FIXED POINTS AND HOMOTOPY RESULTS FOR ĆIRIĆ-TYPE MULTIVALUED OPERATORS ON A SET WITH TWO METRICS

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Abstract. The purpose of this paper is to present some fixed point results for nonself multivalued operators on a set with two metrics. In addition, a homotopy result for multivalued operators on a set with two metrics is given. The data dependence and the well-posedness of the fixed point problem are also discussed.

1. Introduction

Throughout this paper, standard notations and terminologies in nonlinear analysis (see [6], [12], [13]) are used. For the convenience of the reader we recall some of them here.

Let \( (X, d) \) be a metric space. In the sequel we will use the following symbols:

\[
P(X) := \{Y \subset X \mid Y \text{ is nonempty}\}, \quad P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\},
\]

\[
B_{d}(x_{0}, r) := \{x \in X \mid d(x_0, x) < r\}.
\]

If \( d' \) is another metric on \( X \), we will denote by \( \overline{B}_{d'}(x_0, r) \) the closure of \( B_{d}(x_0, r) \) in \( (X, d') \).

Let \( A \) be nonempty subset of the metric \( (X, d) \) and \( x_0 \in X \). Then \( D_{d}(x_0, A) = D(\{x_0\}, A) \) is called the distance from the point \( x_0 \) to the set \( A \).

The Pompeiu-Hausdorff generalized distance between the nonempty closed subsets \( A \) and \( B \) of the metric space \( (X, d) \) is defined by the following formula:

\[
H_{d}(A, B) := \max_{a \in A} \inf_{b \in B} d(a, b), \quad \sup_{b \in B} \inf_{a \in A} d(a, b).
\]

The symbol \( T : X \to X \) means \( T : X \to P(X) \), i.e., \( T \) is a multivalued operator from \( X \) to \( X \). We will denote by \( G(T) := \{(x, y) \in X \times X \mid y \in T(x)\} \) the graph of \( T \). The multivalued operator \( T \) is said to be closed if \( G(T) \) is closed in \( X \times X \).

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For \( T : X \to P(X) \) the symbol \( F_T := \{ x \in X \mid x \in T(x) \} \) denotes the fixed point set, while \( (SF)_T := \{ x \in X \mid \{ x \} = T(x) \} \) is the strict fixed point set of the multivalued operator \( T \).

The aim of this paper is to present some fixed point results for nonself multivalued operators on a set with two metrics. In addition, a homotopy result for multivalued operators on a set with two metrics is given. The data dependence and the well-posedness of the fixed point problem are also discussed. Our results complement and extend some previous theorems given by R. P. Agarwal, D. O’Regan [1], R. P. Agarwal, J. H. Dshalalow, D. O’Regan [2], L. Ćirić [3], M. Frigon, A. Granas [5], S. Reich [10], etc.

2. Fixed points and homotopy results for Ćirić-type multivalued operators on a set with two metrics

Let \( (X, d) \) be a metric space and \( T : X \to P_{cl}(X) \) be a multivalued operator. For \( x, y \in X \), let us denote

\[
M_d^T(x, y) := \max\{d(x, y), D_d(x, T(x)), D_d(y, T(y)), \frac{1}{2}[D_d(x, T(y)) + D_d(y, T(x))]\}.
\]

A slight modified variant of Ćirić’s theorem (see [3]) is the following:

**Theorem 2.1.** Suppose that the metric space \( (X, d) \) is complete and the multivalued operator \( T : X \to P_{cl}(X) \) satisfies the following condition:

there exists \( \alpha \in [0, 1] \) such that \( H_d(T(x), T(y)) \leq \alpha \cdot M_d^T(x, y) \) for each \( x, y \in X \).

Then \( F_T \neq \emptyset \) and for each \( x \in X \) and each \( y \in T(x) \) there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) such that

1. \( x_0 = x, \ x_1 = y; \)
2. \( x_{n+1} \in T(x_n), \ \ n \in \mathbb{N}; \)
3. \( x_n \xrightarrow{d} x^* \in T(x^*), \ \ as \ n \to \infty; \)
4. \( d(x_n, x^*) \leq \left( \frac{\alpha p^n}{1-\alpha p} \right) \cdot d(x_0, x_1) \) for each \( n \in \mathbb{N} \) (where \( p \in ]1, \frac{1}{\alpha}[, \ \frac{1}{\alpha} \) is arbitrary).

A data dependence result for Ćirić-type multivalued operators is the following theorem.

**Theorem 2.2.** Let \( (X, d) \) be a complete metric space and \( T_1, T_2 : X \to P_{cl}(X) \) be two multivalued operators. Suppose that

(i) there exists \( \alpha_i \in [0, 1] \) such that

\[
H_d(T_i(x), T_i(y)) \leq \alpha_i \cdot M_d^{T_i}(x, y), \ \ for \ each \ x, y \in X \ for \ i \in \{1, 2\};
\]

(ii) there exists \( \eta > 0 \) such that \( H_d(T_1(x), T_2(x)) \leq \eta \) for each \( x \in X \).

Then

\[
F_{T_1} \neq \emptyset \neq F_{T_2} \text{ and } H_d(F_{T_1}, F_{T_2}) \leq \frac{\eta}{1 - \max\{\alpha_1, \alpha_2\}}.
\]
Proof. From Ćirić’s theorem we have that $F_{T_1} \neq \emptyset \neq F_{T_3}$.

For our second conclusion, denote $\Upsilon := \frac{\eta}{1 - \max\{\alpha_1, \alpha_2\}}$. For our purpose it’s enough to prove that for any $u \in F_{T_1}$ there exists $v \in F_{T_2}$ such that $d(u, v) \leq \Upsilon$ and a similar relation with the roles of $F_{T_1}$ and $F_{T_2}$ reversed.

Let $u \in F_{T_1}$ be arbitrary. From (ii) for every $q > 1$ there exists $x_1 \in T_2(u)$ such that $d(u, x_1) \leq qH(T_1(u), T_2(u)) \leq q\eta$.

Using (4) for $T_2$ and taking $n := 0$, $x_0 := u$ and $x_1$ as above we have, by Theorem 2.1, that there exists $x_2^* \in F_{T_2}$ such that

$$d(u, x_2^*) \leq \frac{1}{1 - (\alpha_2 p)} \cdot d(u, x_1) \leq \frac{1}{1 - (\alpha_2 p)} \cdot q\eta.$$ Letting $p \downarrow 1$ we get that

$$d(u, x_2^*) \leq \frac{1}{1 - \alpha_2} \cdot q\eta.$$ 

By interchanging the roles of $T_1$ and $T_2$, for each $v \in F_{T_2}$, each $q' > 1$ and each $x'_1 \in T_1(v)$ such that $d(v, x_1) \leq q'H(T_2(v), T_1(v)) \leq q'\eta$ we have that

$$d(v, x'_1) \leq \frac{1}{1 - \alpha_1} \cdot q'\eta,$$ where $x'_1$ is the fixed point of $T_1$ given by Theorem 2.1. Thus

$$H_d(F_{T_1}, F_{T_2}) \leq \frac{\eta}{1 - \max\{\alpha_1, \alpha_2\}} \cdot \max\{q, q'\}.$$ The conclusion follows now by letting $q, q' \downarrow 1$. 

We continue the section with a local version of Ćirić’s theorem on a set with two metrics.

**Theorem 2.3.** Let $X$ be a nonempty set, $x_0 \in X$ and $r > 0$. Suppose that $d$, $\rho$ are two metrics on $X$ and $T : \overline{B}_{\rho}d(x_0, r) \to P(X)$ is a multivalued operator. We suppose that

(i) $(X, d)$ is a complete metric space;
(ii) there exists $c > 0$ such that $d(x, y) \leq c\rho(x, y)$ for each $x, y \in X$;
(iii) if $d \neq \rho$ then $T : \overline{B}_{\rho}d(x_0, r) \to P(X^d)$ is closed, while if $d = \rho$ then $T : \overline{B}_{\rho}d(x_0, r) \to Pcl(X^d)$;
(iv) there exists $\alpha \in [0, 1[$ such that $H_\rho(T(x), T(y)) \leq \alpha M_\rho(x, y)$ for each $x, y \in \overline{B}_{\rho}d(x_0, r)$;
(v) $D_\rho(x_0, T(x_0)) < (1 - \alpha)r$.

Then

(A) there exists $x^* \in \overline{B}_{\rho}d(x_0, r)$ such that $x^* \in T(x^*)$;
(B) if $(SF)_T \neq \emptyset$ and $(x_n)_{n \in \mathbb{N}} \subseteq \overline{B}_{\rho}d(x_0, r)$ is such that $H_\rho(x_n, T(x_n)) \to 0$ as $n \to +\infty$, then $x_n \stackrel{\rho}{\to} x \in (SF)_T$ as $n \to +\infty$ (i.e., the fixed point problem is well-posed in the generalized sense for $T$ with respect to $H_\rho$, see [7], [9]).
Proof. (A) From (v) there exists \( x_1 \in T(x_0) \) such that \( \rho(x_0, x_1) < (1 - \alpha)r \). Clearly \( x_1 \in \overline{B}_\rho^d(x_0, r) \). We have

\[
H_\rho(T(x_0), T(x_1)) \\
\leq \alpha \max\{\rho(x_0, x_1), D_\rho(x_0, T(x_0)), D_\rho(x_1, T(x_1)), \frac{1}{2}[D_\rho(x_0, T(x_1)) + D_\rho(x_1, T(x_0))])
\]

\[
\leq \alpha \max\{\rho(x_0, x_1), D_\rho(x_1, T(x_1)), \frac{1}{2}[\rho(x_0, x_1) + D_\rho(x_1, T(x_1))])
\]

\[
\leq \alpha \max\{\rho(x_0, x_1), D_\rho(x_1, T(x_1))\}.
\]

We claim that \( \max\{\rho(x_0, x_1), D_\rho(x_1, T(x_1))\} = \rho(x_0, x_1) \). If

\[\max\{\rho(x_0, x_1), D_\rho(x_1, T(x_1))\} = D_\rho(x_1, T(x_1)),\]

then we get the following contradiction \( H_\rho(T(x_0), T(x_1)) \leq \alpha D_\rho(x_1, T(x_1)) \leq \alpha H_\rho(T(x_0), T(x_1)) \). Thus

\[H_\rho(T(x_0), T(x_1)) \leq \alpha \rho(x_0, x_1) \].

Hence \( H_\rho(T(x_0), T(x_1)) < \alpha(1 - \alpha)r \). Thus, there exists \( x_2 \in T(x_1) \) such that \( \rho(x_1, x_2) < \alpha(1 - \alpha)r \). Moreover, \( \rho(x_0, x_2) \leq \rho(x_0, x_1) + \rho(x_1, x_2) < (1 - \alpha)r + \alpha(1 - \alpha)r = (1 - \alpha^2)r < r \). Hence, \( x_2 \in \overline{B}_\rho^d(x_0, r) \). Using this procedure, we obtain the sequence \( (x_n)_{n \in \mathbb{N}} \subset \overline{B}_\rho^d(x_0, r) \) having the following properties:

- (a) \( x_{n+1} \in T(x_n), n \in \mathbb{N} \);
- (b) \( \rho(x_{n-1}, x_n) \leq \alpha^{n-1}(1 - \alpha)r, n \in \mathbb{N}^* \);
- (c) \( \rho(x_0, x_n) \leq (1 - \alpha^n)r, n \in \mathbb{N}^* \).

From (b) we get that the sequence \( (x_n)_{n \in \mathbb{N}} \) is Cauchy in \((X, \rho)\). From (ii) the sequence \( (x_n)_{n \in \mathbb{N}} \) is Cauchy in \((X, d)\) too. Taking into account (i) it follows that there exists \( x^* \in \overline{B}_\rho^d(x_0, r) \) such that \( x_n \xrightarrow{d} x^* \). If \( d \neq \rho \), since \( T : \overline{B}_\rho^d(x_0, r) \rightarrow P_\rho^d(X^d) \) is closed, we immediately get that \( x^* \in T(x^*) \), as \( n \to \infty \). If \( d = \rho \) the conclusion follows as in the proof of Ćirić's theorem (see [3], Theorem 2 as well as [2]).

(B) Let \( x \in (SF)_T \). Thus we have:

\[\rho(x_n, x) \]

\[\leq D_\rho(x_n, T(x_n)) + H_\rho(T(x_n), T(x)) \leq D_\rho(x_n, T(x_n)) + \alpha M_\rho^T(x_n, x)\]

\[\leq D_\rho(x_n, T(x_n)) + \alpha \max\{ho(x_n, x), D_\rho(x_n, T(x_n)), \frac{1}{2}[D_\rho(x_n, T(x)) + D_\rho(x, T(x_n))])\}
\]

\[\leq D_\rho(x_n, T(x_n)) + \alpha \max\{\rho(x_n, x), D_\rho(x_n, T(x_n)), \rho(x_n, x) + \frac{1}{2}D_\rho(x_n, T(x_n))\}
\]

\[\leq D_\rho(x_n, T(x_n)) + \alpha \max\{D_\rho(x_n, T(x_n)), \rho(x_n, x) + \frac{1}{2}D_\rho(x_n, T(x_n))\}.
\]

Hence, we get that

\[\rho(x_n, x) \leq \max\{1 + \alpha, \frac{\alpha}{2(1 - \alpha)}\} D_\rho(x_n, T(x_n)) \lor 0 \text{ as } n \to \infty.\]
The proof is complete. \(\square\)

\textbf{Remark 2.1.} Theorem 2.3 holds if the condition (ii) is replaced by:

(ii') if \(\rho \not\geq d\) then for each \(\epsilon > 0\) there exists \(\delta > 0\) such that for each \(x, y \in \overline{B}_d(x_0, r)\) with \(\rho(x, y) < \delta\) we have \(d(u, v) < \epsilon\), for each \(u \in T(x)\) and \(v \in T(y)\).

A homotopy result for Ćirić-type multivalued operators on a set with two metrics is the following theorem.

\textbf{Theorem 2.4.} Let \((X, d)\) be a complete metric space and \(\rho\) another metric on \(X\) such that there exists \(c > 0\) with \(d(x, y) \leq c\rho(x, y)\) for each \(x, y \in X\). Let \(U\) be an open subset of \((X, \rho)\) and \(V\) be a closed subset of \((X, d)\), with \(U \subset V\). Let \(G : V \times [0, 1] \to P(X)\) be a multivalued operator such that the following conditions are satisfied:

(a) \(x \notin G(x, t)\) for each \(x \in V \setminus U\) and each \(t \in [0, 1]\);

(b) there exists \(\alpha \in [0, 1]\), such that for each \(t \in [0, 1]\) and each \(x, y \in V\) we have:

\[H_\rho(G(x, t), G(y, t)) \leq \alpha M_\rho^{G_\cdot, t}(x, y);\]

(c) there exists a continuous increasing function \(\phi : [0, 1] \to \mathbb{R}\) such that

\[H_\rho(G(x, t), G(x, s)) \leq |\phi(t) - \phi(s)|\] for all \(t, s \in [0, 1]\) and each \(x \in V\);

(d) \(G : V \times [0, 1] \to P((X, d))\) is closed.

Then \(G(\cdot, 0)\) has a fixed point if and only if \(G(\cdot, 1)\) has a fixed point.

\textbf{Proof.} Suppose \(G(\cdot, 0)\) has a fixed point \(z\). From (a) we have that \(z \in U\). Define

\[Q := \{(t, x) \in [0, 1] \times U \mid x \in G(x, t)\}.\]

Clearly \(Q \neq \emptyset\), since \((0, z) \in Q\). Consider on \(Q\) a partial order defined as follows:

\[(t, x) \leq (s, y)\] if and only if \(t \leq s\) and \(\rho(x, y) \leq \frac{2}{1 - \alpha} \cdot |\phi(s) - \phi(t)|\).

Let \(M\) be a totally ordered subset of \(Q\) and consider \(t^* := \sup\{t \mid (t, x) \in M\}\). Consider a sequence \((t_n, x_n)_{n \in \mathbb{N}^*} \subset M\) such that \((t_n, x_n) \leq (t_{n+1}, x_{n+1})\) and \(t_n \to t^*, \) as \(n \to +\infty.\) Then

\[\rho(x_m, x_n) \leq \frac{2}{1 - \alpha} \cdot |\phi(t_m) - \phi(t_n)|\] for each \(m, n \in \mathbb{N}^*, m > n.\)

When \(m, n \to +\infty\) we obtain \(\rho(x_m, x_n) \to 0\) and so \((x_n)_{n \in \mathbb{N}^*}\) is \(\rho\)-Cauchy. Thus \((x_n)_{n \in \mathbb{N}^*}\) is \(d\)-Cauchy too. Denote by \(x^* \in (X, d)\) its limit. Since \(x_n \in G(x_n, t_n), n \in \mathbb{N}^*\) and \(G\) is \(d\)-closed we have \(x^* \in G(x^*, t^*).\) Also, from (a) we have \(x^* \in U.\) Hence \((t^*, x^*) \in Q\). Since \(M\) is totally ordered we get \((t_n, x_n) \leq (t^*, x^*)\) for each \((t, x) \in M\). Thus \((t^*, x^*)\) is an upper bound of \(M\). Hence Zorn’s Lemma applies and \(Q\) admits a maximal element \((t_0, x_0) \in Q.\) We claim that \(t_0 = 1.\) This will finish the first part of the proof.
Suppose $t_0 < 1$. Choose $r > 0$ and $t \in [t_0, 1]$ such that $B_\rho(x_0, r) \subset U$ and $r := \frac{2}{1-\alpha} \cdot [\phi(t) - \phi(t_0)]$. Then

$$D_\rho(x_0, G(x_0, t)) \leq D_\rho(x_0, G(x_0, t_0)) + H_\rho(G(x_0, t_0), G(x_0, t))$$

$$\leq [\phi(t) - \phi(t_0)] = \frac{(1-\alpha)r}{2} < (1-\alpha)r.$$

Since $\overline{B}_\rho(x_0, r) \subset V$, the multivalued operator $G(\cdot, t) : \overline{B}_\rho(x_0, r) \to P_{cl}(X)$ satisfies, for all $t \in [0, 1]$, the assumptions of Theorem 2.3. Hence, for all $t \in [0, 1]$, there exists $x \in \overline{B}_\rho(x_0, r)$ such that $x \in G(x, t)$. Thus $(t, x) \in Q$. Since

$$\rho(x_0, x) \leq r = \frac{2}{1-\alpha} \cdot [\phi(t) - \phi(t_0)],$$

we immediately get $(t_0, x_0) < (t, x)$. This is a contradiction with the maximality of $(t_0, x_0)$.

Conversely, if $G(\cdot, 1)$ has a fixed point, then putting $t := 1 - t$ and using first part of the proof we get the conclusion. □

A special case of Theorem 2.4 is when $d = \rho$.

**Corollary 2.1.** Let $(X, d)$ be a complete metric space, $U$ be an open subset of $X$ and $V$ be a closed subset of $X$, with $U \subset V$. Let $G : V \times [0, 1] \to P(X)$ be a closed multivalued operator such that the following conditions are satisfied:

(a) $x \notin G(x, t)$, for each $x \in V \setminus U$ and each $t \in [0, 1]$;

(b) there exists $\alpha \in [0, 1]$, such that for each $t \in [0, 1]$ and each $x, y \in V$ we have

$$H_d(G(x, t), G(y, t)) \leq \alpha M^G(\cdot, t)(x, y);$$

(c) there exists a continuous increasing function $\phi : [0, 1] \to \mathbb{R}$ such that

$$H_d(G(x, t), G(x, s)) \leq |\phi(t) - \phi(s)|$$

for all $t, s \in [0, 1]$ and each $x \in V$.

Then $G(\cdot, 0)$ has a fixed point if and only if $G(\cdot, 1)$ has a fixed point.

**Remark 2.2.** Usually in Corollary 2.1 we take $Q = \overline{U}$. Notice that in this case condition (a) becomes:

(a') $x \notin G(x, t)$, for each $x \in \partial U$ and each $t \in [0, 1]$.

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