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WEIGHTED ENDPOINT INEQUALITIES FOR MULTILINEAR MARCINKIEWICZ INTEGRAL OPERATOR

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Abstract. We prove a sharp inequality for some multilinear operator related to Marcinkiewicz integral operator. As application, we obtain the weighted norm inequality and $L \log L$ type estimate for the multilinear operator.

1. Introduction and results

Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip$_\gamma$ condition on $S^{n-1}(0 < \gamma \leq 1)$, i.e.,

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} \Omega(x')dx' = 0$.

Let m be a positive integer and A be a function on $\mathbb{R}^n$. We denote $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ and the characteristic of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz integral operator is defined by

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z|\leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)dz$$

and

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\leq m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

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Set

$$F_t(f)(y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

We also define that

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2},$$

which is the Marcinkiewicz integral operator (see [16]).

Let H be the Hilbert space

$$H = \left\{ h : ||h|| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2} < \infty \right\}.$$

Then for each fixed $x, y \in \mathbb{R}^n$, $F_t^A(f)(x,y)$ and $F_t(f)(x)$ may be viewed as a mapping from the spaces of measurable functions to H, and it is clear that

$$\mu_S^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right| \text{ and } \mu_S(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|.$$

Note that when $m = 0$, μ_S^A is just the commutator of Marcinkiewicz integral operator (see [10], [16]), while when $m > 0$, it is a non-trivial generalization of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-5]). In [8], authors establish a variant sharp estimate for some multilinear singular integral operators. The main purpose of this paper is to establish a sharp estimate for the multilinear Marcinkiewicz operator, then the weighted norm inequalities and the $L \log L$ type endpoint estimate for the multilinear operator are obtained by using the sharp estimate. We point out that some of our ideas come from [8] and [11]. First, let us introduce some notations (see [7], [11], [13]).

For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy,$$

where, and in what follows, Q will denote a cube with sides parallel to the axes, and $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that

$$f^{\#}(x) = \sup_{x\in Q} \inf_{c\in C} \frac{1}{|Q|} \int_Q |f(y) - c|dy.$$

We say that f belongs to $BMO(\mathbb{R}^n)$ if $f^{\#}$ belongs to $L^\infty(\mathbb{R}^n)$. For $0 < r < \infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator, that is,

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

we write that $M_k f = (M(f^k))^{1/k}$ for $k \in \mathbb{N}$, we denote by M^k the operator M iterated k times, i.e., $M^1 f(x) = Mf(x)$ and

$$M^k f(x) = M(M^{k-1} f)(x) \quad \text{when} \quad k \geq 2.$$

Let B be a Young function and $\tilde{B}$ be the complementary associated to B, we denote that, for a function f

$$||f||_{B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B(\frac{|f(y)|}{\lambda})dy \leq 1\right\}$$

and the maximal function by

$$M_B f(x) = \sup_{x\in Q} ||f||_{B,Q};$$

The main Young function to be used in this paper is $B(t) = t(1+\log^+ t)$ and its complementary $\tilde{B}(t) = \exp t$, the corresponding maximal denoted by $M_{L\log L}$ and $M_{\exp L}$. We have the generalized Hölder's inequality

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \leq ||f||_{B,Q}||g||_{\tilde{B},Q}$$

and the following inequality (in fact they are equivalent), for any $x \in \mathbb{R}^n$,

$$M_{L\log L} f(x) \leq CM^2 f(x)$$

and the following inequalities, for all cubes Q and any $b \in BMO(\mathbb{R}^n)$

$$||b - b_Q||_{\exp L,Q} \leq C||b||_{BMO} \text{ and } |b_{2^{k+1}Q} - b_{2Q}| \leq 2k||b||_{BMO}.$$

We denote the Muckenhoupt weights by A_p for $1 \leq p < \infty$ (see [7]).

Now we state the results in this paper as follows.

Theorem 1. *Let $D^\alpha A \in BMO(\mathbb{R}^n)$ for all α with $|\alpha| = m$. Then for any $0 < r < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$,*

$$(\mu_S^A(f))_r^{\#}(x) \leq C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2 f(x).$$

Theorem 2. *Let $1 < p < \infty$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for all α with $|\alpha| = m$, $w \in A_p$. Then μ_S^A is bounded on $L^p(w)$, that is,*

$$||\mu_S^A(f)||_{L^p(w)} \leq C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}||f||_{L^p(w)}.$$

Theorem 3. *Let $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$, $w \in A_1$. Then there exists a constant $C > 0$ such that for each $\lambda > 0$,*

$$w(\{x \in \mathbb{R}^n : \mu_S^A(f)(x) > \lambda\})$$
$$\leq C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}\int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx.$$

As in [11], Theorem 2 and 3 follow from Theorem 1 and the boundedness of μ_S with M. So we only need to prove Theorem 1.

2. Proof of theorems

We begin with some preliminary lemmas.

Lemma 1 (Kolmogorov, [7, p.485]). *Let $0 < p < q < \infty$ and for any measurable function $f \geq 0$. We define that*

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : f(x) > \lambda\}|^{1/q},$$
$$N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r}, (1/r = 1/p - 1/q),$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} ||f||_{WL^q}.$$

Lemma 2 ([3, p.448]). *Let A be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 3 ([11, p.165]). *Let $w \in A_1$. Then there exists a constant $C > 0$ such that for any function f and for all $\lambda > 0$,*

$$w(\{y \in \mathbb{R}^n : M^2 f(y) > \lambda\}) \leq C\lambda^{-1} \int_{\mathbb{R}^n} |f(y)|(1 + \log^+(\lambda^{-1}|f(y)|))w(y)dy.$$

Lemma 4. *Let $1 < p < \infty$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for all α with $|\alpha| = m$, $1 < r \leq \infty$, $1/q = 1/p - 1/r$. Then μ_S^A is bound from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, that is,*

$$||\mu_S^A(f)||_{L^q} \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^p}.$$

Proof. By Minkowski's inequality and note that $|x - z| \leq 2t$, $|y - z| \geq |x - z| - |x - y| \geq |x - z| - t$ when $|x - y| \leq t$, $|y - z| \leq t$, we have

$$\mu_S^A(f)(x)$$
$$\leq \int_{\mathbb{R}^n} \left[\int\int_{|x-y|\leq t} \left(\frac{|\Omega(y-z)||R_{m+1}(A; x, z)||f(z)|}{|y-z|^{n-1}|x-z|^m} \right)^2 \chi_{\Gamma(z)}(y,t) \frac{dydt}{t^{n+3}} \right]^{1/2} dz$$
$$\leq C \int_{\mathbb{R}^n} \frac{|R_{m+1}(A; x, z)||f(z)|}{|x-z|^m} \left[\int\int_{|x-y|\leq t} \frac{\chi_{\Gamma(z)}(y,t)t^{-n-3}}{(|x-z| - 3t)^{2n-2}} dydt \right]^{1/2} dz$$

$$\le C\int_{\mathbb{R}^n} \frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+3/2}} \left[\int_{|x-z|/2}^{\infty} \frac{dt}{(|x-z|-t)^{2n-2}}\right]^{1/2} dz$$

$$\le C\int_{\mathbb{R}^n} \frac{|R_{m+1}(A;x,y)|}{|x-z|^{m+n}}|f(z)|dz,$$

thus, the lemma follows from [4], [5]. □

We first prove Theorem 1.

Proof of Theorem 1. Fix $\tilde{x} \in \mathbb{R}^n$. Let $Q = Q(x_0, l)$ be a cube centered at x_0 and having side length l such that $\tilde{x} \in Q$. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |\mu_S^A(f)(x) - C_0|^r dx\right)^{1/r} \le CM^2 f(\tilde{x}).$$

Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_{m+1}(A;x,y) = R_{m+1}(\tilde{A};x,y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus\tilde{Q}}$,

$$\begin{aligned}
F_t^A(f)(x,y) &= \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{R_{m+1}(\tilde{A};x,z)}{|x-z|^m} f(z)dz \\
&= \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{R_{m+1}(\tilde{A};x,z)}{|x-z|^m} f_2(z)dz \\
&\quad + \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{R_m(\tilde{A};x,z)}{|x-z|^m} f_1(z)dz \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{(x-z)^\alpha}{|x-z|^m} D^\alpha \tilde{A}(z) f_1(z)dz,
\end{aligned}$$

then

$$\begin{aligned}
&\left|\mu_S^A(f)(x) - \mu_S^{\tilde{A}}(f_2)(x_0)\right| \\
&= \left| ||\chi_{\Gamma(x)} F_t^A(f)(x,y)|| - ||\chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0,y)|| \right| \\
&\le \left|\left|\chi_{\Gamma(x)} F_t\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m} f_1\right)(y)\right|\right| + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left|\left|\chi_{\Gamma(x)} F_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1\right)(y)\right|\right| \\
&\quad + \left|\left|\chi_{\Gamma(x)} F_t^{\tilde{A}}(f)(x,y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0,y)\right|\right| \\
&= I(x) + II(x) + III(x),
\end{aligned}$$

thus,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|\mu_S^A(f)(x) - \mu_S^{\tilde{A}}(f_2)(x_0)\right|^r dx\right)^{1/r} \\
&\leq \left(\frac{C}{|Q|}\int_Q I(x)^r dx\right)^{1/r} + \left(\frac{C}{|Q|}\int_Q II(x)^r dx\right)^{1/r} + \left(\frac{C}{|Q|}\int_Q III(x)^r dx\right)^{1/r} \\
&= I + II + III.
\end{aligned}$$

Now, let us estimate I, II, and III, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, using Lemma 2, we get

$$R_m(\tilde{A}; x, y) \leq C|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO},$$

thus, by Lemma 1 and the weak type (1,1) of μ_S (see [6], [14]), we obtain

$$\begin{aligned}
I &\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}|Q|^{-1}\frac{||\mu_S(f_1)\chi_Q||_{L^r}}{||\chi_Q||_{L^{r/(1-r)}}} \\
&\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}|Q|^{-1}||\mu_S(f_1)||_{WL^1} \\
&\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}|\tilde{Q}|^{-1}\int_{\tilde{Q}}|f(y)|dy \leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(\tilde{x});
\end{aligned}$$

For II, similar to the proof of I, we get

$$\begin{aligned}
II &\leq C \sum_{|\alpha|=m} |Q|^{-1}\frac{||\mu_S(D^\alpha \tilde{A} f_1)\chi_Q||_{L^r}}{||\chi_Q||_{L^{r/(1-r)}}} \\
&\leq C \sum_{|\alpha|=m} |Q|^{-1}||\mu_S(D^\alpha \tilde{A} f_1)||_{WL^1} \\
&\leq C \sum_{|\alpha|=m} |Q|^{-1}\int_{\tilde{Q}}|D^\alpha \tilde{A}(y)||f(y)|dy \\
&\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\exp L, \tilde{Q}}||f||_{L\log L, \tilde{Q}} \\
&\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M_{L\log L} f(\tilde{x}) \\
&\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2 f(\tilde{x});
\end{aligned}$$

To estimate III, we write

$$\begin{aligned}
&\chi_{\Gamma(x)}(y,t)F_t^{\tilde{A}}(f_2)(x,y) - \chi_{\Gamma(x_0)}(y,t)F_t^{\tilde{A}}(f_2)(x_0,y) \\
&= \int_{|y-z|\leq t}\chi_{\Gamma(x)}(y,t)\left[\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m}\right]\frac{\Omega(y-z)R_m(\tilde{A};x,z)f_2(z)}{|y-z|^{n-1}}dz
\end{aligned}$$

$$+\int_{|y-z|\le t}\frac{\chi_{\Gamma(x)}(y,t)\Omega(y-z)f_2(z)}{|x_0-z|^m|y-z|^{n-1}}[R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)]dz$$
$$+\int_{|y-z|\le t}(\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t))\frac{\Omega(y-z)R_m(\tilde{A};x_0,z)}{|y-z|^{n-1}|x_0-z|^m}f_2(z)dz$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{|y-z|\le t}\left[\frac{\chi_{\Gamma(x)}(y,t)(x-z)^\alpha}{|x-z|^m}-\frac{\chi_{\Gamma(x_0)}(y,t)(x_0-z)^\alpha}{|x_0-z|^m}\right]$$
$$\times\frac{\Omega(y-z)D^\alpha\tilde{A}(z)}{|y-z|^{n-1}}f_2(z)dz$$
$$=III_1+III_2+III_3+III_4.$$

Note that $|x-z|\sim|x_0-z|$ for $x\in Q$ and $z\in\mathbb{R}^n\setminus\tilde{Q}$. By Lemma 3 and the following inequality (see [15])

$$|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)||b||_{BMO}\quad \text{for } Q_1\subset Q_2,$$

we know that, for $x\in Q$ and $z\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$|R_m(\tilde{A};x,z)|\le C|x-z|^m\sum_{|\alpha|=m}(||D^\alpha A||_{BMO}+|(D^\alpha A)_{\tilde{Q}(x,z)}-(D^\alpha A)_{\tilde{Q}}|)$$
$$\le Ck|x-z|^m\sum_{|\alpha|=m}||D^\alpha A||_{BMO};$$

For III_1, by the condition on Ω and similar to the proof of Lemma 4, we get

$$\frac{1}{|Q|}\int_Q||III_1||dx\le\frac{C}{|Q|}\int_Q\left(\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|x-x_0|}{|x_0-z|^{m+n+1}}|R_m(\tilde{A};x,z)||f(z)|dz\right)dx$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=0}^\infty k\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|Q|^{1/n}}{|x_0-z|^{n+1}}|f(z)|dz$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k2^{-k}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(z)|dz$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k2^{-k}M(f)(\tilde{x})$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(\tilde{x});$$

For III_2, by the formula (see [3]):

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-z)^\beta$$

and Lemma 3, we have

$$
\begin{aligned}
&|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)| \\
&\le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-z|^{|\beta|}||D^\alpha A||_{BMO} \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|x-x_0||x-z|^{m-1},
\end{aligned}
$$

thus, similar to the proof of Lemma 4

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q ||III_2||dx \\
&\le \frac{C}{|Q|}\int_Q\left(\int_{\mathbb{R}^n\setminus\tilde{Q}}\frac{|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n}}|f(z)|dz\right)dx \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|Q|^{1/n}}{|x_0-z|^{n+1}}|f(z)|dz \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(z)|dz \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(\tilde{x});
\end{aligned}
$$

For III_3, similar to the proof of Lemma 4, we obtain

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q ||III_3||dx \\
&\le \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m} \\
&\quad\cdot\left(\int\int_{\mathbb{R}^{n+1}_+}\frac{\chi_{\Gamma(z)}(y,t)|\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\right)^{1/2}dzdx \\
&\le \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m} \\
&\quad\cdot\left|\int\int_{\Gamma(x)}\frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2}t^{n+3}}-\int\int_{\Gamma(x_0)}\frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2}t^{n+3}}\right|^{1/2}dzdx \\
&\le \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m} \\
&\quad\cdot\left(\int\int_{|x+y-z|\le t}\left|\frac{1}{|x+y-z|^{2n-2}}-\frac{1}{|x_0+y-z|^{2n-2}}\right|\frac{dydt}{t^{n+3}}\right)^{1/2}dzdx
\end{aligned}
$$

$$\le \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}$$
$$\cdot\left(\int\int_{|y|\le t,|x+y-z|\le t}\frac{|x-x_0|dydt}{|x+y-z|^{2n-1}t^{n+3}}\right)^{1/2}dzdx$$
$$\le \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n}\frac{|f_2(z)||R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2}}dzdx$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k2^{-k/2}\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(z)|dz$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(\tilde{x});$$

For III_4, similar to the proof of III_1 and III_3, we get

$$||III_4||$$
$$\le C\sum_{|\alpha|=m}\int_{\mathbb{R}^n}\left(\frac{|x-x_0|}{|x_0-z|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2}}\right)|D^\alpha\tilde{A}(y)||f_2(z)|dz$$
$$\le C\sum_{|\alpha|=m}\sum_{k=1}^\infty(2^{-k}+2^{-k/2})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(z)||D^\alpha A(z)-(D^\alpha A)_{\tilde{Q}}|dz$$
$$\le C\sum_{|\alpha|=m}\sum_{k=1}^\infty k(2^{-k}+2^{-k/2})(||D^\alpha A||_{\exp L,2^k\tilde{Q}}||f||_{L\log L,2^k\tilde{Q}}+||D^\alpha A||_{BMO}M(f)(\tilde{x}))$$
$$\le C\sum_{|\alpha|=m}\sum_{k=1}^\infty k(2^{-k}+2^{-k/2})||D^\alpha A||_{BMO}M_{L\log L}(f)(\tilde{x})$$
$$\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(\tilde{x}).$$

Thus,

$$III\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(\tilde{x}).$$

This completes the proof of Theorem 1. □

From Theorem 1 and the weighted boundedness of μ_S and M, we may obtain the conclusion of Theorem 2.

From Theorem 1 and Lemma 3, we may obtain the conclusion of Theorem 3.

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