EXISTENCE OF NONOSCILLATORY SOLUTIONS OF
HIGHER-ORDER DIFFERENCE EQUATIONS WITH
POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. In this paper, we investigate nonoscillatory solutions of a
class of higher order neutral nonlinear difference equations with positive
and negative coefficients
\[ \Delta^m(x(n) + p(n)x(\tau(n))) + f_1(n, x(\sigma_1(n))) - f_2(n, x(\sigma_2(n))) = 0, \ n \geq n_0. \]
Some sufficient conditions for the existence of nonoscillatory solutions are
obtained.

1. Introduction

Consider the higher-order neutral nonlinear difference equations with posi-
tive and negative coefficients
\begin{align}
\Delta^m(x(n) + p(n)x(\tau(n))) \\
+ f_1(n, x(\sigma_1(n))) - f_2(n, x(\sigma_2(n))) = 0, \ n \geq n_0, \\
\end{align}
\begin{align}
\Delta^m(x(n) + p(n)x(\tau(n))) \\
+ f_1(n, x(\sigma_1(n))) - f_2(n, x(\sigma_2(n))) = g(n), \ n \geq n_0, \\
\end{align}
\begin{align}
\Delta^m(x(n) + p(n)x(\tau(n))) + \sum_{i=1}^{l} b_i(n)x(\sigma_i(n)) = 0, \ n \geq n_0, \\
\end{align}
where \(\tau(n), \sigma_i(n)\) are sequences of positive integers, \(\tau(n) \leq n, \ \lim_{n \to \infty} \tau(n) = \infty, \ \lim_{n \to \infty} \sigma_i(n) = \infty, \ i = 1, 2, p(n), g(n), b_i(n)\) are sequences of real numbers,
\(f_i(n, x)\) is continuous for \(x, x f_i(n, x) > 0\) holds for \(x \neq 0, i = 1, 2.\) Further
\begin{align}
|f_i(n, x) - f_i(n, y)| \leq q_i(n)|x - y|, \\
\end{align}
where \(q_i(n)\) is a sequence of positive real numbers, \(i = 1, 2.\) We also have

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\begin{align}
(1.5) \quad & \sum_{s=n}^{\infty} (s-n)^{(m-1)}q_i(s) < \infty, \\
(1.6) \quad & \sum_{s=n}^{\infty} (s-n)^{(m-1)}|g(s)| < \infty, \\
(1.7) \quad & \sum_{s=n}^{\infty} (s-n)^{(m-1)}|b_j(s)| < \infty.
\end{align}

Recently, there has been an increasing interest in the study of the oscillation and existence for solutions of differential and difference equations. The papers [2, 3, 7] discussed the existence of nonoscillatory solutions of differential equations. The papers [1, 4, 5] discussed the oscillation of difference equations. But there are relatively few which guarantee the existence of nonoscillatory solutions of difference equations, see [6].

This paper is motivated by recent paper [8], where the authors gave some sufficient conditions for the existence of nonoscillatory solutions of the first-order neutral delay differential equations. The purpose of this paper is to present some new criteria for the existence of nonoscillatory solution of (1.1)-
 
(1.3).

A solution of Eq.(1.1)((1.2),(1.3)) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory.

2. Main results

To obtain our main results, we need the following lemma.

**Lemma 1** ([1]). \textit{The space} \( l^\infty \) \textit{is the Banach space of all bounded real sequence. Let} \( K \) \textit{be a closed bounded and convex subset of} \( l^\infty \). \textit{Suppose} \( \Gamma \) \textit{is a continuous map such that} \( \Gamma(K) \subseteq K \), \textit{and suppose further that} \( \Gamma(K) \) \textit{is uniformly Cauchy. Then} \( \Gamma \) \textit{has a fixed point in} \( K \).

**Theorem 1.** \textit{Assume that} \( 1 < p_1 \leq p(n) \leq p_2 \), (1.4) \textit{and} (1.5) \textit{hold, then (1.1) has a bounded nonoscillatory solution which is bounded away from zero.}

**Proof.** \textit{We choose} \( N_1 > n_0 \), \textit{such that}
\[
N_0 = \min \{ \inf_{n \geq N_1} \{ \tau(n) \}, \inf_{n \geq N_1} \{ \sigma_1(n) \}, \inf_{n \geq N_1} \{ \sigma_2(n) \} \} \geq n_0.
\]

\textit{Let} \( BC \) \textit{be the bounded real sequence of Banach space} \( l^\infty \) \textit{and} \( ||x(n)|| = \sup_{n \geq N_1} |x(n)| \). \textit{Define a set} \( X \subseteq BC \text{ as follows:

\[ X = \left\{ x(n) \in BC, \quad \Delta x(n) \leq 0, \quad 0 < M_1 \leq x(n) \leq p_1 M_1, \quad x(n) \equiv x(N_1), \quad n \geq N_1 \right\}. \]

\textit{Then} \( X \) \textit{is a closed bounded and convex subset of} \( BC \).}
Let $c = \min\{\frac{\alpha - M_1}{p_1 M_1}, \frac{p_1 M_1 - \alpha}{p_1 M_1}\}$, where $M_1 < \alpha < p_1 M_1$. We choose $N \geq N_1$, such that for $n \geq N$, \[
\sum_{s=n}^{\infty} \frac{(s-n+1)(m-1)}{(m-1)!} q_i(s) \leq c.\] For any $x \in X$, define:
\[
\psi(n) = \begin{cases} \sum_{i=1}^{n} \frac{(-1)^{i-1} x(\tau^{-i}(n))}{H_i(\tau^{-i}(n))}, & n \geq N \\
\psi(N), & N_0 \leq n \leq N,
\end{cases}
\]
where $\tau^0(n) = n$, $\tau^i(n) = \tau(\tau^{i-1}(n))$, $\tau^{-i}(n) = \tau^{-1}(\tau^{-(i-1)}(n))$, $H_0(n) = 1$, $H_i(n) = \prod_{j=0}^{i-1} p(\tau^j(n))$, $i = 1, 2, \ldots$. From $M_1 \leq x(n) \leq p_1 M_1$, we know $0 < \psi(n) \leq p_1 M_1$, $n \geq N$.

Define a mapping $\Gamma$ on $X$ as follows:
\[
\Gamma x(n) = \begin{cases} \alpha + (-1)^{m-1} \sum_{s=n}^{\infty} \frac{(s-n+1)(m-1)}{(m-1)!} \left[ f_1(s, \psi(\sigma_1(s))) - f_2(s, \psi(\sigma_2(s))) \right], & n \geq N, \\
\Gamma x(N), & N_0 \leq n \leq N.
\end{cases}
\]
$\Gamma$ satisfies the following conditions:

(a) $\Gamma(X) \subseteq X$.

In fact, for any $x \in X$, $\Gamma x(n) \geq \alpha - p_1 M_1 c \geq M_1$, $\Gamma x(n) \leq \alpha + p_1 M_1 c \leq p_1 M_1$.

(b) $\Gamma$ is continuous.

Let $\{x_k(n)\}$ be a sequence in $X$, such that $\lim_{k \to \infty} ||x_k - x|| = 0$.

Since $X$ is a closed set, we know $x \in X$. For any $\varepsilon > 0$, we can choose $n_2 > N$, such that
\[
\sum_{s=n_2}^{\infty} \frac{(s-n_0+1)(m-1)}{(m-1)!} q_i(s) < \varepsilon, \quad i = 1, 2.
\]
\[
|\Gamma x_k(n) - \Gamma x(n)| 
\leq \sum_{s=n}^{n_2-1} \frac{(s-n+1)(m-1)}{(m-1)!} \left( \sum_{i=1}^{2} q_i(s) |\psi_k(\sigma_i(s)) - \psi(\sigma_i(s))| \right) 
+ \sum_{s=n_2}^{\infty} \frac{(s-n_0+1)(m-1)}{(m-1)!} \left[ f_1(s, \psi_k(\sigma_1(s))) - f_2(s, \psi_k(\sigma_2(s))) - f_1(s, \psi(\sigma_1(s))) + f_2(s, \psi(\sigma_2(s))) \right] 
\leq \sum_{s=n}^{n_2-1} \frac{(s-n+1)(m-1)}{(m-1)!} \sum_{i=1}^{2} q_i(s) |\psi_k(\sigma_i(s)) - \psi(\sigma_i(s))| 
+ \frac{2p_1 M_1}{(m-1)!} \sum_{s=n_2}^{\infty} \frac{(s-n_0+1)(m-1)}{(m-1)!} (q_1(s) + q_2(s)).
\]
So \( \lim_{k \to \infty} \|\Gamma x_k - \Gamma x\| = 0. \)

(c) \( \Gamma x \) is uniformly Cauchy.

\[ |\Gamma x(m_1) - \Gamma x(m_2)| \leq \sum_{s=m_2}^{m_1-1} \frac{(s-n+1)(m-1)}{(m-1)!} [q_1(s)\psi(\sigma_1(s)) + q_2(s)\psi(\sigma_2(s))]| \leq \varepsilon. \]

This shows \( \Gamma X \) is uniformly Cauchy.

From Lemma 1, there exists \( x \in X \), such that \( x = \Gamma x \), i.e.,

\[ x(n) = \alpha + (-1)^{m-1} \sum_{s=n}^{\infty} \frac{(s-n+1)(m-1)}{(m-1)!} [f_1(s, \psi(\sigma_1(s)) - f_2(s, \psi(\sigma_2(s)))] , n \geq N. \]

Since \( \psi(n) + p(n)\psi(\tau(n)) = x(n) \), we get

\[ \psi(n) + p(n)\psi(\tau(n)) = \alpha + (-1)^{m-1} \sum_{s=n}^{\infty} \frac{(s-n+1)(m-1)}{(m-1)!} [f_1(s, \psi(\sigma_1(s)) - f_2(s, \psi(\sigma_2(s)))] . \]

So \( \psi(n) \) satisfies (1.1) for \( n \geq N \), and

\[ \frac{p_{n-1}}{p_1p_2} x(\tau^{-1}(n)) \leq \psi(n) \leq x(n). \]

**Theorem 2.** Assume that \( 0 \leq p(n) \leq p < 1 \), (1.4) and (1.5) hold, then (1.1) has a bounded nonoscillatory solution which is bounded away from zero.

**Proof.** We choose \( N > n_0 \), such that

\[ N_0 = \min\{ \inf_{n \geq N} \{\tau(n)\}, \inf_{n \geq N} \{\sigma_1(n)\}, \inf_{n \geq N} \{\sigma_2(n)\} \} \geq n_0. \]

Let BC be the bounded real sequence of Banach space \( l^\infty \) and \( \|x(n)\| = \sup_{n \geq N} |x(n)| \). Define a set \( \Omega \subset BC \) as follows:

\[ \Omega = \{x(n) \in BC, 1 - p \leq x(n) \leq \frac{1}{1-p}\}. \]

Then \( \Omega \) is a closed bounded and convex subset of BC. From (1.5), we know that there exists \( N_1 \), such that for \( n > N_1 \),

\[ \sum_{s=n}^{\infty} \frac{(s-n+1)(m-1)}{(m-1)!} q_i(s) < \frac{p(1-p)}{2} , \quad i = 1, 2. \]

Define two maps \( \Gamma_1 \) and \( \Gamma_2 \) on \( \Omega \) as follows:

\[ (\Gamma_1 x)(n) = \begin{cases} \frac{2-p+p^2}{2(1-p)} - p(n)x(\tau(n)), & n \geq N_1 \\ (\Gamma_1 x)(N_1), & N_0 \leq n \leq N_1 \end{cases} \]
\[(\Gamma_2 x)(n)\]
\[
= \begin{cases}
\frac{(-1)^{m-1}}{(m-1)!} \sum_{s=n}^{\infty} (s-n+1)^{(m-1)} [f_1(s, x(\sigma_1(s))) - f_2(s, x(\sigma_2(s)))] , & n \geq N_1 \\
(\Gamma_2 x)(N_1) , & N_0 \leq n \leq N_1.
\end{cases}
\]

For any \(x, y \in \Omega\),
\[
(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \leq \frac{2 - p + p^2}{2(1-p)} + \frac{p}{2} = \frac{1}{1-p},
\]
\[
(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \geq \frac{2 - p + p^2}{2(1-p)} - \frac{p}{1-p} - \frac{p}{2} = 1 - p.
\]

So \(\Gamma_1 x + \Gamma_2 y \in \Omega\).

Since \(0 \leq p(n) \leq p < 1\), \(\Gamma_1\) is a contraction mapping. It is easy to know that \(\Gamma_2\) is uniformly bounded. We now show that \(\Gamma_2\) is continuous. For any \(\varepsilon > 0\), we can choose \(n_2 > N_1\), such that
\[
\sum_{s=n_2}^{\infty} \frac{(s-n_0+1)^{(m-1)}}{(m-1)!} q_i(s) < \varepsilon.
\]

Let \(\{x_k(n)\}\) be a sequence in \(\Omega\), such that \(\lim_{k \to \infty} ||x_k - x|| = 0\). Since \(\Omega\) is a closed set, we know \(x \in \Omega\) and
\[
|\Gamma_2 x_k(n) - \Gamma_2 x(n)|
\leq \sum_{s=n}^{n_2-1} \frac{(s-n+1)^{(m-1)}}{(m-1)!} |f_1(s, x_k(\sigma_1(s))) - f_1(s, x(\sigma_1(s)))|
\leq \frac{2}{1-p} \sum_{s=n_2}^{\infty} \frac{(s-n_0+1)^{(m-1)}}{(m-1)!} q_j(s)|x_k(\sigma_j(s)) - x(\sigma_j(s))|.
\]

Since \(f_j\) is continuous for \(x\), we get \(\lim_{k \to \infty} ||\Gamma_2 x_k - \Gamma_2 x|| = 0\). So \(\Gamma_2\) is continuous.

\(\forall \varepsilon > 0, \exists N_2\) such that for \(m_1 > m_2 \geq N_2\) and for all \(x(n) \in \Omega\),
\[
|\Gamma_2 x(m_1) - \Gamma_2 x(m_2)|
\leq \sum_{s=m_2}^{m_1-1} \frac{(s-n_0+1)^{(m-1)}}{(m-1)!} |f_1(s, x(\sigma_1(s))) - f_2(s, x(\sigma_2(s)))| \leq \varepsilon.
\]

By discrete Krasnoselskii's fixed point theorem, there exists \(x \in \Omega\), such that \(x = \Gamma_1 x + \Gamma_2 x\), i.e.,
\[
x(n) = \frac{2 - p + p^2}{2(1-p)} - p(n)x(\tau(n))
\]
\[ (+1)^{m-1} \sum_{s=n}^{\infty} \frac{(s-n+1)(m-1)}{(m-1)!} (f_1(s, x(\sigma_1(s))) - f_2(s, x(\sigma_2(s)))) \].

\( x(n) \) is a bounded nonoscillatory solution of (1.1) which is bounded away from zero. \( \square \)

**Theorem 3.** Assume that \(-1 < p \leq p(n) \leq 0, (1.4) \) and (1.5) hold, then (1.1) has a bounded nonoscillatory solution which is bounded away from zero.

**Proof.** Let BC be the bounded real sequence of Banach space \( L^\infty \) and \( \|x(n)\| = \sup |x(n)| \). We choose \( M_1, M_2, \alpha \) such that \( 0 < M_1 < \alpha < (1 + p)M_2 \). Define \( \Omega = \{ x \in BC, M_1 \leq x(n) \leq M_2, n \geq n_0 \} \). Let \( c = \min\{ \frac{\alpha - M_1}{M_2}, \frac{M_2 - \alpha}{M_2} \} \), from (1.5) we know that there exists \( N > n_0 \) such that for \( n \geq N \),

\[ \frac{1}{(m-1)!} \sum_{s=n}^{\infty} (s-n+1)(m-1) (q_1(s) + q_2(s)) \leq c. \]

For any \( x \in \Omega \), define:

\[ \varphi(n) = \left\{ \begin{array}{ll}
            \sum_{i=0}^{k_n-1} (-1)^i p_n^{(i)} x(\tau_n^{(i)}) + (-1)^k_n p_n^{(k_n)} \frac{x_N}{1+p_N}, & n \geq N \\
            x_n, & 0 \leq n \leq N,
          \end{array} \right. \]

where we take \( k_n \) such that \( n_0 \leq \tau_n^{(k_n)} \leq N, \tau_n^{(0)} = n, \tau_n^{(1)} = \tau_n, \tau_n^{(2)} = \tau_n, \ldots, \tau_n^{(k_n)} = \tau_n^{(k_n-1)}, p_n^{(0)} = 1, p_n^{(1)} = p_n, \ldots, p_n^{(s)} = p_n p_{\tau_n} \cdots p_{\tau_n^{(s-1)}} \). It is easy to prove \( x(n) = \varphi(n) + p(n) \varphi(\tau(n)), n \geq N \) and \( M_1 \leq x(n) \leq \varphi(n) \leq M_2 \).

Define a mapping \( \Gamma \) on \( \Omega \) as follows:

(2.2)

\[ \Gamma x(n) = \begin{cases}
\alpha + \sum_{s=n}^{\infty} \frac{(-1)^{m-1}(s-n+1)(m-1)}{(m-1)!} [f_1(s, \varphi(\sigma_1(s))) - f_2(s, \varphi(\sigma_2(s)))] , & n \geq N, \\
\Gamma x(N) , & 0 \leq n \leq N.
\end{cases} \]

Since \( \Gamma x(n) \geq \alpha - cM_2 \geq M_1, \Gamma x(n) \leq \alpha + cM_2 \leq M_2 \), we get \( \Gamma \Omega \subseteq \Omega \). Similar to the proof of Theorem 1, we can obtain \( \Gamma \) is continuous and uniformly Cauchy. So there exists \( x \in \Omega \) such that \( x = \Gamma x \). The proof is complete. \( \square \)

**Theorem 4.** Assume that \( p_1 \leq p(n) \leq p_2 < -1, (1.4) \) and (1.5) hold, then (1.1) has a bounded nonoscillatory solution which is bounded away from zero.

**Proof.** We choose positive constants \( M_1, M_2, \alpha \) such that \( -p_1 M_1 < \alpha < (p_2 - 1)M_2 \). \( BC \) is defined as in Theorem 3. Let \( \Omega = \{ x \in BC, M_1 \leq x(n) \leq M_2, n \geq n_0 \} \), \( c = \min\{ \frac{(M_1 p_1 + \alpha) p_2}{M_2 p_1}, \frac{(p_2 - 1) M_2 - \alpha}{M_2} \} \). Choosing \( N \) sufficiently large such that for \( n \geq N \), we have

\[ \frac{1}{(m-1)!} \sum_{s=n}^{\infty} q_i(s)(s-n+1)(m-1) \leq c, \quad i = 1, 2. \]
Define two maps $\Gamma_1$ and $\Gamma_2$ on $\Omega$ as follows:

$$(\Gamma_1 x)(n) = \begin{cases} 
\frac{-\alpha}{p(\tau^{-1}(n))} - \frac{\alpha x(\tau^{-1}(n))}{p(\tau^{-1}(n))}, & n \geq N \\
(\Gamma_1 x)(N), & n_0 \leq n \leq N 
\end{cases}$$

$$(\Gamma_2 x)(n) = \begin{cases} 
(-1)^{m-1} \frac{(-1)^{m-1}}{(m-1)!p(\tau^{-1}(n))} \times \\
\sum_{s=\tau^{-1}(n)}^{\infty} (s - \tau^{-1}(n) + 1)^{(m-1)}[f_1(s, x(\sigma_1(s))) - f_2(s, x(\sigma_2(s)))] , & n \geq N \\
(\Gamma_2 x)(N), & n_0 \leq n \leq N.
\end{cases}$$

For any $x, y \in \Omega$,

$$(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \geq \frac{-\alpha}{p_1} + \frac{cM_2}{p_2} \geq M_1,$$

$$(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \leq \frac{-\alpha}{p_2} - \frac{M_2}{p_2} - \frac{cM_2}{p_2} \leq M_2,$$

that is $\Gamma_1 x + \Gamma_2 y \in \Omega$.

We also can prove that $\Gamma_1$ is a contraction mapping, $\Gamma_2$ is uniformly bounded and continuous. Further we know $\Gamma_2$ is uniformly Cauchy. So there exists $x \in \Omega$ such that $x = \Gamma_1 x + \Gamma_2 x$. i.e.,

$$x(n) = -\frac{\alpha}{p(\tau^{-1}(n))} - \frac{x(\tau^{-1}(n))}{p(\tau^{-1}(n))} + \frac{(-1)^{m-1}}{(m-1)!p(\tau^{-1}(n))} \times \sum_{s=\tau^{-1}(n)}^{\infty} (s - \tau^{-1}(n) + 1)^{(m-1)}[f_1(s, x(\sigma_1(s))) - f_2(s, x(\sigma_2(s)))].$$

The proof is complete. \hfill \Box

**Theorem 5.** Assume that $p(n)$ satisfies one of the conditions of Theorem 1–Theorem 4, (1.4), (1.5) and (1.6) hold, then (1.2) has a bounded nonoscillatory solution which is bounded away from zero.

**Proof.** Set $g_+(n) = \max \{g(n), 0\}$, $g_-(n) = \max \{-g(n), 0\}$, then $g(n) = g_+(n) - g_-(n)$. (1.2) can be written as follows:

$$\Delta^m (x(n) + p(n)x(\tau(n))) + [f_1(n, x(\sigma_1(n))) + g_-(n)] - [f_2(n, x(\sigma_2(n))) + g_+(n)] = 0.$$ 

Let

$$F_1(n, x(\sigma_1(n))) = f_1(n, x(\sigma_1(n))) + g_-(n),$$

$$F_2(n, x(\sigma_2(n))) = f_2(n, x(\sigma_2(n))) + g_+(n).$$

Similar to the proof of Theorem 1–Theorem 4, we obtain the conclusion. \hfill \Box

**Theorem 6.** Assume that $p(n)$ satisfies one of the conditions of Theorem 1–Theorem 4, (1.7) holds, then (1.3) has a bounded nonoscillatory solution which is bounded away from zero.
Proof. We only prove the case that $0 \leq p(n) \leq p < 1$.

Let $BC$ be the bounded real sequence of Banach space $l^\infty$ and $||x(n)|| = \sup_{n \geq n_0} |x(n)|$. We choose $M_1, M_2, \alpha$ such that $pM_2 + M_1 < \alpha < M_2$. Define $\Omega = \{x \in BC, M_1 \leq x(n) \leq M_2\}$, $c = \min\{\frac{\alpha - pM_2 - M_1}{lM_2}, \frac{M_2 - \alpha}{lM_2}\}$. $N$ is sufficiently large such that for $n \geq N$,

$$\frac{1}{(m - 1)!} \sum_{s=n}^{\infty} (s - n + 1)^{(m-1)}|b_i(s)| \leq c.$$

Define two maps $\Gamma_1$ and $\Gamma_2$ on $\Omega$ as follows:

$$\begin{align*}
(\Gamma_1 x)(n) &= \begin{cases} 
\alpha - p(n)x(\tau(n)), & n \geq N \\
(\Gamma_1 x)(N), & 0 \leq n \leq N,
\end{cases} \\
(\Gamma_2 x)(n) &= \begin{cases} 
(-1)^{m-1} \sum_{s=n}^{\infty} \frac{(s-n+1)^{(m-1)}}{(m-1)!} \sum_{i=1}^{l} b_i(s)x(\sigma_i(s)), & n \geq N \\
(\Gamma_2 x)(N), & 0 \leq n \leq N.
\end{cases}
\end{align*}$$

For any $x, y \in \Omega$,

$$\begin{align*}
(\Gamma_1 x)(n) + (\Gamma_2 y)(n) &\geq \alpha - pM_2 - lM_2c \geq M_1, \\
(\Gamma_1 x)(n) + (\Gamma_2 y)(n) &\leq \alpha + lM_2c \leq M_2,
\end{align*}$$

that is $\Gamma_1 x + \Gamma_2 y \in \Omega$.

$\Gamma_1$ is a contraction mapping, $\Gamma_2$ is continuous and uniformly Cauchy, uniformly bounded. So there exists $x \in \Omega$ such that $x = \Gamma_1 x + \Gamma_2 x$. The proof is complete.

$\square$

References

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