Bull. Korean Math. Soc. **45** (2008), No. 1, pp. 11–21

ON SUPER EDGE-MAGIC LABELING OF SOME GRAPHS

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Abstract. A graph $G = (V, E)$ is called super edge-magic if there exists a one-to-one map λ from $V \cup E$ onto $\{1, 2, 3, \ldots, |V| + |E|\}$ such that $\lambda(V) = \{1, 2, \ldots, |V|\}$ and $\lambda(x) + \lambda(xy) + \lambda(y)$ is constant for every edge xy. In this paper, we investigate whether some families of graphs are super edge-magic or not.

1. Introduction

Throughout this paper, we assume that all graphs are finite, simple and undirected. A graph G has vertex set $V(G)$ and edge set $E(G)$ and we let $|V(G)| = \nu$ and $|E(G)| = \epsilon$. A general reference theoretic notions is West [6].

Given a graph G, $V(G) = V$ and $E(G) = E$. A *labeling* for a graph is a map that takes graph elements to numbers (usually positive or non-negative integers). Various authors have introduced labelings that generalize the idea of magic square.

Kotzig and Rosa [2] defined a magic labeling to be a total labeling on the vertices and edges in which the labels are the integers from 1 to $|V(G)|+|E(G)|$. The sum of labels on an edge and its two endpoints is constant. In 1996 Ringel and Llado [3] redefined this type of labeling as edge-magic. Also, Enomoto *et al* [1] have introduced the name *super edge-magic* for magic labelings in the sense of Kotzig and Rosa, with the added property that the ν vertices receive the smaller labels, $\{1, 2, \ldots, \nu\}$.

A one-to-one map λ from $V \cup E$ onto the integers $\{1, 2, \ldots, \nu + \epsilon\}$ is an *edge-magic labeling* if there is a constant k so that for any edge xy, $\lambda(x) + \lambda(xy) + \lambda(y) = k$. The constant k is called the *edge magic number for* λ. An edge-magic labeling λ is called *super edge-magic* if $\lambda(V) = \{1, 2, \ldots, \nu\}$ and $\lambda(E) = \{\nu + 1, \nu + 2, \ldots, \nu + \epsilon\}$. A graph G is called *edge-magic* (resp. *super edge-magic*) if there exists an edge-magic (resp. super edge-magic) labeling of G.

Received December 6, 2005; Revised September 12, 2006.

2000 *Mathematics Subject Classification.* 05C78, 68R10, 90B18, 90C35.

Key words and phrases. edge magic labeling, super edge-magic graphs, magic number.

This research was supported by the Kyung Hee University Research Fund in 2005.(KHU-20051059).

Kotzig and Rosa [2] and Wallis *et al* [5] showed the edge-magicness for cycles C_n and some variants of C_n. Also, Enomoto *et al* [1] proved that a cycle C_n is super edge-magic if and only if n is odd.

In this paper, we also find super edge-magic variants of C_n. First of all, we show that an $(n, 2)$-kite is super edge-magic if n is even and that the converse is also true. Finally, we prove that the graph $K_2 \cup C_n$ is super edge-magic if n is even. It was motivated by Wallis [4] question to characterize edge-magic graphs.

2. Main results

We take some families of graphs to see whether or not they are super edge-magic. An *(n, t)-kite* is a cycle of length n with a t-edge path (the tail) attached to one vertex. In [5], Wallis *et al.* showed that $(n, 1)$-kite is edge-magic.

The following proposition shows that an $(n, 2)$-kite is super edge-magic if n is even and that the converse is also true.

Proposition 2.1. *Let $G = (n, 2)$-kite. The graph G is super edge-magic if and only if n is even.*

Proof. Let $v_0, v_1, \ldots, v_{n-1}, v_0$ be a vertex sequence of C_n, a vertex v is adjacent to v_0 and a vertex w is adjacent to v. Suppose there exists a super edge-magic labeling λ of G with the magic number k. Set $\lambda(v_0) = \alpha$ and $\lambda(w) = \beta$. Then

$$\begin{aligned} k(n+2) &= \sum_{xy \in E(G)} \{\lambda(x) + \lambda(xy) + \lambda(y)\} \\ &= 2 \sum_{x \in V(G)} \lambda(x) + \lambda(v_0) - \lambda(w) + \sum_{xy \in E(G)} \lambda(xy) \\ &= \frac{(n+2)(5n+13)}{2} + \alpha - \beta. \end{aligned}$$

This implies that $k - \frac{\alpha-\beta}{n+2} = \frac{5n+13}{2}$ is a rational number that is not integer, since $0 < \left|\frac{\alpha-\beta}{n+2}\right| < 1$. Thus n must be an even. Let $n = 2m + 2$ for a nonnegative integer m. $V(G) = \{v_0, v_1, \ldots, v_{n-1}, v, w\}$ and $E(G) = \{v_i v_{i+1} \mid 0 \le i \le n - 2\} \cup \{v_{n-1}v_0\} \cup \{v_0 v\} \cup \{vw\}$. We define a labeling $\lambda : V \cup E \longrightarrow \{1, 2, \ldots, 4m+8\}$ as follows:

Case where m is odd.

$$\lambda(v_i) = \begin{cases} (m-i+1)/2 & \text{if } i = 0, 2, \ldots, m-3, m-1 \\ (3m-i+4)/2 & \text{if } i = 1, 3, \ldots, m-2, m \\ (3m-i+3)/2 & \text{if } i = m+1, m+3, \ldots, 2m \\ (5m-i+10)/2 & \text{if } i = m+2, m+4, \ldots, 2m+1, \end{cases}$$

$$\lambda(v) = (3m+7)/2,$$
$$\lambda(w) = (3m+5)/2,$$

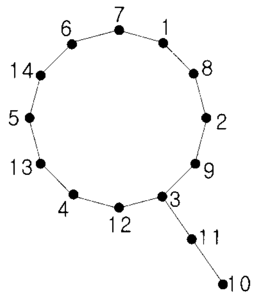

FIGURE 1. An illustration for the labeling given in the proof of Proposition 2.1.

$$\lambda(v_iv_{i+1}) = \begin{cases} 3m+i+9 & \text{if } 0 \le i \le m-1 \\ 3m+8 & \text{if } i = m \\ m+i+5 & \text{if } m+1 \le i \le 2m, \end{cases}$$

$$\lambda(v_{n-1}v_0) = 3m+6,$$
$$\lambda(v_0v) = 3m+7,$$
$$\lambda(vw) = 2m+5.$$

(See Figure 1 for illustration.) It is easily seen that λ is a super edge-magic labeling of G with the edge magic number $5m+11$.

Case where m is even.

$$\lambda(v_i) = \begin{cases} (m+i)/2 & \text{if } i = 0,2,\ldots,m-2 \\ (3m+i+9)/2 & \text{if } i = 1,3,\ldots,m-1 \\ m & \text{if } i = m+1 \\ (m+i+2)/2 & \text{if } i = m, m+2,\ldots,2m \\ (i-m-1)/2 & \text{if } i = m+3, m+5, \ldots, 2m-1 \\ (3m+6)/2 & \text{if } i = 2m+1, \end{cases}$$

$$\lambda(v) = (3m+8)/2,$$
$$\lambda(w) = (3m+4)/2,$$

$$\lambda(v_iv_{i+1}) = \begin{cases} 3m-i+6 & \text{if } i = 0,1,\ldots,m-2 \\ 2m+6 & \text{if } i = m-1 \\ 4m-i+10 & \text{if } i = m, m+1 \\ 5m-i+10 & \text{if } i = m+2, m+3, \ldots, 2m-1 \\ 2m+7 & \text{if } i = 2m, \end{cases}$$

$$\lambda(v_{n-1}v_0) = 3m+8,$$
$$\lambda(v_0v) = 3m+7,$$
$$\lambda(vw) = 2m+5.$$

(See Figure 2 for illustration.) It is easily seen that λ is a super edge-magic labeling of G with the edge magic number $5m+11$. Hence, the graph G is super edge-magic. □

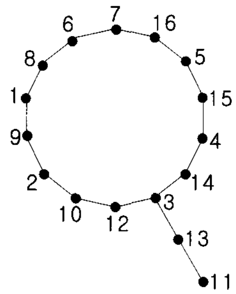

FIGURE 2. An illustration for the labeling given in the proof of Proposition 2.1.

A *complete graph* is a simple graph in which every pair of vertices forms an edge. We denote by K_n the complete graph with n vertices. Also, we denote by $K_2 \cup C_n$ the graph obtained from union of disjoint graphs K_2 and C_n. In [4], W. D. Wallis proved that $K_2 \cup C_3$ is not edge-magic, but $K_2 \cup C_4$ is edge-magic. Then Wallis [4] proposed the following problem: For which values of n, is $K_2 \cup C_n$ edge-magic?

The following theorem shows that $K_2 \cup C_n$ is super edge-magic if n is even $(n \neq 10)$.

Theorem 2.2. *Let $G = K_2 \cup C_n$. The graph G is super edge-magic if n is even $(n \neq 10)$.*

Proof. Let $v_0, v_1, \ldots, v_{n-1}, v_0$ be a vertex sequence of C_n and let u and w be the vertices of K_2. Let $n = 2m$ for a nonnegative integer m. $V(G) = \{v_0, v_1, \ldots, v_{n-1}, u, w\}$ and $E(G) = \{v_i v_{i+1} \mid 0 \leq i \leq n-2\} \cup \{v_{n-1}v_0\} \cup \{uw\}$. We define a labeling $\lambda : V \cup E \longrightarrow \{1, 2, \ldots, \nu + \epsilon\}$ as follows:

If $n = 4$, then $\lambda(V(C_4)) = (2, 3, 5, 4)$ and $\lambda(V(K_2)) = \{1, 6\}$. Thus λ is a super edge-magic labeling of $K_2 \cup C_4$ with the edge magic number 16. Now, consider $n \geq 6$.

Case $m \equiv 0 \pmod 6$

$$\lambda(v_i) = \begin{cases} m + 2 + i/2 & \text{if } i = 0, 2, \ldots, m-2 \\ (i+1)/2 & \text{if } i = 1, 3, \ldots, m-1 \\ (3m+8)/2 & \text{if } i = m. \end{cases}$$

If $m + 1 \leq i \leq n - 1$, then

$$\lambda(v_i) = \begin{cases} (i+3)/2 & \text{if } i \equiv 1 \pmod 6 \\ m + 2 + i/2 & \text{if } i \equiv 2 \pmod 6 \\ (i+5)/2 & \text{if } i \equiv 3 \pmod 6 \\ m + 3 + i/2 & \text{if } i \equiv 4 \pmod 6 \\ (i+1)/2 & \text{if } i \equiv 5 \pmod 6 \\ m + 4 + i/2 & \text{if } i \equiv 0 \pmod 6), \end{cases}$$

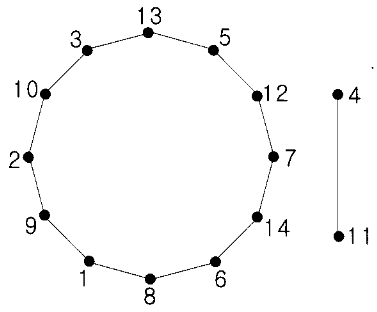

FIGURE 3. An illustration for the labeling given in the proof of Theorem 2.2.

$$\lambda(u) = (m+2)/2,$$
$$\lambda(w) = (3m+4)/2,$$
$$\lambda(v_i v_{i+1}) = \begin{cases} 4m - i + 3 & \text{if } 0 \le i \le m-2, \\ 3m+2 & \text{if } i = m-1. \end{cases}$$

If $m \le i \le n-2$, then

$$\lambda(v_i v_{i+1}) = \begin{cases} 4m + 2 - i & \text{if } i \equiv 1 \pmod 3 \\ 4m + 1 - i & \text{if } i \equiv 2 \pmod 3 \\ 4m - i & \text{if } i \equiv 0 \pmod 3, \end{cases}$$

$$\lambda(v_0 v_{n-1}) = 3m+4,$$
$$\lambda(uw) = 3m+3.$$

(See Figure 3 for illustration.) It is easily seen that λ is a super edge-magic labeling of G with the edge magic number $5m+6$.

Case $m \equiv 3 \pmod 6$

$$\lambda(v_i) = \begin{cases} m + 2 + i/2 & \text{if } i = 0, 2, \ldots, m-1 \\ (i+1)/2 & \text{if } i = 1, 3, \ldots, m-2 \\ (m+5)/2 & \text{if } i = m. \end{cases}$$

If $m+1 \le i \le n-1$, then

$$\lambda(v_i) = \begin{cases} (i+3)/2 & \text{if } i \equiv 1 \pmod 6 \\ m + 2 + i/2 & \text{if } i \equiv 2 \pmod 6 \\ (i+5)/2 & \text{if } i \equiv 3 \pmod 6 \\ m + 3 + i/2 & \text{if } i \equiv 4 \pmod 6 \\ (i+1)/2 & \text{if } i \equiv 5 \pmod 6 \\ m + 4 + i/2 & \text{if } i \equiv 0 \pmod 6, \end{cases}$$

$$\lambda(u) = (m+1)/2,$$
$$\lambda(w) = (3m+5)/2,$$
$$\lambda(v_i v_{i+1}) = \begin{cases} 4m - i + 3 & \text{if } 0 \le i \le m-2 \\ 3m+2 & \text{if } i = m-1. \end{cases}$$

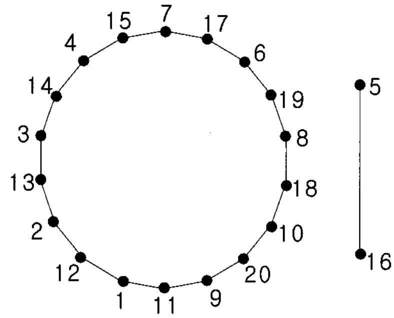

FIGURE 4. An illustration for the labeling given in the proof of Theorem 2.2.

If $m \le i \le n-2$, then

$$\lambda(v_i v_{i+1}) = \begin{cases} 4m+2-i & \text{if } i \equiv 1 \pmod 3 \\ 4m+1-i & \text{if } i \equiv 2 \pmod 3 \\ 4m-i & \text{if } i \equiv 0 \pmod 3, \end{cases}$$

$$\lambda(v_0 v_{n-1}) = 3m+4,$$

$$\lambda(uw) = 3m+3.$$

(See Figure 4 for illustration.) It is easily seen that λ is a super edge-magic labeling of G with the edge magic number $5m+6$.

Case $m \equiv 1 \pmod 6$

$$\lambda(v_i) = \begin{cases} m+2+i/2 & \text{if } i = 0, 2, \ldots, m-1 \\ (i+1)/2 & \text{if } i = 1, 3, \ldots, m-2 \\ (m+5)/2 & \text{if } i = m. \end{cases}$$

If $m+1 \le i \le n-1$, then

$$\lambda(v_i) = \begin{cases} (i+1)/2 & \text{if } i \equiv 1 \pmod 6 \\ m+4+i/2 & \text{if } i \equiv 2 \pmod 6 \\ (i+3)/2 & \text{if } i \equiv 3 \pmod 6 \\ m+2+i/2 & \text{if } i \equiv 4 \pmod 6 \\ (i+5)/2 & \text{if } i \equiv 5 \pmod 6 \\ m+3+i/2 & \text{if } i \equiv 0 \pmod 6 \\ (m+3)/2 & \text{if } i = m+2, \end{cases}$$

$$\lambda(u) = (m+1)/2,$$

$$\lambda(w) = (3m+5)/2,$$

$$\lambda(v_i v_{i+1}) = \begin{cases} 4m-i+3 & \text{if } 0 \le i \le m-2 \\ 3m+2 & \text{if } i = m-1 \\ 2m+i-1 & \text{if } m \le i \le m+2. \end{cases}$$

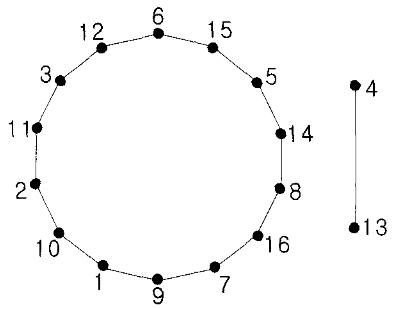

FIGURE 5. An illustration for the labeling given in the proof of Theorem 2.2.

If $m+3 \le i \le n-2$, then

$$\lambda(v_i v_{i+1}) = \begin{cases} 4m+1-i & \text{if } i \equiv 1 \pmod 3 \\ 4m-i & \text{if } i \equiv 2 \pmod 3 \\ 4m+2-i & \text{if } i \equiv 0 \pmod 3), \end{cases}$$

$$\lambda(v_0 v_{n-1}) = 3m+4,$$

$$\lambda(uw) = 3m+3.$$

(See Figure 5 for illustration.) It is easily seen that λ is a super edge-magic labeling of G with the edge magic number $5m+6$.

Case $m \equiv 4 \pmod 6$

$$\lambda(v_i) = \begin{cases} m+2+i/2 & \text{if } i = 0, 2, \ldots, m-2 \\ (i+1)/2 & \text{if } i = 1, 3, \ldots, m-1 \\ 3m/2+4 & \text{if } i = m. \end{cases}$$

If $m+1 \le i \le n-1$, then

$$\lambda(v_i) = \begin{cases} (i+1)/2 & \text{if } i \equiv 1 \pmod 6 \\ m+4+i/2 & \text{if } i \equiv 2 \pmod 6 \\ (i+3)/2 & \text{if } i \equiv 3 \pmod 6 \\ m+2+i/2 & \text{if } i \equiv 4 \pmod 6 \\ (i+5)/2 & \text{if } i \equiv 5 \pmod 6 \\ m+3+i/2 & \text{if } i \equiv 0 \pmod 6 \\ 3m/2+3 & \text{if } i = m+2, \end{cases}$$

$$\lambda(u) = (m+2)/2,$$

$$\lambda(w) = (3m+4)/2,$$

$$\lambda(v_i v_{i+1}) = \begin{cases} 4m-i+3 & \text{if } 0 \le i \le m-2 \\ 3m+2 & \text{if } i = m-1 \\ 2m+i-1 & \text{if } m \le i \le m+2. \end{cases}$$

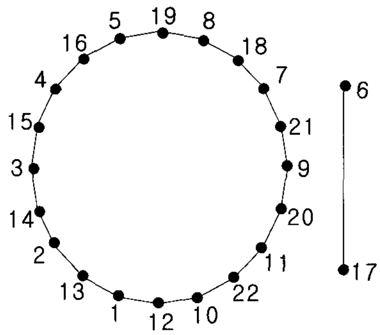

FIGURE 6. An illustration for the labeling given in the proof of Theorem 2.2.

If $m+3 \leq i \leq n-2$, then

$$\lambda(v_i v_{i+1}) = \begin{cases} 4m+1-i & \text{if } i \equiv 1 \pmod 3 \\ 4m-i & \text{if } i \equiv 2 \pmod 3 \\ 4m+2-i & \text{if } i \equiv 0 \pmod 3), \end{cases}$$

$$\lambda(v_0 v_{n-1}) = 3m+4$$

$$\lambda(uw) = 3m+3.$$

(See Figure 6 for illustration.) It is easily seen that λ is a super edge-magic labeling of G with the edge magic number $5m+6$.

Case $m \equiv 2 \pmod 6$

$$\lambda(v_i) = \begin{cases} m+2+i/2 & \text{if } i = 0, 2, \ldots, m-2 \\ (i+1)/2 & \text{if } i = 1, 3, \ldots, m-1 \\ 3m/2+4 & \text{if } i = m \\ (2m-i+7)/2 & \text{if } i = m+1, m+3 \\ 3m/2+3 & \text{if } i = m+2 \\ 2m-i/2+8 & \text{if } i = m+4, m+6. \end{cases}$$

If $i = m+5$, $m+7 \leq i \leq n-1$, then

$$\lambda(v_i) = \begin{cases} (i+5)/2 & \text{if } i \equiv 1 \pmod 6 \\ m+3+i/2 & \text{if } i \equiv 2 \pmod 6 \\ (i+1)/2 & \text{if } i \equiv 3 \pmod 6 \\ m+4+i/2 & \text{if } i \equiv 4 \pmod 6 \\ (i+3)/2 & \text{if } i \equiv 5 \pmod 6 \\ m+2+i/2 & \text{if } i \equiv 0 \pmod 6), \end{cases}$$

$$\lambda(u) = (m+2)/2,$$

$$\lambda(w) = (3m+4)/2,$$

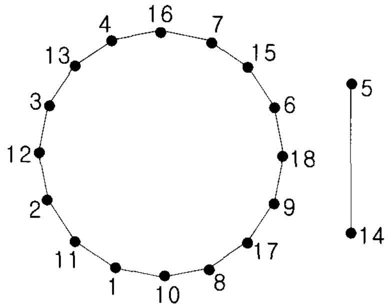

FIGURE 7. An illustration for the labeling given in the proof of Theorem 2.2.

$$\lambda(v_iv_{i+1}) = \begin{cases} 4m-i+3 & \text{if } 0 \le i \le m-2 \\ 3m+2 & \text{if } i = m-1 \\ 2m+i-1 & \text{if } m \le i \le m+2 \\ 3m-2 & \text{if } i = m+3 \\ 2m+i-9 & \text{if } m+4 \le i \le m+6. \end{cases}$$

If $m+7 \le i \le n-2$, then

$$\lambda(v_iv_{i+1}) = \begin{cases} 4m-i & \text{if } i \equiv 1 \pmod 3 \\ 4m+2-i & \text{if } i \equiv 2 \pmod 3 \\ 4m+1-i & \text{if } i \equiv 0 \pmod 3, \end{cases}$$

$$\lambda(v_0v_{n-1}) = 3m+4,$$
$$\lambda(uw) = 3m+3.$$

(See Figure 7 for illustration.) It is easily seen that λ is a super edge-magic labeling of G with the edge magic number $5m+6$.

Case $m \equiv 5 \pmod 6$

$$\lambda(v_i) = \begin{cases} m+2+i/2 & \text{if } i = 0,2,\ldots,m-1 \\ (i+1)/2 & \text{if } i = 1,3,\ldots,m-2 \\ (m+5)/2 & \text{if } i = m \\ (4m-i+10)/2 & \text{if } i = m+1, m+3 \\ (m+3)/2 & \text{if } i = m+2 \\ (2m-i+13)/2 & \text{if } i = m+4, m+6. \end{cases}$$

If $i = m+5$, $m+7 \le i \le n-1$, then

$$\lambda(v_i) = \begin{cases} (i+5)/2 & \text{if } i \equiv 1 \pmod 6 \\ m+3+i/2 & \text{if } i \equiv 2 \pmod 6 \\ (i+1)/2 & \text{if } i \equiv 3 \pmod 6 \\ m+4+i/2 & \text{if } i \equiv 4 \pmod 6 \\ (i+3)/2 & \text{if } i \equiv 5 \pmod 6 \\ m+2+i/2 & \text{if } i \equiv 0 \pmod 6, \end{cases}$$

$$\lambda(u) = (m+1)/2,$$

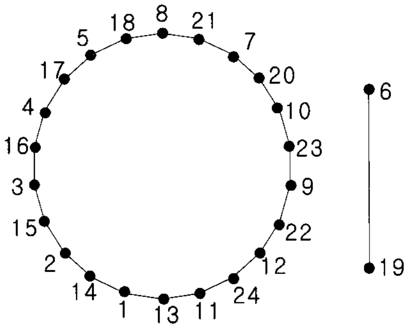

FIGURE 8. An illustration for the labeling given in the proof of Theorem 2.2.

$$\lambda(w) = (3m+5)/2,$$

$$\lambda(v_i v_{i+1}) = \begin{cases} 4m - i + 3 & \text{if } 0 \leq i \leq m-2 \\ 3m+2 & \text{if } i = m-1 \\ 2m+i-1 & \text{if } m \leq i \leq m+2 \\ 3m-2 & \text{if } i = m+3 \\ 2m+i-9 & \text{if } m+4 \leq i \leq m+6. \end{cases}$$

If $m+7 \leq i \leq n-2$, then

$$\lambda(v_i v_{i+1}) = \begin{cases} 4m - i & \text{if } i \equiv 1 \pmod 3 \\ 4m+2-i & \text{if } i \equiv 2 \pmod 3 \\ 4m+1-i & \text{if } i \equiv 0 \pmod 3, \end{cases}$$

$$\lambda(v_0 v_{n-1}) = 3m+4,$$

$$\lambda(uw) = 3m+3.$$

(See Figure 8 for illustration.) It is easily seen that λ is a super edge-magic labeling of G with the edge magic number $5m+6$. Thus, the graph G is super edge-magic. □

3. Closing remark

We have shown that graph $K_2 \cup C_n$ is super edge-magic if n is even ($n \neq 10$). It would be interesting to see if the graph is super edge-magic if n is odd. In fact, it seems to be very difficult to find families of graphs that are super edge-magic or are not.

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