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AUGMENTATION QUOTIENTS OF INTEGRAL GROUP RINGS

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ABSTRACT. The structure of I_G^n/I_G^{n+1} for finite abelian group G is investigated.

1. Two questions

Let G be a finite abelian group, $\mathbb{Z}[G]$ the integral group ring of G and I_G the augmentation ideal of $\mathbb{Z}[G]$, i.e., the kernel of the map from $\mathbb{Z}[G]$ to $\mathbb{Z}$ sending each group element to 1. We define $Q_n(G) = I_G^n/I_G^{n+1}$ for $n \geq 1$ and set $Q_0(G) = \mathbb{Z}$.

It has been proved in [1] that there exist an integer n_0 such that $Q_n(G) \cong Q_{n+1}(G)$ for all $n \geq n_0$. The eventually stabilized group $Q_{n_0}(G)$ is called the augmentation terminal of G and is denoted by $Q_\infty(G)$.

Define $Q_G = \varinjlim H$ where direct limit is taken over all cyclic subgroups H of G with respect to inclusion maps. Hales has proved in [3] that $Q_\infty(G) \cong Q_G$ for all finite abelian group G. His proof is of computational nature, and no isomorphism between $Q_\infty(G)$ and Q_G is given. Hales also determined the structure of the associated graded ring

$$\mathrm{gr}(\mathbb{Z}[G]) = \bigoplus_{n=0}^{\infty} Q_n(G).$$

One may attempt to construct a natural isomorphism between $Q_\infty(G)$ and Q_G in the following way. Using duality of finite abelian groups, one may view $Q_G = \varprojlim G/H$ where (and from now on) inverse limit is taken over all cyclic quotient groups G/H of G with respect to natural projections. It is simple to see that $Q_n(G)$ is cyclic with order $|G|$ when G is cyclic, hence $Q_G \cong \varprojlim Q_n(G/H)$ for all n. Now let $\phi_n : Q_n(G) \to \varprojlim Q_n(G/H)$ which is induced from natural map.

One may ask

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Question 1. *Is ϕ_n bijective for $n \geq n_0$?*

A related question, raised by C. Popescu and D. Burns, is

Question 2. *Is ϕ_n injective for all n?*

The affirmative answer to Question 2 would give us a simple proof of a conjecture of Gross (cf. [2]) on special values of Abelian L-functions. The conjecture predicts a congruence relation which relates the Stickelberger element in $\mathbb{Z}[G]$ with the class number and the generalized regulator in $\mathbb{Z}[G]$ modulo I_G^n for some suitable n. Both sides of the conjecture behaves functorially with respect to natural projection from G to G/H, and the conjecture is fully proved for cyclic extensions. Therefore, one can say that the conjecture is true up to the kernel of ϕ_n. In fact, the conjecture is fully proved for function fields, and for number field case it is enough to prove the conjecture when G is a 2-group. Therefore we focus on the 2-groups in this paper.

Both questions have affirmative answers when G is cyclic (for trivial reason) and more interestingly when G is elementary (i.e., of prime exponent), as shown by Passi in [5]. Unfortunately, in general both have negative answers – Hayward was the first (cf. [4]) to observe this fact, using a computer program to compute the kernel of ϕ_n. His computation suggests that the size of kernel of ϕ_n is quite small in general. The author finds it odd that the natural map between two isomorphic groups is not an isomorphism.

2. An example

Here we discuss an example which shows that ϕ_n is not injective in general.

Let $G = \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z} \cong \langle\sigma\rangle \times \langle\tau\rangle$ and let $\alpha = \sigma - 1, \beta = \tau - 1$ in $\mathbb{Z}[G]$. Let $\eta = \alpha^8\beta^4 - \alpha^4\beta^8 = \alpha^4\beta^4(\alpha^4 - \beta^4)$. We will show that $\overline{\eta} \neq 0 \in Q_{12}(G)$ and that $\phi_{12}(\overline{\eta}) = 0$.

Hales showed (cf. [3], Corollary 2) that $\mathrm{gr}(\mathbb{Z}[G]) \cong \mathbb{Z}[\alpha,\beta]/J$, where J is generated by 16α, 16β and $2^k(\alpha^{2^{4-k}}\beta^{2^{3-k}} - \alpha^{2^{3-k}}\beta^{2^{4-k}})$ for $k = 0, 1, 2, 3$. Considering modulo 2, we conclude that $\eta \notin J$ hence $\overline{\eta} \neq 0 \in Q_{12}(G)$.

In order to show $\phi_{12}(\overline{\eta}) = 0$, one needs to show that for every projection $\psi : G \to \mathbb{Z}/16\mathbb{Z}$, the induced map $\overline{\psi} : \mathrm{gr}(\mathbb{Z}[G]) \to \mathrm{gr}(\mathbb{Z}/16\mathbb{Z})$ kills $\overline{\eta}$. Choose a generator π of $\mathbb{Z}/16\mathbb{Z}$, and set $\gamma = \pi - 1 \in \mathbb{Z}[G]$. It is easy to see that $\mathrm{gr}(\mathbb{Z}/16\mathbb{Z}) \cong \mathbb{Z}[\gamma]/(16\gamma)$. If ψ sends σ to π^a and τ to π^b, the induced map $\overline{\psi}$ sends α to $a\gamma$ and β to $b\gamma$. Therefore, it eventually boils down to showing that $a^4b^4(a^4 - b^4) \equiv 0 \pmod{16}$ for all $a, b \in \mathbb{Z}$. The congruence relation clearly holds if either a or b is even, and when both a and b are odd, $a^4 \equiv b^4 \equiv 1 \pmod{16}$, hence the relation also holds.

3. A result

Let $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. We show that ϕ_n is injective for all n.

By abuse of notation we will adopt the same notations as in previous section, i.e., $G \cong \langle\sigma\rangle \times \langle\tau\rangle$, $\alpha = \sigma - 1$, $\beta = \tau - 1$ etc. In this case, the result of

Hales states that $\mathrm{gr}(\mathbb{Z}[G]) \cong \mathbb{Z}[\alpha, \beta]/J$ where J is generated by $(\alpha^4\beta^2 - \alpha^2\beta^4)$, $2(\alpha^2\beta - \alpha\beta^2)$, 4α and 4β.

Using the same method as in previous section, it is sufficient to show the following

Claim. *Suppose $f(\alpha, \beta)$ is a homogeneous polynomial of degree n with integer coefficients. If $f(a, b) \equiv 0 \pmod 4$ for all $a, b \in \mathbb{Z}$, then $f(\alpha, \beta) \in J$.*

Assume $n \geq 6$. As our goal is to show that $f \equiv 0 \pmod J$, we have the freedom to modify f by elements in J. Using the relation $\alpha^4\beta^2 \equiv \alpha^2\beta^4 \pmod J$, we may assume that $f(\alpha, \beta) = c_0\beta^n + c_1\alpha\beta^{n-1} + c_2\alpha^2\beta^{n-2} + c_3\alpha^3\beta^{n-3}$ for some $c_i \in \mathbb{Z}, i = 0, 1, 2, 3$.

Letting $(a, b) = (0, 1), (2, 1), (1, 1), (-1, 1)$ respectively, we get

$$c_0 \equiv 0 \pmod 4, \quad c_1 \equiv 0 \pmod 2, \quad c_1 + c_2 + c_3 \equiv c_1 - c_2 + c_3 \equiv 0 \pmod 4.$$

From these equations, we conclude that both c_2 and c_3 are even. Therefore, using the relation $2\alpha^2\beta \equiv 2\alpha\beta^2 \pmod J$ we may assume that $f(\alpha, \beta) = c_1\alpha\beta^{n-1}$, from which the conclusion follows readily.

The essentially same method can be applied individually when $n < 6$ to prove the claim in full generality.

Corollary 1. *The conjecture of Gross is true for extensions with Galois group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.*

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