Bull. Korean Math. Soc. **45** (2008), No. 1, pp. 33–37

TANGENTIAL REPRESENTATIONS AT ISOLATED FIXED POINTS OF ODD-DIMENSIONAL G-MANIFOLDS

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Abstract. Let G be a compact abelian Lie group, and M an odd-dimensional closed smooth G-manifold. If the fixed point set $M^G \neq \emptyset$ and $\dim M^G = 0$, then G has a subgroup H with $G/H \cong \mathbb{Z}_2$, the cyclic group of order 2. The tangential representation $\tau_x(M)$ of G at $x \in M^G$ is also regarded as a representation of H by restricted action. We show that the number of fixed points is even, and that the tangential representations at fixed points are pairwise isomorphic as representations of H.

1. Introduction

Let M be a smooth G-manifold, G a compact Lie group. M^G denotes the fixed point set of M. The G-action on M induces a linear action on the tangent space $\tau_x(M)$ at $x \in M^G$, called the *tangential representation* of G.

In his paper [4], P. A. Smith raised a question: if a finite group G acts smoothly on an n-dimensional sphere S^n with exactly two fixed points x and y, is it true that $\tau_x(S^n) \cong \tau_y(S^n)$ as representations of G? Since then, there are published vast literature concerning this question. Some of them are affirmative to the question, and some of them are negative. Among such results, S. E. Cappell and J. L. Shaneson [2] gave counterexamples to the question. In fact, if $G = \mathbb{Z}_{4k}$ with $k \geq 2$, the cyclic group of order $4k$, they constructed smooth actions of G on S^n with odd $n \geq 9$ and with exactly two fixed points x and y such that $\tau_x(S^n) \ncong \tau_y(S^n)$ as representations of G. But their examples show that $\tau_x(S^n) \cong \tau_y(S^n)$ as representations of $\mathbb{Z}_{2k}$ ($\subset \mathbb{Z}_{4k}$) if we restrict the action to $\mathbb{Z}_{2k}$.

In this paper we will show that this phenomenon occurs in more general setting if G is a compact abelian Lie group and M is an odd-dimensional closed smooth G-manifold with $\dim M^G = 0$, i.e., the fixed points are isolated. In this case, since $\tau_x(M)$ is odd-dimensional and the fixed point set $\tau_x(M)^G$ consists of only zero vector, G has a subgroup H with $G/H \cong \mathbb{Z}_2$. This follows

Received May 27, 2006.

2000 *Mathematics Subject Classification.* 57S15, 57S17.

Key words and phrases. tangential representations, Smith equivalent, isolated fixed points.

easily from an elementary representation theory. $\tau_x(M)$ is also regarded as a representation of H by restricted H-action.

For any subgroup H of G with $G/H \cong \mathbb{Z}_2$ and any representation V of H with $\dim V = \dim M$, let

$$M^G_{(H,V)} = \{x \in M^G \mid \tau_x(M) \cong V \text{ as representations of } H\}.$$

Then $M^G = \bigcup_{(H,V)} M^G_{(H,V)}$. This is not necessarily a disjoint union.

The main results are:

Theorem 1.1. *Let G be a compact abelian Lie group and M an odd-dimensional closed smooth G-manifold with $\dim M^G = 0$. Then*

(1) *$M^G = \bigcup_{(H,V)} M^G_{(H,V)}$, where (H,V) runs over the pairs of a subgroup H of G with $G/H \cong \mathbb{Z}_2$ and a representation V of H with $\dim V = \dim M$ and $\dim V^H =$ odd, and*

(2) *$\#M^G_{(H,V)}$ is even for any such (H,V) as above, and $\#M^G$ is also even, where $\#A$ denotes the number of points of a finite set A.*

Corollary 1.2. *Let G and M be as above.*

(1) *If there uniquely exists a subgroup H of G with $G/H \cong \mathbb{Z}_2$, then M^G is divided into a disjoint union of pairs x and y, $M^G = \coprod\{x,y\}$, such that $\tau_x(M) \cong \tau_y(M)$ as representations of H.*

(2) *If, in particular, $G = \mathbb{Z}_2$ or $\mathbb{Z}_4$, then the isomorphism $\tau_x(M) \cong \tau_y(M)$ in (1) can be taken as representations of G.*

(3) *If $G = \mathbb{Z}_2 \oplus H$, H has no subgroup isomorphic to $\mathbb{Z}_2$, and if $M^{\mathbb{Z}_2}$ is discrete, i.e., $\dim M^{\mathbb{Z}_2} = 0$, then the same result as in (2) also follows.*

Remark 1. If G is a cyclic group of order $2k$, k odd, then G is a group satisfying the assumption in Corollary 1.2 (3). For this group Suh [5] considered the question of Smith. As well as in our argument the index 2 subgroup plays a crucial role in [5].

2. Preliminaries

Let M be a G-manifold with $\dim M^G = 0$. For any closed subgroup H of G, M^H denotes the fixed point set of M by restricted H-action. The tangent space $\tau_x(M)$ to M at $x \in M^H$ becomes a representation of H, and decomposes, as representations of H, into a direct sum $\tau_x(M) = \tau_x(M^H) \oplus \nu_x(M^H)$, where $\tau_x(M^H)$ is the tangent space to M^H and $\nu_x(M^H)$ is the normal space of M^H in M at x. Note that $\tau_x(M)^H = \tau_x(M^H)$ and $\nu_x(M^H)^H = \{0\}$.

Let W be a representation of H with $W^H = \{0\}$, and define

$$M^{(H,W)} = \{x \in M^H \mid \nu_x(M^H) \cong W\}.$$

If G is abelian, this is a G-invariant submanifold of M with $\dim M^{(H,W)} = \dim M - \dim W$, and becomes a G/H-manifold. We easily see that

$$(M^{(H,W)})^{G/H} \subset M^G \text{ and } (M^{(H,W)})^{G/H} = M^G_{(H,V)} \text{ for } V = \mathbb{R}^\ell \oplus W,$$

where $\mathbb{R}^\ell$ is an ℓ-dimensional trivial representation with $\ell = \dim M - \dim W$.

If G has a subgroup H with $G/H \cong \mathbb{Z}_2$, then G has a one-dimensional nontrivial representation $\mathbb{R}_H$ induced from

$$G \times \mathbb{R}_H \to G/H \times \mathbb{R}_H \cong \mathbb{Z}_2 \times \mathbb{R}_H \to \mathbb{R}_H,$$

where the first map is induced from the projection $G \to G/H$ and the last map is induced from the multiplication by ± 1. Any one-dimensional nontrivial representation of G is given in this way.

For any $x \in M^G$, $\tau_x(M)$ decomposes as representation of G into a direct sum

$$(*) \qquad \tau_x(M) \cong \mathbb{R}_{H_1}^{\ell_1} \oplus \cdots \oplus \mathbb{R}_{H_t}^{\ell_t} \oplus U,$$

where H_i $(1 \le i \le t)$ are the distinct subgroups of G with $G/H_i \cong \mathbb{Z}_2$, $\mathbb{R}_H^\ell$ denotes the direct sum of ℓ copies of $\mathbb{R}_H$, U does not contain $\mathbb{R}_{H_i}$, $\dim U$ is even and $U^{H_i} = \{0\}$.

3. Proof of the results

Proof of Theorem 1.1. (1) It is clear that $M^G \supset \bigcup_{(H,V)} M^G_{(H,V)}$. To see the reversed inclusion, take any point $x \in M^G$ and consider the decomposition of $\tau_x(M)$ as in $(*)$. Since $\dim \tau_x(M)$ is odd, ℓ_j is odd for some j with $1 \le j \le t$. Assume ℓ_1 is odd, and let $\ell = \ell_1$, $H = H_1$ and

$$W = \mathbb{R}_{H_2}^{\ell_2} \oplus \cdots \oplus \mathbb{R}_{H_t}^{\ell_t} \oplus U.$$

Regarding W as a representation of H, we see $x \in M^G_{(H,V)}$ where $V = \mathbb{R}^\ell \oplus W$, $\dim V = \dim M$, and $\dim V^H = \ell$ is odd.

(2) Let (H, V) be a pair as in Theorem 1.1 (1), and decompose V into the direct sum $V = \mathbb{R}^\ell \oplus W$ such that $W^H = \{0\}$ and ℓ is odd. As noted in the preceding section, $M^{(H,W)}$ is a closed $G/H(\cong \mathbb{Z}_2)$-manifold of dimension ℓ. So we obtain

$$\chi(M^{(H,W)}) \equiv \chi((M^{(H,W)})^{G/H}) \mod 2,$$

where $\chi(\ )$ denotes the Euler characteristic. (See for example, Bredon [1, Chapter III] or Kawakubo [3, Chapter 5].) Since $\dim M^G = 0$, $\chi((M^{(H,W)})^{G/H})$ is just the number of points of $(M^{(H,W)})^{G/H} = M^G_{(H,V)}$. So we see that $\# M^G_{(H,V)}$ is even, since $\dim M^{(H,W)}$ is odd and hence $\chi(M^{(H,W)}) = 0$.

The number of j's with ℓ_j = odd in the decomposition $(*)$ is odd, since $\dim M$ is odd and $\dim U$ is even. As is easily seen from the above argument, the number of such ℓ_j's is the same as the number of (H, V)'s with $x \in M^G_{(H,V)}$. So we can assume for any $x \in M^G$ that the number of (H, V)'s with $x \in M^G_{(H,V)}$ is $2n_x + 1$ for some integer $n_x \ge 0$. Then we have from Theorem 1.1 (1),

$$\sum_{x \in M^G} (2n_x + 1) = \sum_{(H,V)} \# M^G_{(H,V)}.$$

The left-hand side of this equation is equal to

$$2\sum_{x\in M^G} n_x + \#M^G,$$

and the right-hand side is even from the fact just proved. This shows that $\#M^G$ is even. $\square$

Proof of Corollary 1.2. (1) If there uniquely exists a subgroup H of G with $G/H \cong \mathbb{Z}_2$, then the union $M^G = \bigcup_{(H,V)} M^G_{(H,V)}$ is the disjoint union $M^G = \coprod_V M^G_{(H,V)}$, where V runs over the representations of H with $\dim V = \dim M$ and $\dim V^H = \text{odd}$. Since $\#M^G_{(H,V)}$ is even, this implies the required result.

(2) If $G = \mathbb{Z}_2$ then $H = \{1\}$, and if $G = \mathbb{Z}_4$ then $H = \mathbb{Z}_2$. In these cases, for any representation V of H with $\dim V = \mathit{odd}$ and $\dim V^H = \mathit{odd}$, the H-action on V uniquely extends to a G-action on V such that $V^G = \{0\}$. This implies the result.

(3) It is sufficient to show that for $x, y \in M^G_{(H,V)}$, $\tau_x(M) \cong \tau_y(M)$ as representations of G. Let $\mathbb{R}_-$ be the nontrivial real 1-dimensional (irreducible) representation of $\mathbb{Z}_2$ given by the multiplication by ± 1, and $\mathbb{R}_+$ the trivial real 1-dimensional representation of $\mathbb{Z}_2$. Let $\{U_1, U_2, \ldots, U_k\}$ be a complete set of real irreducible representations of H. For any i with $1 \le i \le k$, let $\bar{U}_i = \mathbb{R}_+ \otimes U_i$ and $\bar{U}_{-i} = \mathbb{R}_- \otimes U_i$. Then $\{\bar{U}_i \mid 1 \le |i| \le k\}$ gives a complete set of real irreducible representations of $G = \mathbb{Z}_2 \oplus H$. Thus we have

$$\tau_x(M) \cong \bigoplus_{1\le|i|\le k} \bar{U}_i^{a_i}, \quad \tau_y(M) \cong \bigoplus_{1\le|i|\le k} \bar{U}_i^{b_i}$$

for some nonnegative integers a_i, b_i, where $\bar{U}_i^{a_i}$ denotes the direct sum of a_i copies of $\bar{U}_i$. Since $U_i \cong \bar{U}_i \cong \bar{U}_{-i}$ as representations of H, we have, as representations of H,

$$\tau_x(M) \cong \bigoplus_{i=1}^{k} U_i^{a_i + a_{-i}}, \quad \tau_y(M) \cong \bigoplus_{i=1}^{k} U_i^{b_i + b_{-i}}.$$

Since $x, y \in M^G_{(H,V)}$, we have $\tau_x(M) \cong V \cong \tau_y(M)$ as representations of H. Thus we have

$$(**) \qquad a_i + a_{-i} = b_i + b_{-i} \quad (1 \le i \le k).$$

Since $\bar{U}_i^{\mathbb{Z}_2} \cong U_i$ for $i > 0$, and $\bar{U}_i^{\mathbb{Z}_2} = \{0\}$ for $i < 0$, we have

$$\tau_x(M^{\mathbb{Z}_2}) \cong \tau_x(M)^{\mathbb{Z}_2} \cong \bigoplus_{i=1}^{k} U_i^{a_i}, \quad \tau_y(M^{\mathbb{Z}_2}) \cong \tau_y(M)^{\mathbb{Z}_2} \cong \bigoplus_{i=1}^{k} U_i^{b_i}.$$

Since $M^{\mathbb{Z}_2}$ is discrete by the assumption, we have $a_i = b_i = 0$ for $i > 0$, and hence $(**)$ implies $a_i = b_i$ for any i with $1 \le |i| \le k$. This shows $\tau_x(M) \cong \tau_y(M)$ as representations of G. $\square$

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