g \in f \) if there exists a morphism \( h : B \to A \) such that \( f \circ h = g \) where \( f : A \to D \) and \( g : B \to D \) are monomorphisms.

**Lemma 2.7** ([4]). For any topos \( \mathcal{E} \), the following statements are equivariant:

1. \( \mathcal{E} \) is Boolean.
2. \( \text{Sub}(\Omega) \) is a Boolean algebra.
3. \( \top : 1 \to \Omega \) has a complement in \( \text{Sub}(\Omega) \).
4. \( \bot : 1 \to \Omega \) is the complement of \( \top \) in \( \text{Sub}(\Omega) \).
5. \( \top \cup \bot \cong 1_{\Omega} \) in \( \text{Sub}(\Omega) \).
6. \( \mathcal{E} \) is classical.
7. \( i_1 : 1 \to 1 + 1 \) is a subobject classifier.

**Example 2.8.** If \( M \) is non-trivial monoid and not a group, then the category \( M - \text{Set} \) is a non-Boolean topos.

For the proof see Goldblatt [4], Ebahimi and Mahmoudi [3] and Madanshekaf
and Tavakoli [6].

**Lemma 2.9.** In any topos \( \mathcal{E} \), \( (ES) \) implies \( (SG) \).

**Proof.** Let \( f, g : X \to Y \) be two distinct morphisms. Then there is an equalizer \((E, e)\) of \( f \) and \( g \). Since a topos satisfying \( (AC) \) is Boolean, there is the complement of \( E \), denoted by \( \neg E \), such that \( m : \neg E \to X \) is a monomorphism. Consider \( \sigma_1(\neg E) \) where \( \sigma \) is a support functor and \( 1 = \{ \ast \} \) is a terminal object. Then \( \sigma_1(\neg E) \) is nonzero and \( s : \neg E \to \sigma_1(\neg E) \) is an epimorphism. By hypothesis, there is a morphism \( t : \sigma_1(\neg E) \to \neg E \) such that \( s \circ t = i_{\sigma_1(\neg E)} \). For \( m \circ t : \sigma_1(\neg E) \to X \), the equalizer of \( f \circ m \circ t \) and \( g \circ m \circ t \) is \( (m \circ t)^*(E \to X) \) which is isomorphic to \( 0 \to \sigma_1(\neg E) \) where \( 0 \) is an initial object. Therefore \( f \circ m \circ t \neq g \circ m \circ t \). □

**Definition 2.10.** A topos is called **well-pointed** if it satisfies the extensionality principle for morphisms, i.e., If \( f, g : A \to B \) are a pair of distinct parallel morphisms, then there is an element \( a : 1 \to A \) of \( A \) such that \( f \circ a \neq g \circ a \).

**Lemma 2.11 ([7]).** In a well-pointed topos, \( (ES) \), \( (DES) \) and \( (ASC) \) are equivalent.

### 3. Main Results

In this section, we show that \( (IAC) \) is not equivalent to \( (ASC) \) in some topoi, but \( (IAC) \) is equivalent to \( (ASC) \) in a well-pointed topos. Also we show that \( (AC) \) is not equivalent to \( (WO) \) in some topoi, but \( (AC) \) is equivalent to \( (WO) \) in a well-pointed topos. Therefore we know that \( (ES) \), \( (AC) \), \( (WO) \), \( (DES) \), \( (ASC) \) and \( (IAC) \) are equivalent in a well-pointed topos.

**Theorem 3.1.** There is a topos satisfying \( (IAC) \) and not satisfying \( (ASC) \).

**Proof.** Consider the topos \( G - \text{Set} \) where \( G = \{ e, g \} \) is a group with identity \( e \). We claim that, for any object \( X \in G - \text{Set} \), \((X) : G - \text{Set} \to G - \text{Set} \) preserves epimorphisms. Let \( h : A \to B \) be an epimorphism in \( G - \text{Set} \). Since the forgetful functor \( U : G - \text{Set} \to \text{Set} \) preserves epimorphisms, \( U(h) : U(A) \to U(B) \) is an epimorphism in \( \text{Set} \). Since \( U(h)^{U(X)} : U(A)^{U(X)} \to U(B)^{U(X)} \) is an epimorphism in \( \text{Set} \) and the forgetful functor \( U : G - \text{Set} \to \text{Set} \) preserves exponentials, \( U(h^X) : U(A^X) \to U(B^X) \) is an epimorphism in \( \text{Set} \) ([4]). Since the forgetful functor \( U : G - \text{Set} \to \text{Set} \) retracts epimorphisms, \( h^X : A^X \to B^X \) is an epimorphism in \( G - \text{Set} \).
Thus \( G - \text{Set} \) satisfies (IAC). But this topos does not satisfy (ASC) because there is not an absorbing element in \( G \) ([7]).

**Proposition 3.2.** In a topos \( \mathcal{E} \) which satisfies (SS), (IAC) implies (ASC).

**Proof.** Let \( f : X \to Y \) be an epimorphism in \( \mathcal{E} \). Since \( \mathcal{E} \) has pull-backs, there is a pull-back functor \( Y^* : \mathcal{E} \to \mathcal{E}/Y \) defined by \( Y^*(W) = W \times Y \to Y \) such that makes the following diagram a pull-back.

\[
\begin{array}{ccc}
W \times Y & \xrightarrow{\pi_1} & W \\
\downarrow & & \downarrow \\
Y & \longrightarrow & 1
\end{array}
\]

Since \( \mathcal{E} \) is a finitely complete category with exponentiation, a pull-back functor \( Y^* : \mathcal{E} \to \mathcal{E}/Y \) has a right adjoint \( \Pi_Y : \mathcal{E}/Y \to \mathcal{E} \) for every morphism \( Y \) with codomain \( 1 \). Since \( Y \) is internally projective (Johnstone [5]), \( f^Y : X^Y \to Y^Y \) is an epimorphism. Hence there is a morphism \( u : \Pi_Y(f) \to 1 \) which is a pull-back of \( f^Y : X^Y \to Y^Y \). Also, by the property of pull-back, \( u : \Pi_Y(f) \to 1 \) is an epimorphism. By hypothesis, \( u : \Pi_Y(f) \to 1 \) has a right inverse \( v : 1 \to \Pi_Y(f) \) such that \( u \circ v = i_1 \). Since the pull-back functor has a right adjoint, there is a morphism \( w : Y^*(1) \to f \) such that \( f \circ w = i_Y \). Thus every epimorphism in \( \mathcal{E} \) has a right inverse. Thus \( \mathcal{E} \) is Boolean (Diaconescu [2]). Hence every separated epimorphism is a section by Lemma 2.7.(4). Therefore \( \mathcal{E} \) satisfies (ASC).

**Theorem 3.3.** There is a topos satisfying (ASC) and not satisfying (IAC).

**Proof.** Consider the topos \( M_2 - \text{Set} \) where \( M_2 = \{ e, a \} \) is a monoid with identity \( e \), in which \( a \) has no inverse. Since \( a \) is an absorbent element in \( M_2 \), \( M_2 - \text{Set} \) satisfies (ASC) (Mawanda [7]). But this topos does not satisfy (IAC). Assume \( M_2 - \text{Set} \) satisfies (IAC). By Lemma 2.4, \( M_2 - \text{Set} \) satisfies (ES). Also, by Lemma 2.9, \( M_2 - \text{Set} \) satisfies (SG). But this is a contradiction. Since we can construct the two different morphism \( c, i : \{ X, \psi \} \to \{ X, \phi \} \) where \( X = \{ x, y \} \), \( c(x) = c(y) = x \) and \( i(x) = x, i(y) = y \). Also \( \psi : M_2 \times X \to X \) is an action of \( M_2 \) on \( X \) with \( \psi(e, x) = x, \psi(e, y) = y, \psi(a, x) = x, \psi(a, y) = x \) and \( \phi : M_2 \times X \to X \) is an action of \( M_2 \) on \( X \) with \( \phi(e, x) = x, \phi(e, y) = y, \phi(a, x) = x, \phi(a, y) = y \). Then \( c, i : \{ X, \psi \} \to \{ X, \phi \} \) are action preserving morphisms. And let \( h : \{ \{ \ast \} \} \to \{ X, \psi \} \) where \( h(\ast) = x \) and \( \nu : M_2 \times \{ \ast \} \to \{ \ast \} \) with trivial action. Since \( h(\ast) = x \) is the unique action preserving morphism, we have \( c \circ h = i \circ h \).
Proposition 3.4. In a Boolean topos $\mathcal{E}$, (ASC) implies (IAC).

Proof. By Lemma 2.7 and hypothesis, $\mathcal{E}$ satisfies (ASC) if and only if every epimorphism in $\mathcal{E}$ is a retraction. Let $e : X \to Y$ be an epimorphism in $\mathcal{E}$. We claim that, for any object $K$ in $\mathcal{E}$, $e^K : X^K \to Y^K$ is an epimorphism. For any morphism $s : K \to Y$ in $\mathcal{E}$, since the epimorphism $e : X \to Y$ is a retraction by hypothesis, there is a morphism $r : Y \to X$ in $\mathcal{E}$ such that $e \circ r = i_Y$. Hence we get a morphism $r \circ s : K \to X$ in $\mathcal{E}$ such that $e^K(r \circ s) = e \circ r \circ s = s$. Thus the morphism $e^K : X^K \to Y^K$ is an epimorphism. \qed

Proposition 3.5 ([7]). There is a topos satisfying (AC) and not satisfying (WO).

Proposition 3.6. In a Boolean topos $\mathcal{E}$, (AC) implies (WO).

Proof. Let $X_0$ be a noninitial object in $\mathcal{E}$. By (AC), there is a morphism $\psi : NX_0 \to X_0$ such that $\psi \circ g_i \in g'_i$ where $NX_0$ is the object of noninitial subobjects of $X_0$ with the usual ordering, $g_i : 1 \to NX_0$ is a morphism and $g'_i : X'_0 \to X_0$ is a monomorphism ([8]). By hypothesis, for $g_0$ such that $g_0(*) = X_0$ we get that $-(\psi \circ g_0) \equiv g_1$ which is the complement of $(\psi \circ g_0)$ in $g_0$, where the pullback of $\psi \circ g_0$ and $-(\psi \circ g_0)$ is the initial object, $-(\psi \circ g_1) \equiv g_2$ which is the complement of $(\psi \circ g_1)$ in $g_1$, where the pullback of $\psi \circ g_1$ and $-(\psi \circ g_1)$ is the initial object, etc. Generally, we get that $-(\psi \circ g_{n-1}) \equiv g_n$ which is the complement of $(\psi \circ g_{n-1})$ in $g_{n-1}$, where the pullback of $\psi \circ g_{n-1}$ and $-(\psi \circ g_{n-1})$ is the initial object. That is,

$$
\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\downarrow & & \downarrow \psi \circ g_0 \\
1 & \overset{-(\psi \circ g_0)}{\longrightarrow} & X_0
\end{array}
$$

is a pullback square.

Thus we construct $\phi : X_0 \to NX_0$ such that $\text{Im}(\phi)$ is a subobject of $NX_0$ consisting of $g_0, -(\psi \circ g_0), -(\psi \circ g_1), \ldots$ and $-(\psi \circ g_{n-1})$, where $-(\psi \circ g_{n-1})$ is a noninitial object, and $\psi \circ \phi = i_{X_0}$. Then $\text{Im}(\phi)$ is a linear ordered with minimal choice. Since $\phi$ is a monomorphism, $X_0$ has an ordering with minimal choice. \qed

Proposition 3.7 ([7]). There is a topos satisfying (WO) and not satisfying (AC).

Proposition 3.8. In a well-pointed topos $\mathcal{E}$, (WO) implies (AC).

Proof. Since every morphism is epi-monic factorizable in $\mathcal{E}$, for any morphism $f : A \to B$ in $\mathcal{E}$, there exist an epimorphism $e : A \to X$ and a monomorphism $m :$
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\[ X \to B \] such that \( f = m \circ e \). By hypothesis, there exists a morphism \( t : B \to X \) such that \( t \circ m = i_X \). We only show that there is a morphism \( s : X \to A \) such that \( f = f \circ (s \circ t) \circ f = f \). Since \( e : A \to X \) is an epimorphism, there is a morphism \( q : X \to \Omega^A \) which is the interpretation of the term \( \{a|e(a) = x\} \). By definition of (WO), we can find an epimorphism \( r : V \to X \) and a morphism \( n : V \to A \) such that \( n \) is a minimal choice of \( qr \). Since every epimorphism is a coequalizer, there are morphisms \( u, v : W \to V \) such that the following square

\[
\begin{array}{ccc}
W & \longrightarrow & V \\
\downarrow v & \quad & \downarrow r \\
V & \longrightarrow & X
\end{array}
\]

commutes.

Thus we get \( q \circ r \circ u = q \circ r \circ u \). Also \( nu, nv \) are both minimal choice of \( q \circ r \circ u \). By definition of (WO), we have \( nu = nv \). Since every epimorphism is a coequalizer, there is a morphism \( s : X \to A \) such that \( s \circ r = n \). Also there is a morphism \( c : X \to \varepsilon_A \) such that \( k \circ s = c \) where \( k : A \to \varepsilon_A \) and \( \varepsilon_A \) is the subobject classified by \( ev : \Omega^A \times A \to \Omega \). Then we have \( (q, s) = loc = lokos = (qoeos, s) \) where \( l : \varepsilon_A \to \Omega^A \times A \). Since \( q \) is a monomorphism, we have \( f = f \circ (s \circ t) \circ f = f \). \( \square \)

**Corollary 3.9.** In a well-pointed topos \( \mathcal{E} \), (ES), (AC), (WO), (DES), (ASC) and (IAC) are equivalent.

**REFERENCES**

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