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X-LIFTING MODULES OVER RIGHT PERFECT RINGS

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ABSTRACT. Keskin and Harmanci defined the family $\mathbf{B}(M,X) = \{A \leq M \mid \exists Y \leq X, \exists f \in \mathrm{Hom}_R(M, X/Y), \mathrm{Ker}\, f/A \ll M/A\}$. And Orhan and Keskin generalized projective modules via the class $\mathbf{B}(M,X)$.

In this note we introduce X-local summands and X-hollow modules via the class $\mathbf{B}(M,X)$. Let R be a right perfect ring and let M be an X-lifting module. We prove that if every co-closed submodule of any projective module P contains $\mathrm{Rad}(P)$, then M has an indecomposable decomposition. This result is a generalization of Kuratomi and Chang's result [9, Theorem 3.4]. Let X be an R-module. We also prove that for an X-hollow module H such that every non-zero direct summand K of H with $K \in \mathbf{B}(H,X)$, if $H \oplus H$ has the internal exchange property, then H has a local endomorphism ring.

1. Introduction

Extending modules and lifting modules have been studied extensively in recent years by many ring theorists (see, for example, [3], [5]-[14]).

Let M and X be R-modules. In [8], D. Keskin and A. Harmanci defined the family $\mathbf{B}(M,X) = \{A \leq M \mid \exists Y \leq X, \exists f \in \mathrm{Hom}_R(M, X/Y), \mathrm{Ker}\, f/A \ll M/A\}$. They considered the following conditions:

$\mathbf{B}(M,X)$-(D_1): For any $A \in \mathbf{B}(M,X)$, there exists a direct summand $A^* \leq_\oplus M$ such that $A/A^* \ll M/A^*$

$\mathbf{B}(M,X)$-(D_2): For any $A \in \mathbf{B}(M,X)$, if $B \leq_\oplus M$, $M/A \simeq B$ implies $A \leq_\oplus M$

$\mathbf{B}(M,X)$-(D_3): For any $A \in \mathbf{B}(M,X)$ and $B \leq_\oplus M$, if $A \leq_\oplus M$ and $M = A + B$ then $A \cap B \leq_\oplus M$.

They defined that M is said to be *X-discrete* if $\mathbf{B}(M,X)$-(D_1) and $\mathbf{B}(M,X)$-(D_2) hold, and is said to be *X-quasi-discrete* if $\mathbf{B}(M,X)$-(D_1) and $\mathbf{B}(M,X)$-(D_3) hold. Furthermore, M is said to be *X-lifting* if $\mathbf{B}(M,X)$-(D_1) holds. We have just seen that the following implications hold:

$$\text{"}X\textbf{-discrete} \Longrightarrow X\textbf{-quasi-discrete} \Longrightarrow X\textbf{-lifting"}.$$

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Throughout this paper, all rings R considered are associative rings with identity and all R-modules are unital.

Let M be a right R-module and N a submodule of M. The notation $N \leq_{\oplus} M$ means that N is a direct summand of M.

A submodule K of M is called a *small* submodule (or *superfluous* submodule) of M, abbreviated $K \ll M$, in the case when, for every submodule $L \leq M$, $K + L = M$ implies $L = M$.

2. Preliminaries

Let A and P be submodules of M with $P \in \mathbf{B}(M, X)$. P is called an *X-supplement* of A if it is minimal with the property $A + P = M$ equivalently, if $M = A + P$ and $A \cap P \ll P$.

The module M is called *X-amply supplemented* if for any submodules A, B of M with $A \in \mathbf{B}(M, X)$ and $M = A + B$ there exists an X-supplement P of A such that $P \leq B$.

Let $N_1 \leq N_2 \leq M$. N_1 is a *co-essential* submodule of N_2 in M, abbreviated $N_1 \leq_c N_2$ in M, if the kernel of the canonical map $M/N_1 \longrightarrow M/N_2 \longrightarrow 0$ is small in M/N_1, or equivalently, if $M = N_2 + X$ with $N_1 \leq X$ implies $M = X$.

A submodule N of M is said to be *co-closed* in M (or a *co-closed* submodule of M), if N has no proper co-essential submodule in M. i.e., $N' \leq_c N$ in M implies $N = N'$. It is easy to see that any direct summand of a module M is co-closed in M. Note that every X-supplement submodule of M is co-closed in M.

For $N' \leq N \leq M$, N' is called a *co-closure* of N in M if N' is a co-closed submodule of M with $N' \leq_c N$ in M. Any submodule of a module has a closure, however, co-closure does not exist in general.

Lemma 2.1 ([9, Lemma 1.4] and [5, 3.2, 3.7]). *Let $A \leq B \leq M$. Then the following hold:*

(1) $A \leq_c B$ in M if and only if $M = A + K$ for any submodule K of M with $M = B + K$.

(2) If $A \ll M$ and B is co-closed in M, $A \ll B$.

Lemma 2.2 ([16, Lemma 41.14]). *Any projective module satisfies the following condition:*

(D) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 \ll M$ and $M = M_1 + M_2$, then $M = M_1 \oplus M_2$.

Lemma 2.3 ([13, Theorem 3.5]). *If M is a lifting module with the condition (D), then M can be expressed as a direct sum of hollow modules.*

Lemma 2.4 ([1, Lemma 17.17]). *Suppose that M has a projective cover. If P is projective with an epimorphism $\varphi : P \longrightarrow M$, then P has a decomposition $P = P_1 \oplus P_2$ such that $P_1 \leq \operatorname{Ker} \varphi$ and $\varphi|_{P_2} : P_2 \longrightarrow M$ is a projective cover of M.*

Theorem 2.5 ([3, Theorem 1.1.24]). *For an R-module M, the following hold:*

(1) *If M is a quasi-injective module, then M is a fully invariant submodule of $E(M)$.*

(2) *If M is a quasi-injective module, then any direct decomposition $E(M) = E_1 \oplus \cdots \oplus E_n$ induces $M = (M \cap E_1) \oplus \cdots \oplus (M \cap E_n)$.*

(3) *If M is a quasi-projective module with a projective cover $\varphi : P \longrightarrow M$,* Ker *$\varphi$ is a fully invariant submodule of P; whence any endomorphism of P induces an endomorphism of M.*

(4) *If M is a quasi-projective module with a projective cover $\varphi : P \longrightarrow M$, then any direct decomposition $P = P_1 \oplus \cdots \oplus P_n$ induces $M = \varphi(P_1) \oplus \cdots \oplus \varphi(P_n)$.*

A ring R is called *right perfect* if every right R-module has a projective cover.

Proposition 2.6. *The following statements are equivalent:*

(i) *Every cyclic right R-module has a projective cover;*

(ii) *R_R is a lifting module.*

Proof. (i) $\Longrightarrow$ (ii) Let A be a submodule of R_R and let $\varphi : R \longrightarrow R/A$ be the canonical epimorphism. Since R/A has a projective cover, by Lemma 2.4, there exists a decomposition $R_R = eR \oplus (1-e)R$ such that $(\varphi \mid_{eR}) : eR \longrightarrow R/A \longrightarrow 0$ a projective cover and $(1-e)R \leq A$. This implies Ker $(\varphi \mid_{eR}) = A \cap eR \ll eR$. i.e., $R = eR \oplus (1-e)R$ such that $A \cap eR \ll eR$. Thus R_R is lifting.

(ii) $\Longrightarrow$ (i) Suppose that R_R is lifting. We claim that R/A has a projective cover. Since R_R is lifting, for any $A \leq R$, there exists $A^* \leq_c A$ such that $R = A^* \oplus A^{**}$. Then $\pi \mid_{A^{**}} : A^{**} \longrightarrow R/A \longrightarrow 0$ is a projective cover of R/A, where $\pi : R \longrightarrow R/A \longrightarrow 0$ is the canonical epimorphism. □

As corollaries of Proposition 2.6, we obtain the following two results.

Corollary 2.7. *Let P be a projective module. Then the following statements are equivalent:*

(i) *Every factor module of P has a projective cover;*

(ii) *P is lifting.*

Corollary 2.8. *The following statements are equivalent:*

(i) *Every simple right R-module has a projective cover;*

(ii) *R_R satisfies the lifting property for simple factor modules.*

Lemma 2.9 ([9, Lemma 3.1] and [5, 3.2]). *Let $f : M \longrightarrow N$ be an epimorphism. Suppose $K \leq_c K'$ in M. Then $f(K) \leq_c f(K')$ in N.*

Lemma 2.10 ([8, Lemma 2.2]). *Let M, N and X be R-modules. Then the following hold:*

(1) *If $A \in \mathbf{B}(M,X)$ and $B \leq A$ with $A/B \ll M/B$, then $B \in \mathbf{B}(M,X)$.*

(2) *Let* $h : M \longrightarrow N$ *be an epimorphism and* $A \in \mathbf{B}(M, X)$ *with* $\operatorname{Ker} h \leq A$. *Then* $h(A) \in \mathbf{B}(N, X)$. *Conversely, if* $h(A) \in \mathbf{B}(N, X)$ *and* $\operatorname{Ker} h \leq A$, *then* $A \in \mathbf{B}(M, X)$.

(3) *Let* $B \leq A \leq M$. *Then* $A \in \mathbf{B}(M, X)$ *if and only if* $A/B \in \mathbf{B}(M/B, X)$.

(4) *Let* $h : N \longrightarrow M$ *be an epimorphism and* $A \in \mathbf{B}(M, X)$. *Then* $h^{-1}(A) \in \mathbf{B}(N, X)$.

3. Main results

Theorem 3.1. *Let* R *be a ring. The following conditions are equivalent:*

(1) R *is right perfect;*

(2) *Every projective right* R*-module is lifting;*

(3) *Every quasi-projective right* R*-module is lifting;*

(4) *Every countably generated free right* R*-module is lifting.*

Proof. (1) $\Longleftrightarrow$ (2) This follows from Corollary 2.7.

(2) $\Longrightarrow$ (3) Let Q_R be a quasi-projective module and let A be a submodule of Q. Consider the canonical epimorphism $f : Q \longrightarrow Q/A$. We can take a projective module P_R such that Q is a homomorphic image of P, i.e., we have an epimorphism $g : P \longrightarrow Q$. Since P is a lifting module, by Lemma 2.4, there exists a decomposition $P = P_1 \oplus P_2$ such that $P_1 \leq g^{-1}(A)$, $fg\,|_{P_2}$: $P_2 \longrightarrow Q/A$ is a projective cover. As Q is a quasi-projective module, the decomposition $P = P_1 \oplus P_2$ induces a direct decomposition $Q = g(P_1) \oplus g(P_2)$ by Theorem 2.5. Then $g(P_1) \leq A$ and $g(P_2) \cap A \ll g(P_2)$ hold.

(3) $\Longrightarrow$ (2) Obvious.

(1) $\Longrightarrow$ (4) This follows from [1, Theorem 28.4].

(4) $\Longrightarrow$ (1) By (4), R is semiperfect and $R/J(R)$ is semisimple. Since $R^{(\mathbb{N})}$ is lifting, there exists a decomposition $R^{(\mathbb{N})} = X \oplus Y$ such that $X \leq \operatorname{Rad}(R^{(\mathbb{N})})$ and $\operatorname{Rad}(R^{(\mathbb{N})}) \cap Y \ll Y$. Because $\operatorname{Rad}(R^{(\mathbb{N})}) = \operatorname{Rad}(X) \oplus \operatorname{Rad}(Y)$ and $X \leq \operatorname{Rad}(R^{(\mathbb{N})})$, we see $\operatorname{Rad}(X) = X$, which implies $X = 0$ and $R^{(\mathbb{N})}J(R) = \operatorname{Rad}(R^{(\mathbb{N})}) \ll R^{(\mathbb{N})}$. Hence, by [1, Lemma 28.3], $J(R)$ is right T-nilpotent. Thus R is right perfect. □

A family $\{X_\lambda \mid \lambda \in \Lambda\}$ of submodules of a module M with $X_\lambda \in \mathbf{B}(M, X)$ is called an *X-local summand* of M, if $\Sigma_{\lambda \in \Lambda} X_\lambda$ is direct and $\Sigma \oplus_{\lambda \in F} X_\lambda \leq_\oplus M$ for every finite subset $F \subseteq \Lambda$.

By analogy with the proof of [14, Lemma 2.4] or [11, Theorem 2.17], we have the following lemma.

Lemma 3.2. *If every* X*-local summand of a module* M *is a direct summand, then* M *has an indecomposable decomposition.*

By Lemma 2.1(1), we have the following lemma.

Lemma 3.3. *Assume* $P_i \leq_c Q_i$ *in* P *for every* $i \in I$. *Then* $\Sigma \oplus_{i \in I} P_i \leq_c \Sigma \oplus_{i \in I} Q_i$ *in* P.

Lemma 3.4. *Let $\{P_i\}_{i\in I}$ be a set of R-modules. Assume $P_i \in \mathbf{B}(M,X)$ for every $i \in I$. Then $\Sigma \oplus_{i\in I} P_i \in \mathbf{B}(M,X)$.*

Proof. Since $P_i \in \mathbf{B}(M,X)$, there exist a submodule Y of X and a homomorphism $f_i : M \longrightarrow X/Y$ such that Ker $f/P_i \ll M/P_i$. Put $f = \Sigma \oplus_{i\in I} f_i$. Then $f : M \longrightarrow X/Y$ such that Ker $f/\Sigma \oplus_{i\in I} P_i \ll M/\Sigma \oplus_{i\in I} P_i$. Thus $\Sigma \oplus_{i\in I} P_i \in \mathbf{B}(M,X)$. □

Lemma 3.5. *Let X be a right R-module. Suppose that R is a right perfect ring. Then every projective right R-module is X-lifting.*

Proof. Let P be a projective module. For any $A \in \mathbf{B}(P,X)$, consider the canonical epimorphism $\varphi : P \longrightarrow P/A$. Since P/A has a projective cover, by Lemma 2.4, there exists a decomposition $P = P_1 \oplus P_2$ such that $P_1 \leq$ Ker φ and $\varphi\mid_{P_2} : P_2 \longrightarrow P/A$ is a projective cover of P/A. Hence P is X-lifting. □

Proposition 3.6. *Let R be a right perfect ring and let M be an X-lifting module. Then M is X-amply supplemented.*

Proof. Let A, $B \leq M$ such that $B \in \mathbf{B}(M,X)$ and $M = A + B$. Since $M = A + B$ and $B \in \mathbf{B}(M,X)$, there exist $Y \leq X$ and $f : M \longrightarrow X/Y$ such that Ker $f/B \ll M/B$. Consider the isomorphism $\alpha : M/B \longrightarrow A/A \cap B$. Then α(Ker f/B) = Ker $f/A\cap B$. Hence Ker $f/A\cap B \ll M/A\cap B$. Therefore $A \cap B \in \mathbf{B}(M,X)$. As M is X-lifting, there exists a direct summand K of M such that $K \leq_c A\cap B$ in M. Then $A\cap B = K \oplus [K^* \cap (A \cap B)]$, $M = (A\cap B) + K^*$ and $(A \cap B) \cap K^* \ll K^*$. Thus $M = B + (A \cap K^*)$.

Let D be a co-closure of $A \cap K^*$ in M. Then $M = B + D$ and $B \cap D \leq B \cap (A \cap K^*) \ll K^*$. Hence $B \cap D \ll K^*$. Since D is co-closed in M, $B\cap D \leq D$ and $B \cap D \ll M$, $B \cap D \ll D$. Thus D is an X-supplement of B in M such that $D \leq A$. □

Lemma 3.7 ([8, Lemma 3.2]). *Every epimorphic image of an X-amply supplemented R-module is X-amply supplemented.*

Lemma 3.8. *Let M be an X-amply supplemented module and let* Ker $f \ll M \xrightarrow{f} N \longrightarrow 0$. *Suppose K is co-closed in M with* Ker $f \leq K$. *Then $f(K)$ is co-closed in N.*

Proof. By Lemma 3.7, N is X-amply supplemented. Let $L \leq_c f(K)$ in N. We claim that $L = f(K)$. Since f is an epimorphism, there exists a submodule T of K in M with $f(T) = L$. Since N is X-amply supplemented, there exists an X-supplement P of $f(K)$ such that $P \leq N$. i.e., $N = f(K) + P$ and $f(K) \cap P \ll P$. Since f is an epimorphism, there exists a submodule Q of M with $f(Q) = P$. Then $M = K + Q+$Ker f. As Ker $f \ll M$, $M = K + Q$. This implies $N = f(K) + f(Q) = f(K) + P = L + P = f(T) + f(Q)$. Then $M = T + Q +$ Ker $f = T + Q$. Thus $T \leq_c K$ in M by Lemma 2.1(1). As K is co-closed in M, $T = K$. Hence $L = f(T) = f(K)$. Therefore $f(K)$ is co-closed in N. □

Proposition 3.9. *Suppose that M is an X-lifting module. Then every co-closed submodule K of M with $K \in \mathbf{B}(M,X)$ is a direct summand.*

Proof. Since M is X-lifting, there exists a direct summand K^* such that $K^* \leq_c K$ in M. As K is co-closed in M, $K = K^* \leq_\oplus M$. □

Theorem 3.10. *Let R be a right perfect ring and let M be an X-lifting module. Assume that every co-closed submodule of any projective module P contains $Rad(P)$. Then every X-local summand of M is a direct summand.*

Proof. Let M be an X-lifting module and let $\Sigma_{i\in I}X_i$ be an X-local summand of M with $X_i \in \mathbf{B}(M,X)$. Since R is right perfect, M has a projective cover, say Ker $f \ll P \xrightarrow{f} M \to 0$. By Lemma 3.5, P is projective X-lifting. Since $X_i \in \mathbf{B}(M,X)$, $f^{-1}(X_i) \in \mathbf{B}(P,X)$ by Lemma 2.10(4). So there exists a decomposition $P = P_i \oplus P_i^*$ $(i \in I)$ such that $P_i \leq_c f^{-1}(X_i)$ in P. By Lemma 2.9, $f(P_i) \leq_c f(f^{-1}(X_i)) = X_i$ in M. As X_i is co-closed in M, $f(P_i) = X_i$. First we prove that $\Sigma_{i\in I}P_i$ is direct. Let F be a finite subset of $I - \{i\}$. Since $\Sigma \oplus_{i\in I} X_i$ is an X-local summand of M, we see

$$f(P_i + \Sigma_{j\in F}P_j) = X_i \oplus (\Sigma \oplus_{j\in F} X_j) \leq_\oplus M.$$

So there exists a direct summand Y of M such that $M = X_i \oplus (\Sigma \oplus_{j\in F} X_j) \oplus Y$. As P is lifting, there exists a decomposition $P = Q \oplus Q^*$ such that $Q \leq_c f^{-1}(Y)$ in P. Then $f(Q) = Y$. Thus we see

$$P = P_i + \Sigma_{j\in F}P_j + Q + \text{Ker } f = P_i + \Sigma_{j\in F}P_j + Q.$$

Then $P_i \cap (\Sigma_{j\in F}P_j + Q) \subseteq \text{Ker } f \ll P$. Similarly, we see $Q \cap (P_i + \Sigma_{j\in F}P_j) \ll P$ and $P_j \cap (P_i + \Sigma_{l\in F-\{j\}}P_l + Q) \ll P$. By Lemma 2.2, we obtain $P = P_i \oplus (\Sigma_{j\in F}P_j) \oplus Q$. Hence $\Sigma_{i\in I}P_i$ is direct. By the same argument, we see $\Sigma \oplus_{i\in I} P_i$ is an X-local summand of P. By Lemma 2.3, $\Sigma \oplus_{i\in I} P_i \leq_\oplus P$. So $f(\Sigma \oplus_{i\in I} P_i)$ is co-closed in M by Lemma 3.8. Since M is X-lifting, we see

$$\Sigma \oplus_{i\in I} X_i = f(\Sigma \oplus_{i\in I} P_i) \leq_\oplus M.$$

Thus any X-local summand of M is a direct summand. □

By Lemma 3.2 and Theorem 3.10, we obtain the first main theorem.

Theorem 3.11. *Suppose that every co-closed submodule of any projective module P contains $Rad(P)$. Then every X-lifting module over right perfect rings has an indecomposable decomposition.*

Let X be an R-module. A non-zero R-module H is *X-hollow* if for any proper submodule K of H with $K \in \mathbf{B}(H,X)$, $K \ll H$.

Proposition 3.12. *Let H and X be R-modules. Assume that every non-zero direct summand K of H with $K \in \mathbf{B}(H,X)$. Then H is X-hollow if and only if H is indecomposable X-lifting.*

Proof. ($\Longrightarrow$) Assume H is X-hollow. Let $K \in \mathbf{B}(H,X)$ with $K \lneq H$. Since H is X-hollow, $K \ll H$. So there exists a decomposition $H = 0 \oplus H$ such that $0 \leq_c K$ in H. Thus H is X-lifting. Now, assume that $H = H_1 \oplus H_2$, $H_i \neq 0$, $i = 1, 2$. Since H is X-hollow, $H_i \ll H$, $i = 1, 2$. Hence $H_i = 0$. This is a contradiction. Therefore H is indecomposable. ($\Longleftarrow$) Suppose that H is indecomposable X-lifting. Let $K \in \mathbf{B}(H,X)$ with $K \lneq H$. By hypothesis, there exists a decomposition $H = K^* \oplus K^{**}$ such that $K^* \leq_c K$ in H. As H is indecomposable, we have either $K^* = 0$ or $K^{**} = 0$. If $K^* = 0$, then $K \ll H$. In the second case, $H = K$. This is a contradiction. □

A module M is said to have the *(finite) exchange property* if, for any (finite) index set I, whenever $M \oplus N = \oplus_{i \in I} A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\oplus_{i \in I} B_i)$ for some submodules $B_i \leq A_i$. A module M has the *(finite) internal exchange property* if, for any (finite) direct sum decomposition $M = \oplus_{i \in I} M_i$ and any direct summand X of M, there exist submodules $\overline{M_i} \leq M_i$ such that $M = X \oplus (\oplus_{i \in I} \overline{M_i})$.

By Proposition 3.12, we obtain the second main theorem.

Theorem 3.13 ([15, Proposition 1]). *Let X be an R-module and let H be an X-hollow module. Assume that every non-zero direct summand K of H with $K \in \mathbf{B}(H,X)$.*

If $H \oplus H$ has the internal exchange property, then H has a local endomorphism ring.

Corollary 3.14 (cf., [5, 12.2]). *Let X be an R-module and let H be an X-hollow module. Assume that every non-zero direct summand K of H with $K \in \mathbf{B}(H,X)$. Then the following conditions are equivalent:*

(1) *H has a local endomorphism ring;*

(2) *H has the finite exchange property;*

(3) *H has the exchange property.*

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