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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF HIGHER-ORDER DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. In this paper, we investigate nonoscillatory solutions of a class of higher order neutral nonlinear difference equations with positive and negative coefficients

$\triangle^m(x(n)+p(n)x(\tau(n)))+f_1(n,x(\sigma_1(n)))-f_2(n,x(\sigma_2(n)))=0,\ \ n\geq n_0.$

Some sufficient conditions for the existence of nonoscillatory solutions are obtained.

1. Introduction

Consider the higher-order neutral nonlinear difference equations with positive and negative coefficients

$$\begin{aligned}\triangle^m(x(n)&+p(n)x(\tau(n)))\\&+f_1(n,x(\sigma_1(n)))-f_2(n,x(\sigma_2(n)))=0,\ \ n\geq n_0,\end{aligned}\tag{1.1}$$

$$\begin{aligned}\triangle^m(x(n)&+p(n)x(\tau(n)))\\&+f_1(n,x(\sigma_1(n)))-f_2(n,x(\sigma_2(n)))=g(n),\ \ n\geq n_0,\end{aligned}\tag{1.2}$$

$$\triangle^m(x(n)+p(n)x(\tau(n)))+\sum_{i=1}^{l}b_i(n)x(\sigma_i(n))=0,\ \ n\geq n_0,\tag{1.3}$$

where $\tau(n)$, $\sigma_i(n)$ are sequences of positive integers, $\tau(n)\leq n$, $\lim_{n\to\infty}\tau(n)=\infty$, $\lim_{n\to\infty}\sigma_i(n)=\infty$, $i=1,2$, $p(n),g(n),b_j(n)$ are sequences of real numbers, $f_i(n,x)$ is continuous for x, $xf_i(n,x)>0$ holds for $x\neq 0$, $i=1,2$. Further

$$|f_i(n,x)-f_i(n,y)|\leq q_i(n)|x-y|,\tag{1.4}$$

where $q_i(n)$ is a sequence of positive real numbers, $i=1,2$. We also have

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$$\sum_{s=n}^{\infty}(s-n)^{(m-1)}q_i(s) < \infty, \tag{1.5}$$

$$\sum_{s=n}^{\infty}(s-n)^{(m-1)}|g(s)| < \infty, \tag{1.6}$$

$$\sum_{s=n}^{\infty}(s-n)^{(m-1)}|b_j(s)| < \infty. \tag{1.7}$$

Recently, there has been an increasing interest in the study of the oscillation and existence for solutions of differential and difference equations. The papers [2, 3, 7] discussed the existence of nonoscillatory solutions of differential equations. The papers [1, 4, 5] discussed the oscillation of difference equations. But there are relatively few which guarantee the existence of nonoscillatory solutions of difference equations, see [6].

This paper is motivated by recent paper [8], where the authors gave some sufficient conditions for the existence of nonoscillatory solutions of the first-order neutral delay differential equations. The purpose of this paper is to present some new criteria for the existence of nonoscillatory solution of (1.1)-(1.3).

A solution of Eq.(1.1)((1.2),(1.3)) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory.

2. Main results

To obtain our main results, we need the following lemma.

Lemma 1 ([1]). *The space l^∞ is the Banach space of all bounded real sequence. Let K be a closed bounded and convex subset of l^∞. Suppose Γ is a continuous map such that $\Gamma(K) \subset K$, and suppose further that $\Gamma(K)$ is uniformly Cauchy. Then Γ has a fixed point in K.*

Theorem 1. *Assume that $1 < p_1 \le p(n) \le p_2$, (1.4) and (1.5) hold, then (1.1) has a bounded nonoscillatory solution which is bounded away from zero.*

Proof. We choose $N_1 > n_0$, such that

$$N_0 = \min\{\inf_{n\ge N_1}\{\tau(n)\}, \inf_{n\ge N_1}\{\sigma_1(n)\}, \inf_{n\ge N_1}\{\sigma_2(n)\}\} \ge n_0.$$

Let BC be the bounded real sequence of Banach space l^∞ and $||x(n)|| = \sup_{n\ge N_1}|x(n)|$. Define a set $X \subset BC$ as follows:

$$X = \left\{x(n) \in BC,\ \Delta x(n) \le 0,\ \begin{array}{ll} 0 < M_1 \le x(n) \le p_1 M_1, & n \ge N_1 \\ x_{(n)} = x_{(N_1)}, & N_0 \le n \le N_1 \end{array}\right\}.$$

Then X is a closed bounded and convex subset of BC.

Let $c = \min\{\frac{\alpha - M_1}{p_1 M_1}, \frac{p_1 M_1 - \alpha}{p_1 M_1}\}$, where $M_1 < \alpha < p_1 M_1$. We choose $N \geq N_1$, such that for $n \geq N$, $\sum\limits_{s=n}^{\infty} \frac{(s-n+1)^{(m-1)}}{(m-1)!} q_i(s) \leq c$. For any $x \in X$, define:

$$\psi(n) = \begin{cases} \sum\limits_{i=1}^{\infty} \frac{(-1)^{i-1} x(\tau^{-i}(n))}{H_i(\tau^{-i}(n))}, & n \geq N \\ \psi(N), & N_0 \leq n \leq N, \end{cases}$$

where $\tau^0(n) = n,\ \tau^i(n) = \tau(\tau^{i-1}(n)),\ \tau^{-i}(n) = \tau^{-1}(\tau^{-(i-1)}(n)),\ H_0(n) = 1$, $H_i(n) = \prod\limits_{j=0}^{i-1} p(\tau^j(n)),\ i = 1, 2, \ldots$. From $M_1 \leq x(n) \leq p_1 M_1$, we know $0 < \psi(n) \leq p_1 M_1,\ n \geq N$.

Define a mapping Γ on X as follows:

(2.1)

$$\Gamma x(n) = \begin{cases} \alpha + (-1)^{m-1} \sum\limits_{s=n}^{\infty} \frac{(s-n+1)^{(m-1)}}{(m-1)!} [f_1(s, \psi(\sigma_1(s))) - f_2(s, \psi(\sigma_2(s)))], & n \geq N, \\ \Gamma x(N), \quad N_0 \leq n \leq N. \end{cases}$$

Γ satisfies the following conditions:

(a) $\Gamma(X) \subseteq X$.

In fact, for any $x \in X$, $\Gamma x(n) \geq \alpha - p_1 M_1 c \geq M_1$, $\Gamma x(n) \leq \alpha + p_1 M_1 c \leq p_1 M_1$.

(b) Γ is continuous.

Let $\{x_k(n)\}$ be a sequence in X, such that $\lim\limits_{k \to \infty} \|x_k - x\| = 0$.

Since X is a closed set, we know $x \in X$. For any $\varepsilon > 0$, we can choose $n_2 > N$, such that

$$\sum_{s=n_2}^{\infty} \frac{(s - n_0 + 1)^{(m-1)}}{(m-1)!} q_i(s) < \varepsilon,\ i = 1,\ 2.$$

$$\begin{aligned}
&|\Gamma x_k(n) - \Gamma x(n)| \\
&\leq \sum_{s=n}^{n_2 - 1} \frac{(s-n+1)^{(m-1)}}{(m-1)!} \Big(\sum_{i=1}^{2} q_i(s) |\psi_k(\sigma_i(s)) - \psi(\sigma_i(s))|\Big) \\
&\quad + \sum_{s=n_2}^{\infty} \frac{(s-n_0+1)^{(m-1)}}{(m-1)!} |f_1(s, \psi_k(\sigma_1(s))) - f_2(s, \psi_k(\sigma_2(s))) \\
&\quad - f_1(s, \psi(\sigma_1(s))) + f_2(s, \psi(\sigma_2(s)))| \\
&\leq \sum_{s=n}^{n_2 - 1} \frac{(s-n+1)^{(m-1)}}{(m-1)!} \sum_{i=1}^{2} q_i(s) |\psi_k(\sigma_i(s)) - \psi(\sigma_i(s))| \\
&\quad + \frac{2 p_1 M_1}{(m-1)!} \sum_{s=n_2}^{\infty} (s - n_0 + 1)^{(m-1)} (q_1(s) + q_2(s)).
\end{aligned}$$

So $\lim_{k\to\infty} ||\Gamma x_k - \Gamma x|| = 0$.

(c) Γx is uniformly Cauchy.

$\forall\, \varepsilon > 0$, $\exists\, n_2$ such that for $m_1 > m_2 \geq n_2$ and for all $x(n) \in X$,

$$\begin{aligned}|\Gamma x(m_1) - \Gamma x(m_2)| &\leq \sum_{s=m_2}^{m_1-1} \frac{(s-n+1)^{(m-1)}}{(m-1)!}[q_1(s)\psi(\sigma_1(s)) + q_2(s)\psi(\sigma_2(s))] \\ &\leq \varepsilon.\end{aligned}$$

This shows ΓX is uniformly Cauchy.

From Lemma 1, there exists $x \in X$, such that $x = \Gamma x$, i.e.,

$$\begin{aligned}&x(n) \\ &= \alpha + (-1)^{m-1}\sum_{s=n}^{\infty} \frac{(s-n+1)^{(m-1)}}{(m-1)!}[f_1(s,\psi(\sigma_1(s))) - f_2(s,\psi(\sigma_2(s)))], n \geq N.\end{aligned}$$

Since $\psi(n) + p(n)\psi(\tau(n)) = x(n)$, we get

$$\begin{aligned}&\psi(n) + p(n)\psi(\tau(n)) \\ &= \alpha + (-1)^{m-1}\sum_{s=n}^{\infty} \frac{(s-n+1)^{(m-1)}}{(m-1)!}[f_1(s,\psi(\sigma_1(s))) - f_2(s,\psi(\sigma_2(s)))].\end{aligned}$$

So $\psi(n)$ satisfies (1.1) for $n \geq N$, and $\frac{p_1-1}{p_1p_2}x(\tau^{-1}(n)) \leq \psi(n) \leq x(n)$. □

Theorem 2. *Assume that* $0 \leq p(n) \leq p < 1$, (1.4) *and* (1.5) *hold, then* (1.1) *has a bounded nonoscillatory solution which is bounded away from zero.*

Proof. We choose $N > n_0$, such that

$$N_0 = \min\{\inf_{n\geq N}\{\tau(n)\}, \inf_{n\geq N}\{\sigma_1(n)\}, \inf_{n\geq N}\{\sigma_2(n)\}\} \geq n_0.$$

Let BC be the bounded real sequence of Banach space l^∞ and $||x(n)|| = \sup_{n\geq N}|x(n)|$. Define a set $\Omega \subset BC$ as follows:

$$\Omega = \{x(n) \in BC,\, 1-p \leq x(n) \leq \frac{1}{1-p}\}.$$

Then Ω is a closed bounded and convex subset of BC. From (1.5), we know that there exists N_1, such that for $n > N_1$,

$$\sum_{s=n}^{\infty} \frac{(s-n+1)^{(m-1)}}{(m-1)!} q_i(s) < \frac{p(1-p)}{2}, \quad i = 1,2.$$

Define two maps Γ_1 and Γ_2 on Ω as follows:

$$(\Gamma_1 x)(n) = \begin{cases} \frac{2-p+p^2}{2(1-p)} - p(n)x(\tau(n)), & n \geq N_1 \\ (\Gamma_1 x)(N_1), & N_0 \leq n \leq N_1 \end{cases}$$

$$
(\Gamma_2 x)(n) = \begin{cases} \frac{(-1)^{m-1}}{(m-1)!} \sum_{s=n}^{\infty} (s-n+1)^{(m-1)} [f_1(s, x(\sigma_1(s))) - f_2(s, x(\sigma_2(s)))], & n \geq N_1 \\ (\Gamma_2 x)(N_1) \ N_0 \leq n \leq N_1. \end{cases}
$$

For any $x, y \in \Omega$,

$$
(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \leq \frac{2 - p + p^2}{2(1-p)} + \frac{p}{2} = \frac{1}{1-p},
$$

$$
(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \geq \frac{2 - p + p^2}{2(1-p)} - \frac{p}{1-p} - \frac{p}{2} = 1 - p.
$$

So $\Gamma_1 x + \Gamma_2 y \in \Omega$.

Since $0 \leq p(n) \leq p < 1$, Γ_1 is a contraction mapping. It is easy to know that Γ_2 is uniformly bounded. We now show that Γ_2 is continuous. For any $\varepsilon > 0$, we can choose $n_2 > N_1$, such that

$$
\sum_{s=n_2}^{\infty} \frac{(s - n_0 + 1)^{(m-1)}}{(m-1)!} q_i(s) < \varepsilon.
$$

Let $\{x_k(n)\}$ be a sequence in Ω, such that $\lim_{k\to\infty} ||x_k - x|| = 0$. Since Ω is a closed set, we know $x \in \Omega$ and

$$
\begin{aligned}
&|\Gamma_2 x_k(n) - \Gamma_2 x(n)| \\
\leq & \left| \sum_{s=n}^{n_2-1} \frac{(s-n+1)^{(m-1)}}{(m-1)!} (f_1(s, x_k(\sigma_1(s))) - f_1(s, x(\sigma_1(s)))) \right| \\
& + \left| \sum_{s=n}^{n_2-1} \frac{(s-n+1)^{(m-1)}}{(m-1)!} (f_2(s, x_k(\sigma_2(s))) - f_2(s, x(\sigma_2(s)))) \right| \\
& + \sum_{j=1}^{2} \sum_{s=n_2}^{\infty} \frac{(s-n_0+1)^{(m-1)}}{(m-1)!} q_j(s) |x_k(\sigma_j(s)) - x(\sigma_j(s))|.
\end{aligned}
$$

Since f_j is continuous for x, we get $\lim_{k\to\infty} ||\Gamma_2 x_k - \Gamma_2 x|| = 0$. So Γ_2 is continuous. $\forall \varepsilon > 0, \exists N_2$ such that for $m_1 > m_2 \geq N_2$ and for all $x(n) \in \Omega$,

$$
\begin{aligned}
&|\Gamma_2 x(m_1) - \Gamma_2 x(m_2)| \\
\leq & \sum_{s=m_2}^{m_1-1} \frac{(s-n_0+1)^{(m-1)}}{(m-1)!} |f_1(s, x(\sigma_1(s))) - f_2(s, x(\sigma_2(s)))| \leq \varepsilon.
\end{aligned}
$$

By discrete Krasnoselskii's fixed point theorem, there exists $x \in \Omega$, such that $x = \Gamma_1 x + \Gamma_2 x$, i.e.,

$$
x(n) = \frac{2 - p + p^2}{2(1-p)} - p(n) x(\tau(n))
$$

$$+(-1)^{m-1}\sum_{s=n}^{\infty}\frac{(s-n+1)^{(m-1)}}{(m-1)!}(f_1(s,x(\sigma_1(s)))-f_2(s,x(\sigma_2(s)))).$$

$x(n)$ is a bounded nonoscillatory solution of (1.1) which is bounded away from zero. □

Theorem 3. *Assume that $-1 < p \le p(n) \le 0$, (1.4) and (1.5) hold, then (1.1) has a bounded nonoscillatory solution which is bounded away from zero.*

Proof. Let BC be the bounded real sequence of Banach space l^∞ and $||x(n)|| = \sup_{n\ge n_0}|x(n)|$. We choose M_1, M_2, α such that $0 < M_1 < \alpha < (1+p)M_2$. Define $\Omega = \{x \in BC, M_1 \le x(n) \le M_2,\ n \ge n_0\}$. Let $c = \min\{\frac{\alpha-M_1}{M_2}, \frac{M_2-\alpha}{M_2}\}$, from (1.5) we know that there exists $N > n_0$ such that for $n \ge N$,

$$\frac{1}{(m-1)!}\sum_{s=n}^{\infty}(s-n+1)^{(m-1)}(q_1(s)+q_2(s)) \le c.$$

For any $x \in \Omega$, define:

$$\varphi(n) = \begin{cases} \sum\limits_{i=0}^{k_n-1}(-1)^i p_n^{(i)}x(\tau_n^{(i)}) + (-1)^{k_n}p_n^{(k_n)}\frac{x_N}{1+p_N}, & n \ge N \\ \frac{x_N}{1+p_N}, & n_0 \le n \le N, \end{cases}$$

where we take k_n such that $n_0 \le \tau_n^{(k_n)} \le N$, $\tau_n^{(0)} = n$, $\tau_n^{(1)} = \tau_n$, $\tau_n^{(2)} = \tau_{\tau_n}, \ldots, \tau_n^{(k)} = \tau_{\tau_n^{(k-1)}}$, $p_n^{(0)} = 1$, $p_n^{(1)} = p_n, \ldots, p_n^{(s)} = p_n p_{\tau_n}\cdots p_{\tau_n^{(s-1)}}$. It is easy to prove $x(n) = \varphi(n) + p(n)\varphi(\tau(n)),\ \ n \ge N$ and $M_1 \le x(n) \le \varphi(n) \le \frac{M_2}{1+p}$.

Define a mapping Γ on Ω as follows:

(2.2)
$$\Gamma x(n) = \begin{cases} \alpha + \sum\limits_{s=n}^{\infty}\frac{(-1)^{m-1}(s-n+1)^{(m-1)}}{(m-1)!}[f_1(s,\varphi(\sigma_1(s))) - f_2(s,\varphi(\sigma_2(s)))], & n \ge N, \\ \Gamma x(N), \quad n_0 \le n \le N. \end{cases}$$

Since $\Gamma x(n) \ge \alpha - cM_2 \ge M_1$, $\Gamma x(n) \le \alpha + cM_2 \le M_2$, we get $\Gamma\Omega \subseteq \Omega$. Similar to the proof of Theorem 1, we can obtain Γ is continuous and uniformly Cauchy. So there exists $x \in \Omega$ such that $x = \Gamma x$. The proof is complete. □

Theorem 4. *Assume that $p_1 \le p(n) \le p_2 < -1$, (1.4) and (1.5) hold, then (1.1) has a bounded nonoscillatory solution which is bounded away from zero.*

Proof. We choose positive constants M_1, M_2, α such that $-p_1M_1 < \alpha < (-p_2-1)M_2$. BC is defined as in Theorem 3. Let $\Omega = \{x \in BC, M_1 \le x(n) \le M_2,\ n \ge n_0\}$, $c = \min\{\frac{(M_1p_1+\alpha)p_2}{M_2p_1}, \frac{(-p_2-1)M_2-\alpha}{M_2}\}$. Choosing N sufficiently large such that for $n \ge N$, we have

$$\frac{1}{(m-1)!}\sum_{s=n}^{\infty}q_i(s)(s-n+1)^{(m-1)} \le c, \quad i = 1,2.$$

Define two maps Γ_1 and Γ_2 on Ω as follows:

$$(\Gamma_1 x)(n) = \begin{cases} -\frac{\alpha}{p(\tau^{-1}(n))} - \frac{x(\tau^{-1}(n))}{p(\tau^{-1}(n))}, & n \geq N \\ (\Gamma_1 x)(N), & n_0 \leq n \leq N \end{cases}$$

$$(\Gamma_2 x)(n) = \begin{cases} \frac{(-1)^{m-1}}{(m-1)!p(\tau^{-1}(n))} \times \\ \sum\limits_{s=\tau^{-1}(n)}^{\infty} (s-\tau^{-1}(n)+1)^{(m-1)}[f_1(s,x(\sigma_1(s))) - f_2(s,x(\sigma_2(s)))], & n \geq N \\ (\Gamma_2 x)(N), \quad n_0 \leq n \leq N. \end{cases}$$

For any $x, y \in \Omega$,

$$(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \geq \frac{-\alpha}{p_1} + \frac{cM_2}{p_2} \geq M_1,$$

$$(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \leq \frac{-\alpha}{p_2} - \frac{M_2}{p_2} - \frac{cM_2}{p_2} \leq M_2,$$

that is $\Gamma_1 x + \Gamma_2 y \in \Omega$.

We also can prove that Γ_1 is a contraction mapping, Γ_2 is uniformly bounded and continuous. Further we know Γ_2 is uniformly Cauchy. So there exists $x \in \Omega$ such that $x = \Gamma_1 x + \Gamma_2 x$. i.e.,

$$\begin{aligned} x(n) = & -\frac{\alpha}{p(\tau^{-1}(n))} - \frac{x(\tau^{-1}(n))}{p(\tau^{-1}(n))} + \frac{(-1)^{m-1}}{(m-1)!p(\tau^{-1}(n))} \\ & \times \sum_{s=\tau^{-1}(n)}^{\infty} (s-\tau^{-1}(n)+1)^{(m-1)}[f_1(s,x(\sigma_1(s))) - f_2(s,x(\sigma_2(s)))]. \end{aligned}$$

The proof is complete. □

Theorem 5. *Assume that $p(n)$ satisfies one of the conditions of Theorem 1–Theorem 4, (1.4), (1.5) and (1.6) hold, then (1.2) has a bounded nonoscillatory solution which is bounded away from zero.*

Proof. Set $g_+(n) = \max\{g(n), 0\}$, $g_-(n) = \max\{-g(n), 0\}$, then $g(n) = g_+(n) - g_-(n)$. (1.2) can be written as follows:

$$\Delta^m(x(n)+p(n)x(\tau(n)))+[f_1(n,x(\sigma_1(n)))+g_-(n)]-[f_2(n,x(\sigma_2(n)))+g_+(n)]=0.$$

Let

$$F_1(n,x(\sigma_1(n))) = f_1(n,x(\sigma_1(n))) + g_-(n),$$
$$F_2(n,x(\sigma_2(n))) = f_2(n,x(\sigma_2(n))) + g_+(n).$$

Similar to the proof of Theorem 1–Theorem 4, we obtain the conclusion. □

Theorem 6. *Assume that $p(n)$ satisfies one of the conditions of Theorem 1–Theorem 4, (1.7) holds, then (1.3) has a bounded nonoscillatory solution which is bounded away from zero.*

Proof. We only prove the case that $0 \le p(n) \le p < 1$.

Let BC be the bounded real sequence of Banach space l^∞ and $||x(n)|| = \sup_{n \ge n_0} |x(n)|$. We choose M_1, M_2, α such that $pM_2 + M_1 < \alpha < M_2$. Define $\Omega = \{x \in BC, M_1 \le x(n) \le M_2\}$, $c = \min\{\frac{\alpha - pM_2 - M_1}{lM_2}, \frac{M_2 - \alpha}{lM_2}\}$. N is sufficiently large such that for $n \ge N$,

$$\frac{1}{(m-1)!} \sum_{s=n}^{\infty} (s-n+1)^{(m-1)} |b_i(s)| \le c.$$

Define two maps Γ_1 and Γ_2 on Ω as follows:

(2.3)

$$(\Gamma_1 x)(n) = \begin{cases} \alpha - p(n)x(\tau(n)), & n \ge N \\ (\Gamma_1 x)(N), & n_0 \le n \le N, \end{cases}$$

$$(\Gamma_2 x)(n) = \begin{cases} (-1)^{m-1} \sum\limits_{s=n}^{\infty} \frac{(s-n+1)^{(m-1)}}{(m-1)!} \sum\limits_{i=1}^{l} b_i(s) x(\sigma_i(s)), & n \ge N \\ (\Gamma_2 x)(N), & n_0 \le n \le N. \end{cases}$$

For any $x, y \in \Omega$,

$$(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \ge \alpha - pM_2 - lM_2 c \ge M_1,$$

$$(\Gamma_1 x)(n) + (\Gamma_2 y)(n) \le \alpha + lM_2 c \le M_2,$$

that is $\Gamma_1 x + \Gamma_2 y \in \Omega$.

Γ_1 is a contraction mapping, Γ_2 is continuous and uniformly Cauchy, uniformly bounded. So there exists $x \in \Omega$ such that $x = \Gamma_1 x + \Gamma_2 x$. The proof is complete. □

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