Bull. Korean Math. Soc. **45** (2008), No. 1, pp. 95–99

CLASSIFICATION OF TREES EACH OF WHOSE ASSOCIATED ACYCLIC MATRICES WITH DISTINCT DIAGONAL ENTRIES HAS DISTINCT EIGENVALUES

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ABSTRACT. It is known that each eigenvalue of a real symmetric, irreducible, tridiagonal matrix has multiplicity 1. The graph of such a matrix is a path. In this paper, we extend the result by classifying those trees for which each of the associated acyclic matrices has distinct eigenvalues whenever the diagonal entries are distinct.

1. Introduction

Throughout all matrices are real. We refer a reader to [2] for basic graph theoretic terminology. Let $A = [a_{ij}]$ be an n by n symmetric matrix. If A has k distinct eigenvalues with multiplicities $m_1 \geq m_2 \geq \cdots \geq m_k$, then $(m_1, m_2, \ldots, m_k)$ is the *unordered multiplicity list* of the eigenvalues of A. If an eigenvalue λ of A has multiplicity 1, then λ is a *simple* eigenvalue of A. The *graph* of A, denoted by $G(A)$, consists of the vertices $1, 2, \ldots, n$, and the edges $\{i, j\}$ for which $i \neq j$ and $a_{ij} \neq 0$. Note that $G(A)$ does not depend on the diagonal entries of A. If $G(A)$ is a tree, then A is an irreducible, *acyclic* matrix.

Further, for a given graph G on n vertices, define $S(G)$ to be the set of all n by n symmetric matrices with graph G, i.e.,

$$S(G) = \{A_{n\times n} \mid A \text{ is symmetric and } G(A) = G\ \}.$$

The *spectrum of* $S(G)$ is the set of all spectra realized by some matrix in $S(G)$. The *unordered multiplicity list of* $S(G)$ is the set of all unordered multiplicity lists realized by some matrix in $S(G)$.

It is known that for a tree T, some combinatorial properties of T are reflected in the spectrum of $S(T)$, and hence impose restrictions on the unordered multiplicity list of $S(T)$ (see [4, 1]). For example, the maximum multiplicity of an eigenvalue of a matrix in $S(T)$ is the minimum number of disjoint paths in T

Received May 25, 2007.

2000 *Mathematics Subject Classification.* 15A18, 05C50.

Key words and phrases. acyclic matrix, Parter-vertex, simple eigenvalue.

The work of In-Jae Kim was in part supported by a Faculty Research Grant from the Minnesota State University, Mankato.

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covering all of the vertices of T. Thus, for a given tree T, it can be shown that T is a path if and only if the unordered multiplicity list of $S(T)$ is $\{(1,1,\ldots,1)\}$. This implies that if $G(A)$ is a path, then the symmetric, irreducible, tridiagonal matrix A has only simple eigenvalues regardless of the diagonal entries. A problem related to this fact is the existence of other trees T for which each associated acyclic matrix with some restrictions on the diagonal entries has the unordered multiplicity list $(1,1,\ldots,1)$. For a given graph, define

$$SD(G) = \{A = [a_{ij}] \in S(G) \mid a_{ss} \neq a_{tt} \text{ for } s \neq t\ \}.$$

In this note, we classify the trees T such that every matrix in $SD(T)$ has only simple eigenvalues.

2. Main results

We first provide necessary terms and a known result. Let T be a tree, and let v be a vertex of T. Then $T \setminus \{v\}$ is the induced subgraph obtained by deleting vertex v and all incident edges to v. Each connected component of $T \setminus \{v\}$ is called a *branch* of T at v. Note that each branch of T at v is a tree. If the degree of v is k, then there are k branches of T at v. Let $A \in S(T)$, and B be a branch of T at v. Then $A(v)$ denotes the principal submatrix of A whose graph is $T \setminus \{v\}$, and $A[B]$ denotes the principal submatrix of A obtained by retaining rows and columns indexed by the vertices of B. If $B_1, B_2, \ldots, B_k$ are the branches of T at v, then $A(v)$ is, up to permutation similarity, equal to $A[B_1] \oplus A[B_2] \oplus \cdots \oplus A[B_k]$. The multiplicity of an eigenvalue λ of A is denoted by $m_A(\lambda)$. If $\lambda = 0$, then $m_A(\lambda)$ is the nullity $\nu(A)$ of A. It is a simple consequence of linear algebra that $\nu(A) \geq \nu(A(v)) - 1$. Moreover, by the Cauchy Interlacing Theorem (see [3, Theorem 4.3.8]), $m_A(\lambda)$ and $m_{A(v)}(\lambda)$ differ by at most one. If $m_{A(v)}(\lambda) = m_A(\lambda) + 1$, vertex v is a *Parter-vertex* of A for λ (see [5]). If $G(A)$ is a tree, then we have the following result.

Lemma 2.1 (Parter-Wiener Theorem, [5]). *Let A be an irreducible, acyclic matrix, and let λ be an eigenvalue of A with $m_A(\lambda) \geq 2$. Then there exists a Parter-vertex v of A for λ such that λ is an eigenvalue of at least three of the direct summands of $A(v)$.*

The following result along with Lemma 2.4 gives the characterization of the trees T for which each matrix in $SD(T)$ has distinct eigenvalues.

Theorem 2.2. *Let T be a tree that is not a path. Then each matrix in $SD(T)$ has distinct eigenvalues if and only if each vertex of degree 3 or more in T has at most one branch which is not a pendant vertex.*

To prove Theorem 2.2, we first establish the following lemma.

Lemma 2.3. *Let T be a tree on n vertices for $n \geq 2$. Then there exists a singular matrix in $SD(T)$ with all nonzero diagonal entries.*

Proof. Let $L = [\ell_{ij}]$ be the Laplacian matrix of T (see [2]). Then $L \in S(T)$. Since $n \geq 2$ and T is connected, L is singular and $\ell_{ii} > 0$ for each $i = 1, \ldots, n$. Let $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_j \neq 0$ for each $j = 1, \ldots, n$ such that $d_1^2\ell_{11} > d_2^2\ell_{22} > \cdots > d_n^2\ell_{nn} > 0$, and let $A = DLD$. Then A is a singular matrix with nonzero diagonal entries in $SD(T)$. □

Proof of Theorem 2.2. Let T be a tree that is not a path. For sufficiency, assume that each of vertices of degree 3 or more in T has at most one branch which is not a pendant vertex, and $A \in SD(T)$. Suppose to the contrary that there exists an eigenvalue λ of A with $m_A(\lambda) \geq 2$. Then, by Lemma 2.1, there exists a Parter-vertex v of A for λ such that at least three of the direct summands of $A(v)$ have λ as an eigenvalue. This implies that at least two diagonal entries of A are λ. This contradicts that A has distinct diagonal entries.

For necessity, suppose to the contrary that there exists a vertex v of degree 3 or more having at least two branches which are not pendant vertices. Let k denote the degree of v, and $B_1, \ldots, B_k$ be the branches of T at v. Assume that there are exactly two branches of T at v which are not pendant vertices, say B_1, B_2. Then $B_3, \ldots, B_k$ are pendant vertices of T. By Lemma 2.3, there is a singular matrix A_j without any zero diagonal entries in $SD(B_j)$ for each $j = 1, 2$. Moreover, by using the construction in Lemma 2.3, we can construct A_1, A_2 such that the diagonal entries of A_1, A_2 are nonzero, distinct, real numbers. We now construct a matrix $A = [a_{ij}]$ in $SD(T)$ such that $A[B_j] = A_j$ for $j = 1, 2$, $A[B_3] = 0$, $a_{ss} \neq a_{tt}$ for $s \neq t$, and the other entries, not yet assigned any value, are 1. Since there are exactly 3 singular summands $A[B_1], A[B_2]$, and $A[B_3]$ of $A(v)$, it follows that $\nu(A(v)) = 3$. Since $\nu(A) \geq \nu(A(v)) - 1$, it follows that $\nu(A) \geq 2$. Hence, not all of the eigenvalues of A are simple.

Next assume that there are at least three branches of T at v which are not pendant vertices of T, and B_1, B_2, B_3 are three of them. As before, we construct a matrix A in $SD(T)$ such that $A[B_j]$'s are singular matrices with no zero diagonal entries and $A[B_j] \in SD(B_j)$ for $j = 1, 2, 3$, and $a_{ss} \neq a_{tt}$ for $s \neq t$, and the other entries, not yet assigned any value, are 1. Then, since there are at least 3 singular summands $A[B_1], A[B_2]$, and $A[B_3]$ of $A(v)$, $\nu(A(v)) \geq 3$. Since $\nu(A) \geq \nu(A(v)) - 1$, it follows that $\nu(A) \geq 2$. Hence, the result follows. □

We now turn our attention to characterize the trees for which each vertex of degree 3 or more has at most one branch which is not a pendant vertex.

Lemma 2.4. *Let T be a tree. If each vertex of degree 3 or more in T has at most one branch which is not a pendant vertex, then there are at most two vertices of degree 3 or more in T.*

Proof. Suppose to the contrary that T has at least three vertices of degree 3 or more. Let u, v, w be vertices of degree 3 or more in T. Since T is a tree, there

exists a pair of vertices v_1, v_2 in $\{u, v, w\}$ such that the path P connecting v_1 and v_2 does not pass through vertex $v_3 \in \{u, v, w\} \setminus \{v_1, v_2\}$.

Since the branch of T at v_1 (resp. v_2) containing vertex v_2 (resp. vertex v_1) is a non-pendant branch, by the assumption, the path connecting v_1 and v_3 shares a vertex y, which is neither v_1 nor v_2, with the path P. Thus, $\deg(y) \geq 3$ and there exist at least two branches of T at y that are not pendant vertices. This contradicts the assumption. □

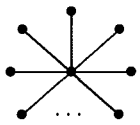
The following is a direct consequence of Theorem 2.2 and Lemma 2.4.

Corollary 2.5. *Let T be a tree. Then each matrix in $SD(T)$ has distinct eigenvalues if and only if T is one of the following trees:*

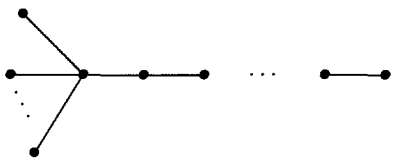
(1) *No vertex of degree* 3 *or more* (*Paths*) :

(2) *One vertex of degree* 3 *or more:*

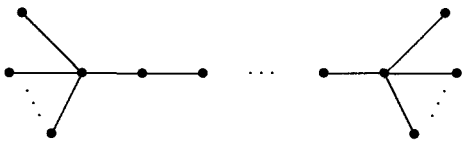
(2a) *No branch which is not a pendant vertex* (*Stars*) :



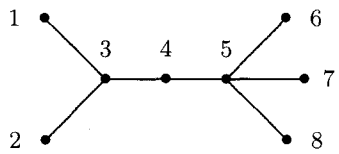
(2b) *One branch which is not a pendant vertex:*



(3) *Two vertices of degree* 3 *or more:*



Example 2.6. Consider the following tree T and $A \in SD(T)$:



$$A = \begin{bmatrix} a_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & a_3 & b_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & a_4 & b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & a_5 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & 0 & b_5 & a_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_6 & 0 & a_7 & 0 \\ 0 & 0 & 0 & 0 & b_7 & 0 & 0 & a_8 \end{bmatrix},$$

where the a_i's are distinct real numbers, and each b_j is a nonzero real number. Then Corollary 2.5 implies that for any nonzero real numbers $b_1, \ldots, b_7$, each eigenvalue of A is simple.

References

[1] A. Leal Duarte and C. R. Johnson, *On the minimum number of distinct eigenvalues for a symmetric matrix whose graph is a given tree*, Math. Inequal. Appl. **5** (2002), no. 2, 175–180.

[2] C. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, 207. Springer-Verlag, New York, 2001.

[3] R. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

[4] C. R. Johnson and A. Leal Duarte, *The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree*, Linear and Multilinear Algebra **46** (1999), no. 1-2, 139–144.

[5] C. R. Johnson, A. Leal Duarte, and C. M. Saiago, *The Parter-Wiener theorem: refinement and generalization*, SIAM J. Matrix Anal. Appl. **25** (2003), no. 2, 352–361.

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