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FIXED POINTS AND HOMOTOPY RESULTS FOR ĆIRIĆ-TYPE MULTIVALUED OPERATORS ON A SET WITH TWO METRICS

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Abstract. The purpose of this paper is to present some fixed point results for nonself multivalued operators on a set with two metrics. In addition, a homotopy result for multivalued operators on a set with two metrics is given. The data dependence and the well-posedness of the fixed point problem are also discussed.

1. Introduction

Throughout this paper, standard notations and terminologies in nonlinear analysis (see [6], [12], [13]) are used. For the convenience of the reader we recall some of them here.

Let (X,d) be a metric space. In the sequel we will use the following symbols:

$$P(X) := \{Y \subset X \mid Y \text{ is nonempty}\},\ P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\},$$

$B_d(x_0,r) := \{x \in X | d(x_0,x) < r\}$. If d' is another metric on X, we will denote by $\overline{B}_d^{d'}(x_0,r)$ the closure of $B_d(x_0,r)$ in (X,d').

Let A be nonempty subset of the metric (X,d) and $x_0 \in X$. Then $D_d(x_0,A) = D(\{x_0\},A)$ is called the distance from the point x_0 to the set A.

The Pompeiu-Hausdorff generalized distance between the nonempty closed subsets A and B of the metric space (X,d) is defined by the following formula:

$$H_d(A,B) := \max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}.$$

The symbol $T : X \multimap X$ means $T : X \to P(X)$, i.e., T is a multivalued operator from X to X. We will denote by $G(T) := \{(x,y) \in X \times X | y \in T(x)\}$ the graph of T. The multivalued operator T is said to be closed if $G(T)$ is closed in $X \times X$.

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For $T : X \to P(X)$ the symbol $F_T := \{x \in X | \ x \in T(x)\}$ denotes the fixed point set, while $(SF)_T := \{x \in X | \ \{x\} = T(x)\}$ is the strict fixed point set of the multivalued operator T.

The aim of this paper is to present some fixed point results for nonself multivalued operators on a set with two metrics. In addition, a homotopy result for multivalued operators on a set with two metrics is given. The data dependence and the well-posedness of the fixed point problem are also discussed. Our results complement and extend some previous theorems given by R. P. Agarwal, D. O'Regan [1], R. P. Agarwal, J. H. Dshalalow, D. O'Regan [2], L. Ćirić [3], M. Frigon, A. Granas [5], S. Reich [10], etc.

2. Fixed points and homotopy results for Ćirić-type multivalued operators on a set with two metrics

Let (X, d) be a metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. For $x, y \in X$, let us denote

$$M_d^T(x,y) := \max\{d(x,y), D_d(x,T(x)), D_d(y,T(y)), \tfrac{1}{2}[D_d(x,T(y)) + D_d(y,T(x))]\}.$$

A slight modified variant of Ćirić's theorem (see [3]) is the following:

Theorem 2.1. *Suppose that the metric space (X, d) is complete and the multivalued operator $T : X \to P_{cl}(X)$ satisfies the following condition:*

there exists $\alpha \in [0,1[$ such that $H_d(T(x),T(y)) \leq \alpha \cdot M_d^T(x,y)$ for each $x, y \in X$.

Then $F_T \neq \emptyset$ and for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that

(1) $x_0 = x, \ x_1 = y$;

(2) $x_{n+1} \in T(x_n), \ n \in \mathbb{N}$;

(3) $x_n \xrightarrow{d} x^* \in T(x^*)$, *as* $n \to \infty$;

(4) $d(x_n, x^*) \leq \frac{(\alpha p)^n}{1-\alpha p} \cdot d(x_0, x_1)$ *for each* $n \in \mathbb{N}$ *(where* $p \in]1, \frac{1}{\alpha}[$ *is arbitrary).*

A data dependence result for Ćirić-type multivalued operators is the following theorem.

Theorem 2.2. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ be two multivalued operators. Suppose that*

(i) *there exists $\alpha_i \in [0,1[$ such that*

$$H_d(T_i(x), T_i(y)) \leq \alpha_i \cdot M_d^{T_i}(x,y), \text{ for each } x, y \in X \text{ for } i \in \{1,2\};$$

(ii) *there exists $\eta > 0$ such that $H_d(T_1(x), T_2(x)) \leq \eta$ for each $x \in X$.*

Then

$$F_{T_1} \neq \emptyset \neq F_{T_2} \text{ and } H_d(F_{T_1}, F_{T_2}) \leq \frac{\eta}{1 - \max\{\alpha_1, \alpha_2\}}.$$

Proof. From Ćirić's theorem we have that $F_{T_1} \neq \emptyset \neq F_{T_2}$.

For our second conclusion, denote $\Upsilon := \frac{\eta}{1-\max\{\alpha_1,\alpha_2\}}$. For our purpose it's enough to prove that for any $u \in F_{T_1}$ there exists $v \in F_{T_2}$ such that $d(u,v) \leq \Upsilon$ and a similar relation with the roles of F_{T_1} and F_{T_2} reversed.

Let $u \in F_{T_1}$ be arbitrary. From (ii) for every $q > 1$ there exists $x_1 \in T_2(u)$ such that $d(u,x_1) \leq qH(T_1(u),T_2(u)) \leq q\eta$.

Using (4) for T_2 and taking $n := 0$, $x_0 := u$ and x_1 as above we have, by Theorem 2.1, that there exists $x_2^* \in F_{T_2}$ such that

$$d(u,x_2^*) \leq \frac{1}{1-(\alpha_2 p)} \cdot d(u,x_1) \leq \frac{1}{1-(\alpha_2 p)} \cdot q\eta.$$

Letting $p \searrow 1$ we get that

$$d(u,x_2^*) \leq \frac{1}{1-\alpha_2} \cdot q\eta.$$

By interchanging the roles of T_1 and T_2, for each $v \in F_{T_2}$, each $q' > 1$ and each $x_1' \in T_1(v)$ such that $d(v,x_1') \leq q' H(T_2(v),T_1(v)) \leq q'\eta$ we have that

$$d(v,x_1^*) \leq \frac{1}{1-\alpha_1} \cdot q'\eta,$$

where x_1^* is the fixed point of T_1 given by Theorem 2.1. Thus

$$H_d(F_{T_1},F_{T_2}) \leq \frac{\eta}{1-\max\{\alpha_1,\alpha_2\}} \cdot \max\{q,q'\}.$$

The conclusion follows now by letting $q, q' \searrow 1$. □

We continue the section with a local version of Ćirić's theorem on a set with two metrics.

Theorem 2.3. *Let X be a nonempty set, $x_0 \in X$ and $r > 0$. Suppose that d, ρ are two metrics on X and $T : \overline{B}^d_\rho(x_0,r) \to P(X)$ is a multivalued operator. We suppose that*

(i) *(X,d) is a complete metric space;*
(ii) *there exists $c > 0$ such that $d(x,y) \leq c\rho(x,y)$ for each $x,y \in X$;*
(iii) *if $d \neq \rho$ then $T : \overline{B}^d_\rho(x_0,r) \to P(X^d)$ is closed, while if $d = \rho$ then $T : \overline{B}^d_d(x_0,r) \to P_{cl}(X^d)$;*
(iv) *there exists $\alpha \in [0,1[$ such that $H_\rho(T(x),T(y)) \leq \alpha M_\rho^T(x,y)$ for each $x,y \in \overline{B}^d_\rho(x_0,r)$;*
(v) *$D_\rho(x_0,T(x_0)) < (1-\alpha)r$.*

Then

(A) *there exists $x^* \in \overline{B}^d_\rho(x_0,r)$ such that $x^* \in T(x^*)$;*

(B) *if $(SF)_T \neq \emptyset$ and $(x_n)_{n\in\mathbb{N}} \subset \overline{B}^d_\rho(x_0,r)$ is such that $H_\rho(x_n,T(x_n)) \to 0$ as $n \to +\infty$, then $x_n \xrightarrow{\rho} x \in (SF)_T$ as $n \to +\infty$ (i.e., the fixed point problem is well-posed in the generalized sense for T with respect to H_ρ, see [7], [9]).*

Proof. (A) From (v) there exists $x_1 \in T(x_0)$ such that $\rho(x_0, x_1) < (1-\alpha)r$. Clearly $x_1 \in \overline{B}^d_\rho(x_0, r)$. We have

$$\begin{aligned}
&H_\rho(T(x_0), T(x_1)) \\
&\leq \alpha \max\{\rho(x_0, x_1), D_\rho(x_0, T(x_0)), D_\rho(x_1, T(x_1)), \\
&\qquad \frac{1}{2}[D_\rho(x_0, T(x_1)) + D_\rho(x_1, T(x_0))]\} \\
&\leq \alpha \max\{\rho(x_0, x_1), D_\rho(x_1, T(x_1)), \frac{1}{2}[\rho(x_0, x_1) + D_\rho(x_1, T(x_1))]\} \\
&\leq \alpha \max\{\rho(x_0, x_1), D_\rho(x_1, T(x_1))\}.
\end{aligned}$$

We claim that $\max\{\rho(x_0, x_1), D_\rho(x_1, T(x_1))\} = \rho(x_0, x_1)$. If

$$\max\{\rho(x_0, x_1), D_\rho(x_1, T(x_1))\} = D_\rho(x_1, T(x_1)),$$

then we get the following contradiction $H_\rho(T(x_0), T(x_1)) \leq \alpha D_\rho(x_1, T(x_1)) \leq \alpha H_\rho(T(x_0), T(x_1))$. Thus

$$H_\rho(T(x_0), T(x_1)) \leq \alpha \rho(x_0, x_1).$$

Hence $H_\rho(T(x_0), T(x_1)) < \alpha(1-\alpha)r$. Thus, there exists $x_2 \in T(x_1)$ such that $\rho(x_1, x_2) < \alpha(1-\alpha)r$. Moreover, $\rho(x_0, x_2) \leq \rho(x_0, x_1) + \rho(x_1, x_2) < (1-\alpha)r + \alpha(1-\alpha)r = (1-\alpha^2)r < r$. Hence, $x_2 \in \overline{B}^d_\rho(x_0, r)$. Using this procedure, we obtain the sequence $(x_n)_{n\in\mathbb{N}} \subset \overline{B}^d_\rho(x_0, r)$ having the following properties:

(a) $x_{n+1} \in T(x_n), n \in \mathbb{N}$;
(b) $\rho(x_{n-1}, x_n) \leq \alpha^{n-1}(1-\alpha)r, n \in \mathbb{N}^*$;
(c) $\rho(x_0, x_n) \leq (1-\alpha^n)r, n \in \mathbb{N}^*$.

From (b) we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, ρ). From (ii) the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d) too. Taking into account (i) it follows that there exists $x^* \in \overline{B}^d_\rho(x_0, r)$ such that $x_n \xrightarrow{d} x^*$. If $d \neq \rho$, since $T : \overline{B}^d_\rho(x_0, r) \to P_{cl}(X^d)$ is closed, we immediately get that $x^* \in T(x^*)$, as $n \to \infty$. If $d = \rho$ the conclusion follows as in the proof of Ćirić's theorem (see [3], Theorem 2 as well as [2]).

(B) Let $x \in (SF)_T$. Thus we have:

$$\begin{aligned}
&\rho(x_n, x) \\
&\leq D_\rho(x_n, T(x_n)) + H_\rho(T(x_n), T(x)) \leq D_\rho(x_n, T(x_n)) + \alpha M^T_\rho(x_n, x) \\
&\leq D_\rho(x_n, T(x_n)) + \alpha \cdot \max\{\rho(x_n, x), D_\rho(x_n, T(x_n)), \tfrac{1}{2}[D_\rho(x_n, T(x)) + D_\rho(x, T(x_n))]\} \\
&\leq D_\rho(x_n, T(x_n)) + \alpha \cdot \max\{\rho(x_n, x), D_\rho(x_n, T(x_n)), \rho(x_n, x) + \tfrac{1}{2}D_\rho(x_n, T(x_n))\} \\
&\leq D_\rho(x_n, T(x_n)) + \alpha \cdot \max\{D_\rho(x_n, T(x_n)), \rho(x_n, x) + \tfrac{1}{2}D_\rho(x_n, T(x_n))\}.
\end{aligned}$$

Hence, we get that

$$\rho(x_n, x) \leq \max\{1+\alpha, \frac{\alpha}{2(1-\alpha)}\} D_\rho(x_n, T(x_n) \searrow 0 \text{ as } n \to \infty.$$

The proof is complete. □

Remark 2.1. Theorem 2.3 holds if the condition (ii) is replaced by:

(ii') if $\rho \not\geq d$ then for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x, y \in \overline{B}^d_\rho(x_0, r)$ with $\rho(x,y) < \delta$ we have $d(u,v) < \epsilon$, for each $u \in T(x)$ and $v \in T(y)$.

A homotopy result for Ćirić-type multivalued operators on a set with two metrics is the following theorem.

Theorem 2.4. *Let (X,d) be a complete metric space and ρ another metric on X such that there exists $c > 0$ with $d(x,y) \leq c\rho(x,y)$ for each $x, y \in X$. Let U be an open subset of (X,ρ) and V be a closed subset of (X,d), with $U \subset V$. Let $G : V \times [0,1] \to P(X)$ be a multivalued operator such that the following conditions are satisfied:*

(a) *$x \notin G(x,t)$ for each $x \in V \setminus U$ and each $t \in [0,1]$;*

(b) *there exists $\alpha \in [0,1[$, such that for each $t \in [0,1]$ and each $x, y \in V$ we have:*

$$H_\rho(G(x,t), G(y,t)) \leq \alpha M_\rho^{G(\cdot,t)}(x,y);$$

(c) *there exists a continuous increasing function $\phi : [0,1] \to \mathbb{R}$ such that*

$$H_\rho(G(x,t), G(x,s)) \leq |\phi(t) - \phi(s)| \text{ for all } t, s \in [0,1] \text{ and each } x \in V;$$

(d) *$G : V \times [0,1] \to P((X,d))$ is closed.*

Then $G(\cdot,0)$ has a fixed point if and only if $G(\cdot,1)$ has a fixed point.

Proof. Suppose $G(\cdot,0)$ has a fixed point z. From (a) we have that $z \in U$. Define

$$Q := \{(t,x) \in [0,1] \times U |\ x \in G(x,t)\}.$$

Clearly $Q \neq \emptyset$, since $(0,z) \in Q$. Consider on Q a partial order defined as follows:

$$(t,x) \leq (s,y) \text{ if and only if } t \leq s \text{ and } \rho(x,y) \leq \frac{2}{1-\alpha} \cdot [\phi(s) - \phi(t)].$$

Let M be a totally ordered subset of Q and consider $t^* := \sup\{t | (t,x) \in M\}$. Consider a sequence $(t_n, x_n)_{n \in \mathbb{N}^*} \subset M$ such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ and $t_n \to t^*$, as $n \to +\infty$. Then

$$\rho(x_m, x_n) \leq \frac{2}{1-\alpha} \cdot [\phi(t_m) - \phi(t_n)] \text{ for each } m, n \in \mathbb{N}^*, m > n.$$

When $m, n \to +\infty$ we obtain $\rho(x_m, x_n) \to 0$ and so $(x_n)_{n \in \mathbb{N}^*}$ is ρ-Cauchy. Thus $(x_n)_{n \in \mathbb{N}^*}$ is d-Cauchy too. Denote by $x^* \in (X,d)$ its limit. Since $x_n \in G(x_n, t_n), n \in \mathbb{N}^*$ and G is d-closed we have $x^* \in G(x^*, t^*)$. Also, from (a) we have $x^* \in U$. Hence $(t^*, x^*) \in Q$. Since M is totally ordered we get $(t,x) \leq (t^*, x^*)$ for each $(t,x) \in M$. Thus (t^*, x^*) is an upper bound of M. Hence Zorn's Lemma applies and Q admits a maximal element $(t_0, x_0) \in Q$. We claim that $t_0 = 1$. This will finish the first part of the proof.

Suppose $t_0 < 1$. Choose $r > 0$ and $t \in]t_0, 1]$ such that $B_\rho(x_0, r) \subset U$ and $r := \frac{2}{1-\alpha} \cdot [\phi(t) - \phi(t_0)]$. Then

$$D_\rho(x_0, G(x_0,t)) \leq D_\rho(x_0, G(x_0,t_0)) + H_\rho(G(x_0,t_0), G(x_0,t))$$
$$\leq [\phi(t) - \phi(t_0)] = \frac{(1-\alpha)r}{2} < (1-\alpha)r.$$

Since $\overline{B}^d_\rho(x_0, r) \subset V$, the multivalued operator $G(\cdot,t) : \overline{B}^d_\rho(x_0,r) \to P_{cl}(X)$ satisfies, for all $t \in [0,1]$, the assumptions of Theorem 2.3. Hence, for all $t \in [0,1]$, there exists $x \in \overline{B}^d_\rho(x_0,r)$ such that $x \in G(x,t)$. Thus $(t,x) \in Q$. Since

$$\rho(x_0, x) \leq r = \frac{2}{1-\alpha} \cdot [\phi(t) - \phi(t_0)],$$

we immediately get $(t_0, x_0) < (t, x)$. This is a contradiction with the maximality of (t_0, x_0).

Conversely, if $G(\cdot, 1)$ has a fixed point, then putting $t := 1-t$ and using first part of the proof we get the conclusion. □

A special case of Theorem 2.4 is when $d = \rho$.

Corollary 2.1. *Let (X,d) be a complete metric space, U be an open subset of X and V be a closed subset of X, with $U \subset V$. Let $G : V \times [0,1] \to P(X)$ be a closed multivalued operator such that the following conditions are satisfied:*

(a) *$x \notin G(x,t)$, for each $x \in V \setminus U$ and each $t \in [0,1]$;*

(b) *there exists $\alpha \in [0,1[$, such that for each $t \in [0,1]$ and each $x, y \in V$ we have*

$$H_d(G(x,t), G(y,t)) \leq \alpha M_d^{G(\cdot,t)}(x,y);$$

(c) *there exists a continuous increasing function $\phi : [0,1] \to \mathbb{R}$ such that*

$$H_d(G(x,t), G(x,s)) \leq |\phi(t) - \phi(s)| \text{ for all } t, s \in [0,1] \text{ and each } x \in V.$$

Then $G(\cdot, 0)$ has a fixed point if and only if $G(\cdot, 1)$ has a fixed point.

Remark 2.2. Usually in Corollary 2.1 we take $Q = \overline{U}$. Notice that in this case condition (a) becomes:

(a') $x \notin G(x,t)$, for each $x \in \partial U$ and each $t \in [0,1]$.

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