

SECOND ORDER REGULAR VARIATION AND ITS APPLICATIONS TO RATES OF CONVERGENCE IN EXTREME-VALUE DISTRIBUTION

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ABSTRACT. The rate of convergence of the distribution of order statistics to the corresponding extreme-value distribution may be characterized by the uniform and total variation metrics. de Haan and Resnick [4] derived the convergence rate when the second order generalized regularly varying function has second order derivatives. In this paper, based on the properties of the generalized regular variation and the second order generalized variation and characterized by uniform and total variation metrics, the convergence rates of the distribution of the largest order statistic are obtained under weaker conditions.

1. Introduction

Let $\{X_i, i = 1, 2, \dots\}$ be independent random variables with common distribution function F . Denote the largest extreme value of X_1, X_2, \dots, X_n by $M_n = \max\{X_1, \dots, X_n\}, n = 1, 2, \dots$. According to Gnedenko [6], if there exist normalizing constants $a_n > 0, b_n \in R$ such that

$$a_n^{-1}(M_n - b_n) \xrightarrow{d} G$$

as $n \rightarrow \infty$, where G is non-degenerate, then G must be of the type of

$$G_\gamma(x) = \exp\{-(1 + \gamma x)^{-1/\gamma}\}, \gamma \in R, 1 + \gamma x \geq 0.$$

Denote $V = (-\log^{-1} F)^\leftarrow$, where $U^\leftarrow : I \rightarrow R$ is the left continuous inverse defined by $U^\leftarrow(x) = \sup\{s : U(s) \leq x\}, x \in I$. Then

$$(1.1) \quad a_n^{-1}(M_n - b_n) \xrightarrow{d} G_\gamma$$

holds for a given $\gamma \in R$ if and only if there exists a function $a(t)$ on R^+ such that

$$(1.2) \quad \lim_{t \rightarrow \infty} a^{-1}(t)[V(tx) - V(t)] = \frac{x^\gamma - 1}{\gamma}.$$

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The rate of convergence of (1.1) is an important part in extreme value theory. There are so many papers devoted to this subject. The purpose of this paper is not to review the literature. We refer the readers to Nadarajah [7] for a comprehensive review of the papers on rates of convergence. However, in relation to this paper, we like to mention the following. Under the second order von Mises condition, de Haan and Resnick [4] derived Edgeworth expansions of $P\{(M_n - b_n)/a_n \leq x\}$ and used related results to obtain the rates of convergence of (1.1) in the uniform metric and the total variation metric when V has second order derivatives. Under weaker conditions, Cheng and Jiang [2] obtained Edgeworth expansions and uniform convergence rates of (1.1).

In this paper, we consider the convergence rates of (1.1) characterized by the uniform and the total variation metrics under some weak conditions. Our main tools are inequalities based on regular variation functions and generalized regular variation of second order. In Section 2, some properties of generalized regularly varying functions and second order regularly varying functions are provided. Some lemmas related to the convergence rates of (1.1) are given in Section 3. In Section 4, the convergence rates of (1.1), characterized by the uniform and the total variation metrics, are established under some weak conditions.

2. Regular variation and its properties

In de Haan [3], a measurable function f on R^+ is said to be regularly varying with index $\gamma \in R$ at infinity, denoted by $f \in Rv(\gamma)$, if f is positive near infinity and

$$\lim_{t \rightarrow \infty} \frac{f(tx)}{f(t)} = x^\gamma, \forall x \in R^+.$$

Following is an extension of Potter bound of regularly varying functions.

Lemma 2.1. *If $f \in Rv(\gamma, a)$, for any $\varepsilon, \delta > 0$, there must exist $t_0 = t_0(\varepsilon, \delta) > 0$ such that for $t > t_0$*

$$(1 - \varepsilon) \min\{x^{\gamma+\delta}, x^{\gamma-\delta}\} < \frac{f(tx)}{f(t)} < (1 + \varepsilon) \max\{x^{\gamma+\delta}, x^{\gamma-\delta}\}.$$

Proof. See de Haan [3]. □

A measurable function f on R^+ is a generalized regular variation function with index γ , denoted by $f \in GR(\gamma, a)$, if there exists a positive auxiliary function $a(t)$ such that

$$\lim_{t \rightarrow \infty} \frac{f(tx) - f(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma}, \forall x \in R^+.$$

Now denote

$$a^*(t) = \begin{cases} \gamma f(t), & \gamma > 0, \\ -\gamma[f(\infty) - f(t)], & \gamma < 0, \\ \hat{f}(t), & \gamma = 0, \end{cases}$$

where $\widehat{f}(t) := f(t) - \frac{1}{t} \int_0^t f(u)du$.

Lemma 2.2. *If $f \in GR(\gamma, a)$, there exists $a^*(t) \sim a(t)$ as $t \rightarrow \infty$. For any $\varepsilon, \delta > 0$, there exists $t_0 = t_0(\varepsilon, \delta) > 0$ such that*

$$\left| \frac{f(tx) - f(t)}{a^*(t)} - \frac{x^\gamma - 1}{\gamma} \right| \leq \varepsilon \max\{x^{\gamma+\delta}, x^{\gamma-\delta}\}, \forall t, tx \geq t_0.$$

Proof. See Cheng and Jiang [2]. □

According to de Haan and Stadtmuller [5], a measurable function f on R^+ is a generalized regular variation function of second order, denoted by $f \in GR_2(\gamma, \rho; a, A)$ if there exist positive function $a(t)$, and function $A(t)$ with constant sign near infinity, satisfying $\lim_{t \rightarrow \infty} A(t) = 0$ and $|A| \in Rv(\rho)$ with $\rho \leq 0$, and function $K(x)$ which is not a multiple of $(x^\gamma - 1)/\gamma$, such that

$$\lim_{t \rightarrow \infty} \frac{1}{A(t)} \left\{ \frac{f(tx) - f(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma} \right\} = K(x), \forall x \in R^+.$$

By a perfect choice of $a(t)$ and $A(t)$, $K(x)$ may take the form of

$$K_{\gamma, \rho}(x) := \begin{cases} \frac{\log^2 x}{2}, & \gamma = 0 = \rho, \\ \frac{x^\gamma \log x}{\gamma} - \frac{1}{\gamma} \cdot \frac{x^\gamma - 1}{\gamma}, & \rho = 0 \neq \gamma, \\ \frac{x^{\gamma+\rho}-1}{\gamma+\rho}, & \rho < 0. \end{cases}$$

In this paper, we suppose that the measurable function f on R^+ satisfies the following properties, i.e., there exist function $A(t)$ with constant sign near infinity, $\lim_{t \rightarrow \infty} A(t) = 0$, and $|A| \in Rv(\rho)$ with $\rho \leq 0$, and function $a(t) > 0$, such that

$$(2.1) \quad \frac{\frac{f'(tx)}{t^{-1}a(t)} - x^{\gamma-1}}{A(t)} \rightarrow K'_{\gamma, \rho}(x),$$

where

$$K'_{\gamma, \rho}(x) = \begin{cases} x^{\gamma-1} \log x, & \rho = 0, \\ x^{\gamma+\rho-1}, & \rho < 0. \end{cases}$$

Note that the condition (2.1) is weaker than the second order von Mises condition in de Haan and Resnick [4] since the latter assumes that the regularly varying function has second order derivatives, see page 104 of de Haan and Resnick [4].

Lemma 2.3. *Let f be a measurable function on R^+ . If there exist functions $a(t) \in Rv(\gamma)$, and $A(t) \in Rv(\rho)$, $\rho < 0$, such that*

$$(2.2) \quad \lim_{t \rightarrow \infty} \frac{\frac{f(tx)}{t^{-1}a(t)} - x^{\gamma-1}}{A(t)} = x^{\gamma+\rho-1}$$

we have

- (1) $c_0 = \lim_{t \rightarrow \infty} [\log f(t) - (\gamma - 1) \log t] < \infty$,
- (2) $c_0 - [\log f(t) - (\gamma - 1) \log t] \in Rv(\rho)$,

$$(3) \lim_{t \rightarrow \infty} t^{-\gamma} a(t) = e^{c_0}.$$

Proof. Notice for $\rho < 0$, $A(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence for $x > 0$, we have

$$\begin{aligned} & \frac{\log \frac{f(tx)}{x^{\gamma-1} t^{-1} a(t)}}{A(t)} \rightarrow x^\rho \\ \Rightarrow & \frac{\log f(tx) - (\gamma - 1) \log x + \log t - \log a(t)}{A(t)} \rightarrow x^\rho \\ \Rightarrow & \frac{\log f(tx) - \log f(t) - (\gamma - 1) \log x}{\rho A(t)} \rightarrow \frac{x^\rho - 1}{\rho} \\ \Rightarrow & \frac{[\log f(tx) - (\gamma - 1) \log tx] - [\log f(t) - (\gamma - 1) \log t]}{\rho A(t)} \rightarrow \frac{x^\rho - 1}{\rho}. \end{aligned}$$

By the well-known property of generalized regularly varying functions, we have $c_0 = \lim_{t \rightarrow \infty} [\log f(t) - (\gamma - 1) \log t]$ and $c_0 - [\log f(t) - (\gamma - 1) \log t] \in Rv(\rho)$. Now (1), (2) and (3) are obvious. \square

Lemma 2.4. *If a measurable function f on R^+ satisfies (2.1), we have*

$$\begin{aligned} \pm t^{1-\gamma} f'(t) & \in GR(0, t^{1-\gamma} f'(t) | A|(t)), \quad \text{if } \rho = 0; \\ f'(t) - ct^{\gamma-1} & \in Rv(\gamma + \rho - 1), \quad \text{if } \rho < 0, \end{aligned}$$

where $c = \lim_{t \rightarrow \infty} t^{1-\gamma} f'(t)$ for $\rho < 0$.

Proof. If $\rho = 0$, we have

$$\begin{aligned} & \frac{\frac{f'(tx)}{t^{-1} a(t)} - x^{\gamma-1}}{A(t)} \rightarrow x^{\gamma-1} \log x \\ \Rightarrow & \frac{f'(tx) - t^{-1} a(t) x^{\gamma-1}}{t^{-1} a(t) A(t)} \rightarrow x^{\gamma-1} \log x \\ \Rightarrow & \frac{x^{1-\gamma} f'(tx) - t^{-1} a(t)}{t^{-1} a(t) A(t)} \rightarrow \log x \\ \Rightarrow & \frac{x^{1-\gamma} f'(tx) - f'(t)}{t^{-1} a(t) A(t)} \rightarrow \log x \\ \Rightarrow & \frac{(tx)^{1-\gamma} f'(tx) - t^{1-\gamma} f'(t)}{t^{1-\gamma} f'(t) A(t)} \rightarrow \log x, \end{aligned}$$

which means $\pm t^{1-\gamma} f'(t) \in GR(0, t^{1-\gamma} f'(t) | A|(t))$. If $\rho < 0$, according to (2.1), we have $A(t) \sim \frac{f'(t)}{t^{-1} a(t)} - 1 \in Rv(\rho)$ and

$$\frac{\frac{f'(tx)}{t^{-1} a(t)} - x^{\gamma-1}}{\frac{f'(t)}{t^{-1} a(t)} - 1} \rightarrow x^{\gamma+\rho-1},$$

i.e.,

$$\frac{(tx)^{1-\gamma} f'(tx) - t^{-\gamma} a(t)}{t^{1-\gamma} f'(t) - t^{-\gamma} a(t)} \rightarrow x^\rho,$$

which implies

$$\frac{(tx)^{1-\gamma}f'(tx) - t^{1-\gamma}f'(t)}{\rho[t^{1-\gamma}f'(t) - t^{-\gamma}a(t)]} \rightarrow \frac{x^\rho - 1}{\rho}.$$

Hence, $t^{1-\gamma}f'(t) \in GR(\rho, \rho[t^{1-\gamma}f'(t) - t^{-\gamma}a(t)])$. By the properties of generalized regularly varying functions, we have

$$c := \lim t^{1-\gamma}f'(t) < \infty$$

and

$$c - t^{1-\gamma}f'(t) \in Rv(\rho)$$

if $\rho < 0$. The result follows. \square

Remark 2.1. If $\rho < 0$ in Lemma 2.4, we have

$$\lim_{t \rightarrow \infty} t^{1-\gamma}f'(t) = \lim_{t \rightarrow \infty} t^{-\gamma}a(t) = c.$$

Lemma 2.5. *If there exist function $A(t)$ with constant sign near infinity, satisfying $\lim_{t \rightarrow \infty} A(t) = 0$ and $|A| \in Rv(\rho)$ with $\rho \leq 0$, and function $a(t) > 0$ such that measurable f on R^+ satisfies (2.1) then $a^*(t) \sim a(t)$ and $A^*(t) \sim A(t)$, where*

$$a^*(t) = \begin{cases} tf'(t), & \rho = 0, \\ ct^\gamma, & \rho < 0 \end{cases}$$

and

$$A^*(t) = \begin{cases} \widehat{(f'(t)t^{1-\gamma})^{-1}f'(t)t^{1-\gamma}}, & \rho = 0, \\ \frac{f'(t)-ct^{\gamma-1}}{ct^{\gamma-1}}, & \rho < 0, \end{cases}$$

where $\widehat{g}(t) := g(t) - \frac{1}{t} \int_0^t g(u)du$. Furthermore, for any $\varepsilon, \delta > 0$, there exists $t_0 = t_0(\varepsilon, \delta) > 0$ such that for $t > t_0$, we have

$$(2.3) \quad \left| \frac{\frac{f'(tx)}{t^{-1}a^*(t)} - x^{\gamma-1}}{A^*(t)} - K'_{\gamma, \rho}(x) \right| \leq \varepsilon \max\{x^{\gamma+\rho-1+\delta}, x^{\gamma+\rho-1-\delta}\}.$$

For $\gamma > 0$, we have

$$(2.4) \quad \left| \frac{\frac{f(tx)-f(t)}{a^*(t)} - \frac{x^{\gamma-1}}{\gamma}}{A^*(t)} - K_{\gamma, \rho}(x) \right| \leq \varepsilon \max\{x^{\gamma+\rho+\delta}, x^{\gamma+\rho-\delta}\}.$$

For $\gamma < 0$, we have

$$(2.5) \quad \left| \frac{\frac{f(tx)-f(\infty)-\gamma^{-1}a^*(t)}{a^*(t)} - \frac{x^{\gamma-1}}{\gamma}}{A^*(t)} - K_{\gamma, \rho}(x) \right| \leq \varepsilon \max\{x^{\gamma+\rho+\delta}, x^{\gamma+\rho-\delta}\},$$

where $f(\infty) := \lim_{t \rightarrow \infty} f(t)$.

Proof. For $\rho = 0$, we have

$$\begin{aligned}
& \left| \frac{\frac{f'(tx)}{t^{-1}a^*(t)} - x^{\gamma-1}}{A^*(t)} - K'_{\gamma,\rho}(x) \right| \\
&= \left| \frac{\frac{f'(tx)}{f'(t)} - x^{\gamma-1}}{A^*(t)} - x^{\gamma-1} \log x \right| \\
&= x^{\gamma-1} \left| \frac{\frac{x^{1-\gamma}f'(tx)}{f'(t)} - 1}{A^*(t)} - \log x \right| \\
&= x^{\gamma-1} \left| \frac{(tx)^{1-\gamma}f'(tx) - t^{1-\gamma}f'(t)}{A^*(t)t^{1-\gamma}f'(t)} - \log x \right| \\
&= x^{\gamma-1} \left| \frac{(tx)^{1-\gamma}f'(tx) - t^{1-\gamma}f'(t)}{t^{1-\gamma}f'(t)\widehat{t^{1-\gamma}f'(t)}[t^{1-\gamma}f'(t)]^{-1}} - \log x \right| \\
&= x^{\gamma-1} \left| \frac{(tx)^{1-\gamma}f'(tx) - t^{1-\gamma}f'(t)}{t^{1-\gamma}\widehat{f'(t)}} - \log x \right| \\
&\leq x^{\gamma-1}\varepsilon_1 \max\{x^{-\delta}, x^\delta\} \\
&= \varepsilon \max\{x^{\gamma-1+\delta}, x^{\gamma-1-\delta}\}.
\end{aligned}$$

For $\rho < 0$, we have

$$\begin{aligned}
& \left| \frac{\frac{f'(tx)}{t^{-1}a^*(t)} - x^{\gamma-1}}{A^*(t)} - K'_{\gamma,\rho}(x) \right| \\
&= \left| \frac{\frac{f'(tx)}{t^{-1}a^*(t)} - x^{\gamma-1}}{A^*(t)} - x^{\gamma+\rho-1} \right| \\
&= \left| \frac{f'(tx) - t^{-1}a^*(t)x^{\gamma-1}}{t^{-1}a^*(t)A^*(t)} - x^{\gamma+\rho-1} \right| \\
&= \left| \frac{f'(tx) - c(tx)^{\gamma-1}}{t^{-1}a^*(t)(a^*(t))^{-1}(tf'(t) - ct^\gamma)} - x^{\gamma+\rho-1} \right| \\
&= \left| \frac{f'(tx) - c(tx)^{\gamma-1}}{f'(t) - ct^{\gamma-1}} - x^{\gamma+\rho-1} \right| \\
&\leq \varepsilon \max\{x^{\gamma+\rho-1+\delta}, x^{\gamma+\rho-1-\delta}\},
\end{aligned}$$

which completes the proof of (2.3). Now consider (2.4) and (2.5). For $\gamma > 0$, we have

$$\left| \frac{\frac{f(tx)-f(t)}{a^*(t)} - \frac{x^{\gamma-1}}{\gamma}}{A^*(t)} - K_{\gamma,\rho}(x) \right|$$

$$\begin{aligned}
&= \left| \frac{\int_1^x \frac{f'(tx_1)}{t^{-1}a^*(t)} dx_1 - \int_1^x x_1^{\gamma-1} dx_1}{A^*(t)} - \int_1^x K'_{\gamma,\rho}(x_1) dx_1 \right| \\
&= \left| \frac{\int_1^x \frac{f'(tx_1)}{t^{-1}a^*(t)} - x_1^{\gamma-1}}{A^*(t)} - \int_1^x K'_{\gamma,\rho}(x_1) dx_1 \right| \\
&= \left| \int_1^x \frac{\frac{f'(tx_1)}{t^{-1}a^*} - x_1^{\gamma-1}}{A^*(t)} - K'_{\gamma,\rho}(x_1) dx_1 \right| \\
&\leq \int_1^x \left| \frac{\frac{f'(tx_1)}{t^{-1}a^*} - x_1^{\gamma-1}}{A^*(t)} - K'_{\gamma,\rho}(x_1) \right| dx_1 \\
&\leq \varepsilon_1 \int_1^x \max \left\{ x_1^{\gamma+\rho-1+\delta}, x_1^{\gamma+\rho-1-\delta} \right\} dx_1 \\
&\leq \varepsilon_1 \max \left\{ \int_1^x x_1^{\gamma+\rho-1+\delta} dx_1, \int_1^x x_1^{\gamma+\rho-1-\delta} dx_1 \right\} \\
&= \varepsilon_1 \max \left\{ \frac{1}{\gamma+\rho+\delta} (x^{\gamma+\rho+\delta} - 1), \frac{1}{\gamma+\rho-\delta} (x^{\gamma+\rho-\delta} - 1) \right\} \\
&\leq \varepsilon \max \{ x^{\gamma+\rho+\delta}, x^{\gamma+\rho-\delta} \}.
\end{aligned}$$

For $\gamma < 0$, we have

$$\begin{aligned}
&\left| \frac{\frac{f(tx)-f(\infty)-\gamma^{-1}a^*(t)}{a^*(t)} - \frac{x^{\gamma}-1}{\gamma}}{A^*(t)} - K_{\gamma,\rho}(x) \right| \\
&= \left| \frac{\frac{f(tx)-f(\infty)}{a^*(t)} - \frac{x^{\gamma}}{\gamma}}{A^*(t)} - K_{\gamma,\rho}(x) \right| \\
&= \left| -\left(\frac{\int_x^\infty \frac{f'(tx_1)}{t^{-1}a^*(t)} dx_1 - \int_x^\infty x_1^{\gamma-1} dx_1}{A^*(t)} - \int_x^\infty K'_{\gamma,\rho}(x_1) dx_1 \right) \right| \\
&= \left| \frac{\int_x^\infty \frac{f'(tx_1)}{t^{-1}a^*(t)} - x_1^{\gamma-1}}{A^*(t)} - \int_x^\infty K'_{\gamma,\rho}(x_1) dx_1 \right| \\
&= \left| \int_x^\infty \frac{\frac{f'(tx_1)}{t^{-1}a^*(t)} - x_1^{\gamma-1}}{A^*(t)} - K'_{\gamma,\rho}(x_1) dx_1 \right| \\
&\leq \int_x^\infty \left| \frac{\frac{f'(tx_1)}{t^{-1}a^*(t)} - x_1^{\gamma-1}}{A^*(t)} - K'_{\gamma,\rho}(x_1) \right| dx_1 \\
&\leq \varepsilon_1 \int_x^\infty \max \{ x_1^{\gamma+\rho-1+\delta}, x_1^{\gamma+\rho-1-\delta} \} dx_1
\end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon_1 \max \left\{ \int_x^\infty x_1^{\gamma+\rho-1+\delta} dx_1, \int_x^\infty x_1^{\gamma+\rho-1-\delta} dx_1 \right\} \\
&= \varepsilon_1 \max \left\{ \frac{1}{\gamma+\rho+\delta} (-x^{\gamma+\rho+\delta}), \frac{1}{\gamma+\rho-\delta} (-x^{\gamma+\rho-\delta}) \right\} \\
&\leq \varepsilon \max \{ x^{\gamma+\rho+\delta}, x^{\gamma+\rho-\delta} \}.
\end{aligned}$$

□

3. Lemmas related to rates of convergence

Let $a_n = a^*(n)$ and $A_n = A^*(n)$ in which a^* and A^* are as defined before, and $f := V$. Denote

$$\begin{aligned}
b_n &= \begin{cases} V(n), & \gamma \geq 0, \\ V(\infty) + \gamma^{-1}a_n - (\gamma + \rho)^{-1}a_nA_n, & \gamma < 0, \rho < 0, \\ V(\infty) + \gamma^{-1}a_n + \gamma^{-2}a_nA_n, & \gamma < 0, \rho = 0, \end{cases} \\
p_{n,\gamma}(x) &= \frac{V(nx) - b_n}{a_n} - \frac{x^\gamma - 1}{\gamma} \quad \forall x \in R^+, \\
p'_{n,\gamma}(x) &= \frac{nV'(nx)}{a_n} - x^{\gamma-1} \quad \forall x \in R^+,
\end{aligned}$$

and

$$H_{\gamma,\rho}(x) = \begin{cases} K_{\gamma,\rho}(x) + \frac{1}{\gamma+\rho}, & \gamma < 0, \rho < 0, \\ K_{\gamma,\rho}(x) - \frac{1}{\gamma^2}, & \gamma < 0, \rho = 0, \\ K_{\gamma,\rho}(x), & \text{otherwise.} \end{cases}$$

Lemma 3.1. *If function V satisfies the conditions in (2.1) then*

$$(3.1) \quad \left| \frac{p_{n,\gamma}(x)}{A_n} - H_{\gamma,\rho}(x) \right| \leq \varepsilon \max \{ x^{\gamma+\rho+\delta}, x^{\gamma+\rho-\delta} \} \quad \forall n, nx > n_0$$

and

$$(3.2) \quad \left| \frac{p'_{n,\gamma}(x)}{A_n} - H'_{\gamma,\rho}(x) \right| \leq \varepsilon \max \{ x^{\gamma+\rho-1+\delta}, x^{\gamma+\rho-1-\delta} \} \quad \forall n, nx > n_0.$$

Proof. For $\gamma < 0, \rho < 0$, write

$$\begin{aligned}
&\left| \frac{p_{n,\gamma}(x)}{A_n} - H_{\gamma,\rho}(x) \right| \\
&= \left| \left(\frac{V(nx) - V(\infty) - \gamma^{-1}a_n + (\gamma + \rho)^{-1}a_nA_n}{a_n} - \frac{x^\gamma - 1}{\gamma} \right) A_n^{-1} \right. \\
&\quad \left. - \left(K_{\gamma,\rho}(x) + \frac{1}{\gamma + \rho} \right) \right| \\
&= \left| \frac{\frac{V(nx) - V(\infty) - \gamma^{-1}a_n}{a_n} - \frac{x^\gamma - 1}{\gamma}}{A_n} - K_{\gamma,\rho}(x) \right|.
\end{aligned}$$

Now (3.1) follows by Lemma 2.5. For $\gamma < 0$, $\rho = 0$, we have

$$\begin{aligned} & \left| \frac{p_{n,\gamma}(x)}{A_n} - H_{\gamma,\rho}(x) \right| \\ &= \left| \left(\frac{V(nx) - V(\infty) - \gamma^{-1}a_n + \gamma^{-2}a_n A_n}{a_n} - \frac{x^\gamma - 1}{\gamma} \right) A_n^{-1} \right. \\ &\quad \left. - \left(K_{\gamma,\rho}(x) - \frac{1}{\gamma^2} \right) \right| \\ &= \left| \frac{\frac{V(nx) - V(\infty) - \gamma^{-1}a_n}{a_n} - \frac{x^\gamma - 1}{\gamma}}{A_n} - K_{\gamma,\rho}(x) \right|. \end{aligned}$$

Using Lemma 2.5 again, (3.1) holds for $\gamma < 0$, $\rho \leq 0$. Similarly for (3.2). \square

Lemma 3.2. *Let $\alpha_n = -\log^{-1} A_n^2$ and $\beta_n = A_n^{-2}$. If V satisfies the conditions in (2.1) for $k \geq 0$ then*

$$(3.2.1) \quad \lim_{n \rightarrow \infty} \sup_{x \in [\alpha_n, \beta_n]} x^{-\gamma-k} G_0(\log x) |A_n^{-1} p_{n,\gamma}(x) - H_{\gamma,\rho}(x)| = 0, \quad k > 0,$$

$$(3.2.2) \quad \lim_{n \rightarrow \infty} \sup_{x \in [\alpha_n, \beta_n]} x^{-\gamma} G'_0(\log x) |A_n^{-1} p'_{n,\gamma}(x) - H'_{\gamma,\rho}(x)| = 0, \quad k \geq 0,$$

$$(3.2.3) \quad \lim_{n \rightarrow \infty} \sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} x^{-2\gamma-k} p_{n,\gamma}^2(x) = 0, \quad k \geq 0,$$

$$(3.2.4) \quad \lim_{n \rightarrow \infty} \sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} x^{-2\gamma-k} p_{n,\gamma}(x) p'_{n,\gamma}(x) = 0, \quad k \geq 0,$$

$$(3.2.5) \quad \lim_{n \rightarrow \infty} \sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} x^{-3\gamma-1} p_{n,\gamma}^3(x) = 0.$$

Proof. For $\rho \leq 0$, $|A^*| \in RV(\rho)$ and $|A^*|^2 \in RV(2\rho)$. According to Lemma 2.1, there exists constant $c > 0$ and integer $n_0 > 0$ such that $A_n^2 \geq cn^{2\beta-1}$ for $n \geq n_0$, which implies $n\alpha_n \geq -n[(2\rho-1)\log n + \log c]^{-1} \rightarrow \infty$ as $n \rightarrow \infty$. Set n such that $nx > n_0$. From Lemma 3.1, for $\delta \in [0, 1)$ we have

$$\begin{aligned} & \sup_{\alpha_n \leq x \leq \beta_n} x^{-(k+\gamma)} \exp(-x^{-1}) |A_n^{-1} p_{n,\gamma}(x) - H_{\gamma,\rho}(x)| \\ & \leq \varepsilon \sup_{x \in R^+} x^{-(k+\gamma)} \exp(-x^{-1}) \max\{x^{\gamma+\rho+\delta}, x^{\gamma+\rho-\delta}\} \\ & = \varepsilon \sup_{x \in R^+} \exp(-x^{-1}) \max\{x^{-k+\rho+\delta}, x^{-k+\rho-\delta}\}. \end{aligned}$$

It is easy to check $\sup_{x \in R^+} \exp(-x^{-1}) \max\{x^{-k+\rho+\delta}, x^{-k+\rho-\delta}\} < \infty$ for $k > 0$ which implies (3.2.1). Similarly, (3.2.2) can be proved. For (3.2.3), we have

$$\begin{aligned} & \sup_{\alpha_n \leq x \leq \beta_n} A_n^{-1} x^{-2\gamma-k} p_{n,\gamma}^2(x) \\ &= \sup_{\alpha_n \leq x \leq \beta_n} A_n x^{-2\gamma-k} (A_n^{-1} p_{n,\gamma}(x))^2 \end{aligned}$$

$$\begin{aligned}
&= \sup_{\alpha_n \leq x \leq \beta_n} A_n x^{-2\gamma-k} [(A_n^{-1} p_{n,\gamma}(x) - H_{\gamma,\rho}(x)) + H_{\gamma,\rho}(x)]^2 \\
&\leq 2 \sup_{\alpha_n \leq x \leq \beta_n} A_n x^{-2\gamma-k} (A_n^{-1} p_{n,\gamma}(x) - H_{\gamma,\rho}(x))^2 + 2 \sup_{\alpha_n \leq x \leq \beta_n} A_n x^{-2\gamma-k} |H_{\gamma,\rho}(x)|^2 \\
&=: 2I_1 + 2I_2.
\end{aligned}$$

Notice

$$\begin{aligned}
I_1 &= A_n \sup_{\alpha_n \leq x \leq \beta_n} x^{-2\gamma-k} (A_n^{-1} p_{n,\gamma}(x) - H_{\gamma,\rho}(x))^2 \\
&\leq A_n \sup_{\alpha_n \leq x \leq \beta_n} \max\{x^{2(\rho-\frac{1}{2}k+\delta)}, x^{2(\rho-\frac{1}{2}k-\delta)}\} \\
&= A_n \sup_{\alpha_n \leq x \leq \beta_n} \max\{x^{2(\rho-\frac{1}{2}k+\delta)}, x^{2(\rho-\frac{1}{2}k-\delta)}\} \\
&\leq A_n \max\{A_n^{-4\delta}, (-\log^{-1} A_n)^{2(\rho-\frac{1}{2}k-\delta)}\} \rightarrow 0
\end{aligned}$$

and

$$I_2 = A_n \sup_{\alpha_n \leq x \leq \beta_n} x^{-2\gamma-k} |H_{\gamma,\rho}(x)|^2 \rightarrow 0$$

hence (3.2.3) follows. Before proving (3.2.4), we need to prove:

$$(3.3) \quad \lim_{n \rightarrow \infty} \sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} x^{-2\gamma-k} (p_{n,\gamma}(x)')^2 = 0.$$

Write

$$\begin{aligned}
&\sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} x^{-2\gamma-k} (p_{n,\gamma}(x)')^2 \\
&= \sup_{x \in [\alpha_n, \beta_n]} A_n x^{-2\gamma-k} (A_n^{-1} p_{n,\gamma}(x)')^2 \\
&= \sup_{x \in [\alpha_n, \beta_n]} A_n x^{-2\gamma-k} (A_n^{-1} p_{n,\gamma}(x)' - H_{\gamma,\rho}(x)' + H_{\gamma,\rho}(x)')^2 \\
&\leq 2A_n \sup_{x \in [\alpha_n, \beta_n]} x^{-2\gamma-k} (A_n^{-1} p_{n,\gamma}(x)' - H_{\gamma,\rho}(x)')^2 + 2A_n \sup_{x \in [\alpha_n, \beta_n]} x^{-2\gamma-k} (H_{\gamma,\rho}(x)')^2 \\
&= : 2I_3 + 2I_4.
\end{aligned}$$

For $\delta \in (0, 1/4)$, we have

$$\begin{aligned}
I_3 &= A_n \sup_{x \in [\alpha_n, \beta_n]} x^{-2\gamma-k} |A_n^{-1} p_{n,\gamma}(x)' - H_{\gamma,\rho}(x)'|^2 \\
&\leq A_n \sup_{x \in [\alpha_n, \beta_n]} \max\{x^{2[(\rho-\frac{1+k}{2})+\delta]}, x^{2[(\rho-\frac{1+k}{2})-\delta]}\} \\
&\leq A_n \max\{A_n^{-4\delta}, (-\log^{-1} A_n)^{2(\rho-\frac{1+k}{2}-\delta)}\} \rightarrow 0.
\end{aligned}$$

It is easy to check $I_4 = A_n \sup_{x \in [\alpha_n, \beta_n]} x^{-2\gamma-k} (H_{\gamma, \rho}(x)')^2 \rightarrow 0$ as $n \rightarrow \infty$. Therefore (3.3) holds. Combining (3.3) and (3.2.3), we have

$$\begin{aligned} & \sup_{x \in [\alpha_n, \beta_n]} A_n^{-2} x^{-4\gamma-2k} p_{n,\gamma}^2(x) (p_{n,\gamma}(x)')^2 \rightarrow 0 \\ \Rightarrow & \sup_{x \in [\alpha_n, \beta_n]} [A_n^{-1} x^{-2\gamma-k} p_{n,\gamma}(x) p_{n,\gamma}(x)']^2 \rightarrow 0 \\ \Rightarrow & \lim_{n \rightarrow \infty} \sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} x^{-2\gamma-k} p_{n,\gamma}(x) p_{n,\gamma}(x)' = 0 \end{aligned}$$

and so (3.2.4) follows. The proof of (3.2.5) is simpler since

$$\begin{aligned} & \sup_{x \in [\alpha_n, \beta_n]} A_n^{-3} x^{-6\gamma-2} p_{n,\gamma}^6(x) \rightarrow 0 \\ \Rightarrow & \sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} x^{-3\gamma-1} p_{n,\gamma}^3(x) \rightarrow 0. \end{aligned}$$

□

Lemma 3.3. Let $J_n(x) = G_0(\log x + q_n(x)) - G_0(\log x) = q_n G'_0(\log x + \theta q_n(x))$ and $J(x) = x^{-(1+\gamma)} G_0(\log x) H_{\gamma, \rho}(x)$, where $q_n(x) = \gamma^{-1} \log\{1 + \gamma a_n^{-1}(V(nx) - b_n)\} - \log x = x^{-\gamma} p_{n,\gamma}(x)[1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-1}$. If V satisfies the conditions in (2.1) then

$$(3.4) \quad \lim_{n \rightarrow \infty} \sup_{x \in [0, \infty]} |A_n^{-1} J_n(x) - J(x)| = 0$$

and

$$(3.5) \quad \lim_{n \rightarrow \infty} \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} J'_n(x) - J'(x)| = 0.$$

Proof. If V satisfies (2.1) then (3.2.1) and (3.2.2) hold. Especially letting $k = 1$, (3.2.1) is tantamount to (2.2) of Cheng and Jiang [2]. (3.2.2) is equal to (2.1) of Cheng and Jiang [2] if $k = 0$ in Lemma 3.2. Now (3.4) may be obtained as in Cheng and Jiang [2]. We pay attention to (3.5). Write

$$\begin{aligned} & \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} J'_n(x) - J'(x)| \\ = & \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} [q'_n(x) G'_0(\log x + \theta q_n(x)) \\ & \quad + q_n(x) G''_0(\log x + \theta q_n(x))(x^{-1} + \theta q'_n(x))] - J'(x)| \\ = & \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} q'_n(x) G'_0(\log x + \theta q_n(x)) \\ & \quad + A_n^{-1} q_n(x) G''_0(\log x + \theta q_n(x))(x^{-1} + \theta q'_n(x)) - J'(x)| \\ = & \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \{p'_{n,\gamma}(x)[x^\gamma + \theta_0 \gamma p_{n,\gamma}(x)]^{-1} G'_0(\log x + \theta q_n(x))\} \\ & \quad + A_n^{-1} \{-p_{n,\gamma}(x)[x^\gamma + \theta_0 \gamma p_{n,\gamma}(x)]^{-2} (\gamma x^{\gamma-1} + \theta_0 \gamma p'_{n,\gamma}(x)) G'_0(\log x + \theta q_n(x))\}| \end{aligned}$$

$$\begin{aligned}
& + A_n^{-1} \{ p_{n,\gamma}(x) [x^\gamma + \theta_0 \gamma p_{n,\gamma}(x)]^{-1} G_0''(\log x + \theta q_n(x)) (x^{-1} + \theta q'_n(x)) \} - J'(x) | \\
= & \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \{ p'_{n,\gamma}(x) [x^\gamma + \theta_0 \gamma p_{n,\gamma}(x)]^{-1} G_0'(\log x + \theta q_n(x)) \} \\
& + A_n^{-1} \{ -p_{n,\gamma}(x) [x^\gamma + \theta_0 \gamma p_{n,\gamma}(x)]^{-2} (\gamma x^{\gamma-1} + \theta_0 \gamma p'_{n,\gamma}(x)) G_0'(\log x + \theta q_n(x)) \} \\
& + A_n^{-1} \{ p_{n,\gamma}(x) [x^\gamma + \theta_0 \gamma p_{n,\gamma}(x)]^{-1} G_0''(\log x + \theta q_n(x)) (x^{-1} + \theta q'_n(x)) \} \\
& - H'_{\gamma, \rho}(x) x^{-\gamma} G_0'(\log x) + \gamma H_{\gamma, \rho}(x) x^{-\gamma-1} G_0'(\log x) \\
& - H_{\gamma, \rho}(x) x^{-\gamma-1} G_0''(\log x) | \\
\leq & \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} \{ p'_{n,\gamma}(x) x^{-\gamma} \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right) G_0'(\log x + \theta q_n(x)) \} \right. \\
& \left. - H'_{\gamma, \rho}(x) x^{-\gamma} G_0'(\log x) \right| \\
+ & \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} \{ p_{n,\gamma}(x) x^{-2\gamma} \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right)^2 (\gamma x^{\gamma-1} + \theta_0 \gamma p'_{n,\gamma}(x)) \right. \\
& \times G_0'(\log x + \theta q_n(x)) \} - \gamma H_{\gamma, \rho}(x) x^{-\gamma-1} G_0'(\log x) \Big| \\
+ & \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \{ p_{n,\gamma}(x) [x^\gamma + \theta_0 \gamma p_{n,\gamma}(x)]^{-1} G_0''(\log x + \theta q_n(x)) \\
& (x^{-1} + \theta_0 q'_n(x)) \} - H_{\gamma, \rho}(x) x^{-\gamma-1} G_0''(\log x) | \\
= & : A + B + C,
\end{aligned}$$

where

$$\begin{aligned}
A \leq & \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} \{ p'_{n,\gamma}(x) x^{-\gamma} \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right) G_0'(\log x + \theta q_n(x)) \} \right. \\
& \left. - A_n^{-1} \{ p'_{n,\gamma}(x) x^{-\gamma} G_0'(\log x + \theta q_n(x)) \} \right| \\
& + \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \{ p'_{n,\gamma}(x) x^{-\gamma} G_0'(\log x + \theta q_n(x)) \} - H'_{\gamma, \rho}(x) x^{-\gamma} G_0'(\log x) | \\
= & A_1 + A_2.
\end{aligned}$$

Let $M = \max\{\sup_{x \in R^+} G_0'(\log x), \sup_{x \in R^+} G_0''(\log x), \sup_{x \in R^+} G_0'''(\log x)\}$. We have

$$A_1 \leq M \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} x^{-2\gamma} p_{n,\gamma}(x) p'_{n,\gamma}(x)| \left| \frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right|$$

and $A_1 \rightarrow 0$ as $n \rightarrow \infty$ by Lemma 3.2. For A_2 , we have

$$\begin{aligned}
A_2 & \leq \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \{ p'_{n,\gamma}(x) x^{-\gamma} G_0'(\log x + \theta q_n(x)) \} - A_n^{-1} \{ p'_{n,\gamma}(x) x^{-\gamma} G_0'(\log x) \} | \\
& \leq \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \{ p'_{n,\gamma}(x) x^{-\gamma} G_0'(\log x + \theta q_n(x)) \} - A_n^{-1} \{ p'_{n,\gamma}(x) x^{-\gamma} G_0'(\log x) \} |
\end{aligned}$$

$$\begin{aligned}
& + \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \{p'_{n,\gamma}(x) x^{-\gamma} G'_0(\log x)\} - H'_{\gamma, \rho}(x) x^{-\gamma} G'_0(\log x)| \\
& \leq \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} p'_{n,\gamma}(x) x^{-\gamma} G''_0(\log x + \theta_1 q_n(x)) q_n(x)| \\
& \quad + \sup_{x \in [\alpha_n, \beta_n]} x^{-\gamma} G'_0(\log x) |A_n^{-1} p'_{n,\gamma}(x) - H'_{\gamma, \rho}(x)| \\
& \leq \sup_{x \in [\alpha_n, \beta_n]} M |A_n^{-1} x^{-2\gamma} p_{n,\gamma}(x) p'_{n,\gamma}(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-1}| \\
& \quad + \sup_{x \in [\alpha_n, \beta_n]} x^{-\gamma} G'_0(\log x) |A_n^{-1} p'_{n,\gamma}(x) - H'_{\gamma, \rho}(x)|,
\end{aligned}$$

where $0 \leq \theta_1 \leq 1$. According to Lemma 3.2, $A_2 \rightarrow 0$ as $n \rightarrow \infty$. Hence $A \rightarrow 0$ as $n \rightarrow \infty$.

Write

$$\begin{aligned}
B &= \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} \{p_{n,\gamma}(x) x^{-2\gamma} \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right)^2 (\gamma x^{\gamma-1} + \theta_0 \gamma p'_{n,\gamma}(x)) \right. \\
&\quad \times G'_0(\log x + \theta q_n(x)) \left. - \gamma H'_{\gamma, \rho}(x) x^{-\gamma-1} G'_0(\log x) \right| \\
&= \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} \{p_{n,\gamma}(x) x^{-\gamma-1} \gamma \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right)^2 (1 + \theta_0 x^{1-\gamma} p'_{n,\gamma}(x)) \right. \\
&\quad \times G'_0(\log x + \theta q_n(x)) \left. - \gamma H'_{\gamma, \rho}(x) x^{-\gamma-1} G'_0(\log x) \right| \\
&\leq \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} \{p_{n,\gamma}(x) x^{-\gamma-1} \gamma \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right)^2 (1 + \theta_0 x^{1-\gamma} p'_{n,\gamma}(x)) \right. \\
&\quad \times G'_0(\log x + \theta q_n(x)) \left. - A_n^{-1} \{p_{n,\gamma}(x) x^{-\gamma-1} \gamma G'_0(\log x + \theta q_n(x))\} \right| \\
&\quad + \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} \{p_{n,\gamma}(x) x^{-\gamma-1} \gamma G'_0(\log x + \theta q_n(x))\} \right. \\
&\quad \left. - A_n^{-1} \{p_{n,\gamma}(x) x^{-\gamma-1} \gamma G'_0(\log x)\} \right| \\
&\quad + \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} \{p_{n,\gamma}(x) x^{-\gamma-1} \gamma G'_0(\log x)\} - \gamma H'_{\gamma, \rho}(x) x^{-\gamma-1} G'_0(\log x) \right| \\
&=: B_1 + B_2 + B_3,
\end{aligned}$$

where

$$\begin{aligned}
B_1 &\leq M \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1} \gamma \frac{1 + \theta_0 x^{1-\gamma} p'_{n,\gamma}(x) - (1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x))^2}{(1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x))^2} \right| \\
&= M \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1} \gamma \right. \\
&\quad \times \left. \frac{\theta_0 x^{1-\gamma} p'_{n,\gamma}(x) - 2\theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x) - \theta_0^2 \gamma^2 x^{-2\gamma} p_{n,\gamma}^2(x)}{(1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x))^2} \right|
\end{aligned}$$

$$\begin{aligned}
&\leq M \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} p_{n,\gamma}(x) p'_{n,\gamma}(x) x^{-2\gamma} \theta_0 \frac{1}{(1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x))^2} \right| \\
&+ M \sup_{x \in [\alpha_n, \beta_n]} \left| 2A_n^{-1} p_{n,\gamma}^2(x) x^{-2\gamma-1} \gamma^2 \theta_0 \frac{1}{(1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x))^2} \right| \\
&+ M \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} p_{n,\gamma}^3(x) x^{-3\gamma-1} \gamma^3 \theta_0^2 \frac{1}{(1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x))^3} \right|.
\end{aligned}$$

According to Lemma 3.2, $B_1 \rightarrow 0$ as $n \rightarrow \infty$. Notice

$$\begin{aligned}
B_2 &\leq M \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1} \gamma \theta q_n(x)| \\
&= M \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} p_{n,\gamma}^2(x) x^{-2\gamma-1} \theta \frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right|.
\end{aligned}$$

Using Lemma 3.2 again, $B_2 \rightarrow 0$ as $n \rightarrow \infty$. Similarly, $B_3 \rightarrow 0$ as $n \rightarrow \infty$. Hence $B \rightarrow 0$ as $n \rightarrow \infty$.

Write

$$\begin{aligned}
C &\leq \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1}| \\
&\times \left| \frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} (1 + \theta_0 q'_n(x) x^{-1}) G''_0(\log x + \theta_0 q_n(x)) - G''_0(\log x) \right| \\
&+ \sup_{x \in [\alpha_n, \beta_n]} x^{-\gamma-1} G''_0(\log x) |A_n^{-1} p_{n,\gamma}(x) - H_{\gamma, \rho}(x)| \\
&=: C_1 + C_2.
\end{aligned}$$

Notice $C_2 \rightarrow 0$ as $n \rightarrow \infty$ by Lemma 3.2(3.2.1). Write

$$\begin{aligned}
C_1 &\leq \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1}| \\
&\times \left| \frac{1 + \theta_0 q'_n(x) x^{-1}}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} - 1 \right| G''_0(\log x + \theta_0 q_n(x)) \\
&+ \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1}| |G''_0(\log x + \theta_0 q_n(x)) - G''_0(\log x)| \\
&\leq M \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1}| \left| \frac{\theta_0 q'_n(x) x^{-1} - \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right| \\
&+ M \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1}| q_n(x) \\
&=: C_{11} + C_{12},
\end{aligned}$$

where

$$C_{12} = M \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1}| p_{n,\gamma}(x) x^{-\gamma} \frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \rightarrow 0$$

by Lemma 3.2. Write

$$\begin{aligned} C_{11} &\leq M \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1} \theta_0 q'_n(x) x^{-1} \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right) \right| \\ &\quad + M \sup_{x \in [\alpha_n, \beta_n]} \left| \gamma A_n^{-1} p_{n,\gamma}^2(x) x^{-2\gamma-1} \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right) \right| \\ &=: J_1 + J_2. \end{aligned}$$

Using Lemma 3.2 again, we have $J_2 \rightarrow 0$ as $n \rightarrow \infty$. Write

$$\begin{aligned} J_1 &= M \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-1} \theta_0 q'_n(x) x^{-1} \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right) \right| \\ &= M \sup_{x \in [\alpha_n, \beta_n]} \left| A_n^{-1} p_{n,\gamma}(x) x^{-\gamma-2} \theta_0 q'_n(x) \left(\frac{1}{1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)} \right) \right|. \end{aligned}$$

Since

$$\begin{aligned} q'_n(x) &= -\gamma x^{-\gamma-1} p_{n,\gamma}(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-1} + x^{-\gamma} p'_{n,\gamma}(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-1} \\ &\quad - x^{-\gamma} p_{n,\gamma}(x) \theta_0 [-\gamma^2 x^{-\gamma-1} p_{n,\gamma}(x) + \gamma x^{-\gamma} p'_{n,\gamma}(x)] [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-2} \\ &= -\gamma x^{-\gamma-1} p_{n,\gamma}(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-1} + x^{-\gamma} p'_{n,\gamma}(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-1} \\ &\quad + \theta_0 \gamma^2 x^{-2\gamma-1} p_{n,\gamma}^2(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-2} \\ &\quad - \theta_0 \gamma x^{-2\gamma} p_{n,\gamma}(x) p'_{n,\gamma}(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-2}, \end{aligned}$$

we have

$$\begin{aligned} J_1 &\leq M \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \gamma x^{-2\gamma-3} p_{n,\gamma}^2(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-2}| \\ &\quad + M \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} x^{-2\gamma-2} p_{n,\gamma}(x) p'_{n,\gamma}(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-2}| \\ &\quad + M \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \theta_0 \gamma^2 x^{-3\gamma-3} p_{n,\gamma}^3(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-3}| \\ &\quad + M \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} \theta_0 \gamma x^{-3\gamma-2} p_{n,\gamma}^2(x) p'_{n,\gamma}(x) [1 + \theta_0 \gamma x^{-\gamma} p_{n,\gamma}(x)]^{-3}|. \end{aligned}$$

If $J_1 \rightarrow 0$ as $n \rightarrow \infty$ then the proof of the Lemma will be complete. We need to prove

- (a) $\sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} p_{n,\gamma}^2(x) x^{-2\gamma-3} \rightarrow 0$,
- (b) $\sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} p_{n,\gamma}(x) p'_{n,\gamma}(x) x^{-2\gamma-2} \rightarrow 0$,
- (c) $\sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} p_{n,\gamma}^3(x) x^{-3\gamma-3} \rightarrow 0$,
- (d) $\sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} p_{n,\gamma}^2(x) p'_{n,\gamma}(x) x^{-3\gamma-2} \rightarrow 0$,

where (a) and (b) are obvious by Lemma 3.2. For (c) and (d), by Lemma 3.2, we have

$$\begin{aligned}
& \sup_{x \in [\alpha_n, \beta_n]} A_n^{-2} p_{n,\gamma}^4(x) x^{-4\gamma-4} \rightarrow 0 \\
\Rightarrow & \sup_{x \in [\alpha_n, \beta_n]} A_n^{-\frac{3}{2}} p_{n,\gamma}^3(x) x^{-3\gamma-3} \rightarrow 0 \\
\Rightarrow & \sup_{x \in [\alpha_n, \beta_n]} \frac{A_n^{-1} p_{n,\gamma}^3(x) x^{-3\gamma-3}}{A_n^{\frac{1}{2}}} \rightarrow 0, \\
& \sup_{x \in [\alpha_n, \beta_n]} A_n^{-2} p_{n,\gamma}^4(x) (p'_{n,\gamma}(x))^2 x^{-6\gamma-4} \\
= & \sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} p_{n,\gamma}^2(x) x^{-2\gamma} A_n A_n^{-2} p_{n,\gamma}^2(x) (p'_{n,\gamma}(x))^2 x^{-4\gamma-4},
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{x \in [\alpha_n, \beta_n]} A_n^{-1} p_{n,\gamma}^2(x) x^{-2\gamma} \rightarrow 0, \\
& \sup_{x \in [\alpha_n, \beta_n]} A_n^{-2} p_{n,\gamma}^2(x) (p'_{n,\gamma}(x))^2 x^{-4\gamma-4} \rightarrow 0.
\end{aligned}$$

□

4. Main results

Theorem 4.1. *If V satisfies the conditions in (2.1) then*

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{P\{M_n \leq a_n u + b_n\} - G_\gamma(u)}{A_n} = -J((1 + \gamma u)^{\frac{1}{\gamma}})$$

holds locally uniformly on R .

Proof. Combining (3.4) in Lemma 3.3 and (1.2), the result follows in a similar way to Theorem 3.1 of Cheng and Jiang [2]. □

Remark 4.1. The following Edgeworth expansion of the distribution of the normalized maxima may be obtained by Theorem 4.1:

$$\begin{aligned}
& P\{M_n \leq a_n u + b_n\} \\
& = G_\gamma(u) + A_n G_\gamma(u) H_{\gamma, \rho}(-\log^{-1} G_\gamma(u)) \log^{\gamma+1} G_\gamma(u) + o(A_n).
\end{aligned}$$

Remark 4.2. We may find the uniform convergence rate of (1.1) by Theorem 4.1:

$$\lim_{n \rightarrow \infty} \sup_{u \in R} \left| \frac{P\{M_n \leq a_n u + b_n\} - G_\gamma(u)}{A_n} \right| = \sup_{u \in R} u^{-(\gamma+1)} |H_{\gamma, \rho}(u)| \exp(-u^{-1}).$$

Note that this result is the same as equation (3.4) in de Haan and Resnick [4].

The next theorem considers the rate of convergence with respect to the total variation metric (Billingsley, [1]):

$$\begin{aligned} D_n &:= \sup_{A \in \mathcal{B}(R)} |P[a_n^{-1}(M_n - b_n) \in A] - G_\gamma(A)| \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \left| \frac{d}{dx} F^n(a_n x + b_n) - G'_\gamma(x) \right| dx. \end{aligned}$$

Theorem 4.2. *Let*

$$\begin{aligned} a_n &= a^*(n), \\ b_n &= \begin{cases} V(n), & \gamma \geq 0, \\ V(\infty) + \gamma^{-1} a_n, & \gamma < 0, \end{cases} \\ x_n(u) &= [-n \log F(a_n u + b_n)]^{-1}, \end{aligned}$$

and

$$\frac{nV(nx_n(u))}{dx_n(u)} = a_n du.$$

If V satisfies the conditions in (2.1) for $\gamma \in R$ and $\rho \leq 0$ then

$$(4.2) \quad \lim_{n \rightarrow \infty} \frac{D_n}{|A(n)|} = \frac{1}{2} \int_0^\infty J'(x) dx.$$

Proof. Denote

$$\begin{aligned} Q_{n,\gamma}(u) &= \frac{V(nu) - b_n}{a_n}, \\ Q_{n,\gamma}(u) &= p_{n,\gamma}(u) + \frac{u^\gamma - 1}{\gamma}, \\ Q'_{n,\gamma}(u) &= p'_{n,\gamma}(u) + u^{\gamma-1} \end{aligned}$$

and write

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| \frac{d}{du} F^n(a_n u + b_n) - G'_\gamma(u) \right| du \\ &= \int_0^\infty \left| \frac{dF^n(V(nx_n(u)))}{dx_n(u)} - \frac{nV'(nx_n(u))}{a_n} G'_\gamma\left(\frac{V(nx_n(u)) - b_n}{a_n}\right) \right| dx_n(u) \\ &= \int_0^\infty \left| \frac{dF^n(V(nx_n(u)))}{dx_n(u)} - Q'_{n,\gamma}(x_n(u)) G'_\gamma(Q_{n,\gamma}(x_n(u))) \right| dx_n(u). \end{aligned}$$

Writing x instead of $x_n(u)$, note that

$$\begin{aligned} 2D_n &= \int_0^\infty \left| \frac{dF^n(V(nx))}{dx} - Q'_{n,\gamma}(x) G'_\gamma(Q_{n,\gamma}(x)) \right| dx \\ &= \int_0^\infty \left| \frac{dG_0(\log x)}{dx} - Q'_{n,\gamma}(x) G'_\gamma(Q_{n,\gamma}(x)) \right| dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \left| [G_0(\log x)]' - \frac{d}{dx} G_\gamma(Q_{n,\gamma}(x)) \right| dx \\
&= \int_0^\infty \left| [G_0(\log x)]' - \frac{d}{dx} G_0(\gamma^{-1} \log\{1 + \gamma Q_{n,\gamma}(x)\}) \right| dx \\
&= \int_0^\infty \left| \frac{dG_0(\log x)}{dx} - \frac{d}{dx} G_0(\gamma^{-1} \log\{1 + \gamma Q_{n,\gamma}(x)\}) \right| dx \\
&= \int_0^\infty |J'_n(x)| dx.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\alpha_n}^{\beta_n} |J'(x)| dx - \int_{\alpha_n}^{\beta_n} |A_n^{-1} J'_n(x) - J'(x)| dx \\
&\leq \int_{\alpha_n}^{\beta_n} |A_n^{-1} J'_n(x)| dx \\
&= \int_{\alpha_n}^{\beta_n} |A_n^{-1} J'_n(x) - J'(x) + J'(x)| dx \\
&\leq \int_{\alpha_n}^{\beta_n} |J'(x)| dx + \int_{\alpha_n}^{\beta_n} |A_n^{-1} J'_n(x) - J'(x)| dx
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \sup_{x \in [\alpha_n, \beta_n]} |A_n^{-1} J'_n(x) - J'(x)| = 0 \\
\Rightarrow &\lim_{n \rightarrow \infty} \int_{\alpha_n}^{\beta_n} |A_n^{-1} J'_n(x) - J'(x)| dx = 0,
\end{aligned}$$

we can write

$$\lim_{n \rightarrow \infty} \int_{\alpha_n}^{\beta_n} |A_n^{-1} J'_n(x)| dx = \int_0^\infty |J'(x)| dx.$$

We must deal with the parts of the integral near $\pm\infty$. Write

$$\begin{aligned}
&A_n^{-1} \int_{\beta_n}^\infty |[G_0(\log x)]' - \frac{d}{dx} G_\gamma(Q_{n,\gamma}(x))| dx \\
&\leq A_n^{-1} \{1 - G_0(\log A_n^{-2})\} + A_n^{-1} \{1 - G_\gamma(Q_{n,\gamma}(A_n^{-2}))\} \\
&\leq 2A_n^{-1} \{1 - G_0(-\log A_n^2)\} + A_n^{-1} \{G_\gamma(Q_{n,\gamma}(A_n^{-2})) - G_0(\log A_n^{-2})\}.
\end{aligned}$$

The first part goes to zero by simple operation and the second part goes to zero by the proof of (3.4). Similarly,

$$\begin{aligned}
&A_n^{-1} \int_0^{\alpha_n} \left| [G_0(\log x)]' - \frac{d}{dx} G_\gamma(Q_{n,\gamma}(x)) \right| dx \\
&\leq A_n^{-1} G_0(\log \alpha_n) + A_n^{-1} G_\gamma(Q_{n,\gamma}(\alpha_n))
\end{aligned}$$

$$\begin{aligned}
&\leq 2A_n^{-1}G_0(\log \alpha_n) + A_n^{-1}|G_\gamma(Q_{n,\gamma}(\alpha_n)) - G_0(\log \alpha_n)| \\
&= 2A_n^{-1}G_0(\log -\log^{-1} A_n^2) \\
&\quad + A_n^{-1}|G_\gamma(Q_{n,\gamma}(-\log^{-1} A_n^2)) - G_0(\log -\log^{-1} A_n^2)| \\
&\rightarrow 0.
\end{aligned}$$

The result follows. \square

Remark 4.3. Note that under the second order von Mises condition the integral in (4.2) is essentially the same as that given by Theorem 4.1 of de Haan and Resnick [4].

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