Bull. Korean Math. Soc. **45** (2008), No. 1, pp. 119–131

CERTAIN GENERALIZATIONS OF G-SEQUENCES AND THEIR EXACTNESS

KEE YOUNG LEE*, MOO HA WOO, AND XUEZHI ZHAO

ABSTRACT. In this paper, we generalize the Gottlieb groups and the related G-sequence of those groups, and present some sufficient conditions to ensure the exactness or non-exactness of G-sequences at some terms. We also give some applications of the exactness or non-exactness of G-sequences. Especially, we show that the non-exactness of G-sequences implies the non-triviality of homotopy groups of some function spaces.

1. Introduction

Some subgroups of homotopy groups were introduced in [3] and [4], and were named later as Gottlieb groups. Their properties and structures turned out to be an interesting and active topic because of the connection with other mathematical fields such as the fixed point theory ([2] and [6]), the rational homotopy theory ([10]) and transformation groups ([15]). Unfortunately, concrete calculations of Gottlieb groups are very difficult, even more than those of homotopy groups. Although they are all homotopy type invariant, Gottlieb groups have no functoriality. Many traditional and effective methods in studying homotopy groups can not be applied to the Gottlieb groups.

After a series works in [9], [13], and [14], the Gottlieb groups, together with other "Gottlieb-like" groups, are successfully arranged in some sequences namely, the G-sequences. It gives us a useful tool to access the Gottlieb groups. Since the homomorphisms in G-sequences are the restrictions of those in homotopy sequences, any G-sequence is half exact, i.e., compositions of consecutive homomorphisms are trivial. Some conditions are known under which the G-sequence is exact, but in general it is not.

This paper presents some results in this direction. We generalize Gottlieb groups and related G-sequences. We present some conditions under which the G-sequence is exact or non-exact at the given terms, rather than the exactness of the whole sequence. Furthermore, the relations amongst the exactness of generalized G-sequences and other topological invariants are addressed.

Received July 23, 2007.
2000 *Mathematics Subject Classification.* Primary 55Q32, 55Q52, 55P62.
Key words and phrases. Gottlieb group, G-sequence, homotopy group.
*This work was supported by a Korea University Grant.

This paper is organized as follows. We give the definitions of generalized Gottlieb groups and related G-sequence in Section 2. We bring focus into the dependence of the generalized Gottlieb groups on the choice of base points in the pairs of spaces involved in Section 3. In Section 4, we present some sufficient conditions to ensure the exactness or non-exactness of G-sequence at some terms. Some general applications of our conditions for exactness and non-exactness will be given. In the meanwhile, we get a piece of information of homotopy groups of some function spaces.

In this paper, all spaces are finite CW-complexes, all topological pairs are CW-pairs and all subspaces mentioned contain the same base point as their total spaces. We denote the n-dimensional cube by I^n, its boundary by ∂I^n and the union of all $n-1$ faces of I^n except for the initial face by J^{n-1}.

2. Generalized G-sequences

In this section, we shall generalize the original Gottlieb groups in [4] and the G-sequence introduced in [14].

Let (X, A) and (Y, B) be two pairs of non-empty spaces. We write X^Y for the space consisting of continuous maps from Y to X with compact open topology, and $(X, A)^{(Y,B)}$ for the subspace of X^Y consisting of relative maps from (Y, B) to (X, A). We then have a natural pair of spaces $(X^Y, (X, A)^{(Y,B)})$. Pick a point $b_0 \in B$, $g_0 \in (X, A)^{(Y,B)}$ and set $g_0(b_0) = a_0$. We choose b_0, g_0 and a_0 as base points of (Y, B), $(X^Y, (X, A)^{(Y,B)})$ and (X, A) respectively. Through the whole of this paper, we always choose base points in this manner. Then there is a natural based map:

$$\omega^{b_0} : (X^Y, (X, A)^{(Y,B)}, g_0) \to (X, A, a_0)$$

which is given by $\omega^{b_0}(g) = g(b_0)$ for any $g \in X^Y$, and is said to *the evaluation map at* b_0. Thus, we have the following commutative diagram: [Diagram 1]

$$\begin{array}{ccccccccc}
\xrightarrow{\partial_*} & \pi_n((X,A)^{(Y,B)}, g_0) & \xrightarrow{i_*} & \pi_n(X^Y, g_0) & \xrightarrow{j_*} & \pi_n(X^Y, (X,A)^{(Y,B)}, g_0) & \xrightarrow{\partial_*} \\
& \omega_*^{b_0} \downarrow & & \omega_*^{b_0} \downarrow & & \omega_*^{b_0} \downarrow & \\
\xrightarrow{\partial_*} & \pi_n(A, a_0) & \xrightarrow{i_*} & \pi_n(X, a_0) & \xrightarrow{j_*} & \pi_n(X, A, a_0) & \xrightarrow{\partial_*}
\end{array}$$

where the two horizontal sequences are homotopy exact sequences of the based pairs of spaces $(X^Y, (X, A)^{(Y,B)}, g_0)$ and (X, A, a_0) respectively. If we consider the images of these evaluation homomorphisms $\omega_*^{b_0}$, we get a half exact sequence: [Sequence 2]

$$\to \omega_*^{b_0}\pi_n((X,A)^{(Y,B)}, g_0) \to \omega_*^{b_0}\pi_n(X^Y, g_0) \to \omega_*^{b_0}\pi_n(X^Y, (X,A)^{(Y,B)}, g_0) \to$$

We call this sequence as *G-sequence for $(X^Y, (X, A)^{(Y,B)}, g_0)$ at b_0*.

The first G-sequence defined in [14], which is said to be *the G-sequence for the triple (X, A, a_0)*, is just the G-sequence for $(X^A, (X, A)^{(A,A)}, i_A) =$

(X^A, A^A, i_A) at a_0, where i_A is the inclusion from A to X. Moreover, we have $G_n(X,A,a_0) = \omega_*^{b_0}\pi_n(X^A, i_A)$, $G_n^{Rel}(X,A,a_0) = \omega_*^{b_0}\pi_n(X^A,(X,A)^{(A,A)}, i_A)$.

The G-sequence for a map $f : A \to X$ was introduced in [12]. Let Z_f denote the mapping cylinder of the map f, i.e.,

$$Z_f = (A \times I \cup X)\big/(a,0) \sim f(a).$$

The space A is regarded as a subspace according to the embedding $i_{Z_f}^A : A \to Z_f$ given by $i_{Z_f}^A(a) = (a,1)$. Since $(Z_f^A, (Z_f, A\times 1)^{(A,A)}, i_{Z_f}^A) = (Z_f^A, (A\times 1)^A, i_{Z_f}^A)$, the G-sequence for $(Z_f^A, (Z_f, A\times 1)^{(A,A)}, i_{Z_f}^A)$ at $a_0 \in A$ is the same as the G-sequence for the map $f : A \to X$ (see [12, Theorem 3.3 and Corollary 3.4]).

Finally, we illustrate a property of the relative term in G-sequence. For any subset Y' of Y, there is a natural map $r^{Y,Y'} : X^Y \to X^{Y'}$, which is just the map restriction. Moreover, for a triple (Y, Y', B) , we have a relative map

$$r^{Y,Y'} : (X^Y, (X,A)^{(Y,B)}) \to (X^{Y'}, (X,A)^{(Y',B)}).$$

Clearly, $r^{Y,Y'}$ is not surjective in general. But we have

Proposition 2.1. *Let (X,A) and (Y,B) be finite CW- pairs, and let Y' be a sub-complex of Y with $B \subset Y' \subset Y$. Then, for any $f : (Y,B) \to (X,A)$ as a base point of the pair $(X^Y, (X,A)^{(Y,B)})$, the homomorphism*

$$r_*^{Y,Y'} : \pi_n(X^Y, (X,A)^{(Y,B)}, f) \to \pi_n(X^{Y'}, (X,A)^{(Y',B)}, f|_{Y'})$$

induced by the natural map $r^{Y,Y'}$ is surjective for each positive integer n.

Proof. Pick an arbitrary element in $\pi_n(X^{Y'}, (X,A)^{(Y',B)}, f|_{Y'})$, which is represented by a map $\alpha : (I^n, \partial I^n, J^{n-1}) \to (X^{Y'}, (X,A)^{(Y',B)}, f|_{Y'})$. Then α gives rise to a map $H : (Y' \times I^n, B \times \partial I^n) \to (X, A)$, which is defined by $H(y', s) = \alpha(s)(y')$ for all $y' \in Y'$ and $s \in I^n$. Note that $H(y', s) = f(y')$ for all $y' \in Y'$ and $s \in J^{n-1}$.

Now, we can find an extension $H' : (Y' \times I^n) \cup (Y \times J^{n-1}) \to X$ of H by defining $H(y,s) = f(y)$ for all $y \in Y - Y'$ and $s \in J^{n-1}$. Since $Y' \times I^n \cup Y \times J^{n-1} = Y \times I^{n-1} \times 1 \cup (Y' \times I^{n-1} \cup Y \times \partial I^{n-1}) \times I$ and $Y' \times I^{n-1} \cup Y \times \partial I^{n-1}$ is a sub-complex of $Y \times I^{n-1}$, there is an extension $\bar{H} : (Y \times I^n, B \times \partial I^n) \to (X, A)$ of H' by the absolute homotopy extension property.

Define a map $\beta : I^n \to X^Y$ by $\beta(s) = H(\cdot, s)$ for all $s \in I^n$. It is easy to check that β maps $(I^n, \partial I^n, J^{n-1})$ into $(X^Y, (X,A)^{(Y,B)}, f)$, and that $\alpha = r^{Y,Y'} \circ \beta$. Thus the proof is complete. □

From the following commutative diagram

$$\begin{array}{ccc} (X^Y, (X,A)^{(Y,B)}, f) & \xrightarrow{r^{Y,Y'}} & (X^{Y'}, (X,A)^{(Y',B)}, f|_{Y'}) \\ \omega^{b_0} \downarrow & & \omega^{b_0} \downarrow \\ (X, A, a_0) & = & (X, A, a_0) \end{array}$$

where $b_0 \in B$ and $f(b_0) = a_0$, we have the following corollary.

Corollary 2.2. *Let (X,A), (Y,B) and Y' be the same as in Proposition 2.1. Then for each $f \in (X,A)^{(Y,B)}$ and each $b_0 \in B$,*

$$\omega_*^{b_0}(\pi_n(X^Y,(X,A)^{(Y,B)},f)) = \omega_*^{b_0}(\pi_n(X^{Y'},(X,A)^{(Y',B)},f|_{Y'})).$$

In [14], a relative Gottlieb group $G_n^{Rel}(X,A)$ is equal to

$$\omega_*^{a_0}\pi_n(X^X,(X,A)^{(X,A)},id_X)$$

([14, Theorem 2]). Applying Corollary 2.2 to the case $(X,A) = (Y,B)$ and $Y' = B$, we have

$$G_n^{Rel}(X,A) = \omega_*^{a_0}\pi_n(X^A,(X,A)^{(A,A)},id_A) = \omega_*^{a_0}\pi_n(X^A,A^A,id_A).$$

So, Proposition 2.1 is a generalization of [9, Lemma 3.1].

3. Dependence on base points

In this section, we shall discuss the dependence of the groups on the base points b_0 and g_0 in the G-sequence defined in Section 2 when the base point a_0 of the pair (X,A) is fixed. It sounds an elementary topic, but it is in fact related to the root theory. Such a connection seems to be interesting.

As the remark in the beginning of section 2, we may assume that $g_0^{-1}(a_0)$ is always non-empty. When we choose base points, the condition may not be strong, since we have the following proposition.

Proposition 3.1. *Let $g_0 \in (X,A)^{(Y,B)}$. If the image $g_0(B)$ of B under g_0 meets the path component of A containing a_0, then there is a map g_0' in the path-component of $(X,A)^{(Y,B)}$ containing g_0 and a point $b_0 \in B$ such that $g_0'(b_0) = a_0$.*

Proof. Since the image $g_0(B)$ of B under g_0 meets the path component A_0 of A containing a_0, there is a point $a_1 \in g_0(B) \cap A_0$. Hence, there is a point $b_0 \in B$ with $g_0(b_0) = a_1$ and a path α from a_0 to a_1 in A_0. We then obtain a map $H : (B \times \{0\}) \cup \{b_0\} \times I \to A$ given by $H(b,0) = g_0(b)$ and $H(b_0,t) = \alpha(1-t)$ for all $b \in B$ and $t \in I$. Using the absolute homotopy extension property twice, we can find an extension $H' : (Y \times I, B \times I) \to (X,A)$ of H . The map $H'(\cdot,1)$ is the desired map g_0'. □

First, we prove the following simple fact.

Proposition 3.2. *The final two $\omega_*^{b_0}$'s in Diagram 1 induced by the evaluation map at b_0 are all surjective, i.e.,*

$$\omega_*^{b_0}(\pi_0((X,A)^{(Y,B)},g_0)) = \pi_0(A,a_0)$$

and

$$\omega_*^{b_0}(\pi_0(X^Y,g_0)) = \pi_0(X,a_0).$$

Proof. Consider the last part of Diagram 1:

$$\begin{array}{ccccc} \cdots \xrightarrow{\partial_*} & \pi_0((X,A)^{(Y,B)},g_0) & \xrightarrow{i_*} & \pi_0(X^Y,g_0) \\ & \omega_*^{b_0}\downarrow & & \omega_*^{b_0}\downarrow \\ \cdots \xrightarrow{\partial_*} & \pi_0(A,a_0) & \xrightarrow{i_*} & \pi_0(X,a_0) \end{array}$$

By definition, $\pi_0(A,a_0)$ is a based set consisting of all components of A with base element the component containing a_0. Pick an arbitrary component A_m and a constant map c_{a_m} from Y to X with value a_m in A_m. Then c_{a_m} is an element in $(X,A)^{(Y,B)}$. It is obvious that $\omega^{b_0}(c_{a_m}) = a_m$. It follows that as an element of $\pi_0(A,a_0)$, $A_m = \omega_*^{b_0}(c_{a_m})$, where c_{a_m} is regarded as an element of $\pi_0((X,A)^{(Y,B)},g_0)$. Thus, we obtain that $\omega_*^{b_0}(\pi_0((X,A)^{(Y,B)},g_0)) = \pi_0(A,a_0)$.

By using the same argument, we can prove that $\omega_*^{b_0}\pi_0(X^Y,g_0) = \pi_0(X,a_0)$. □

By Proposition 3.2, the final two terms in G-sequence, which are based sets, not groups, are independent on the choice of g_0 and b_0. In general, the groups in the G-sequence (Sequence 2) are dependent on the choice of g_0 and b_0.

Example. Let $X = Y = S^1 \vee S^1$ be a wedge of two circles, and $A = B = \{a_0\}$ be the wedge point. Write $c_{a_0} : X \to A$ for the constant map at a_0. Then we have that $\omega_*^{a_0}(\pi_1(X^Y, c_{a_0})) = \pi_1(X,a_0)$ (see [16]). Since $\omega_*^{a_0}(\pi_1(X^Y, id_X))$ is contained in the center of $\pi_1(X,a_0)$ and $\pi_1(X,a_0)$ is a free group F_2 with two generators, it follows that $\omega_*^{a_0}(\pi_1(X^Y, id_X))$ is a trivial group.

From this example, we know that the groups in the G-sequence (Sequence 2) maybe quite different if the base points b_0 and g_0 come from different components.

Proposition 3.3. *Let g_0 and g_1 be in the same component of $(X,A)^{(Y,B)}$. If b_0 and b_1 are in the same path component of B and $g_0(b_0) = g_1(b_1)$, then we obtain following three types of isomorphisms for each positive integer n:*

$$\omega_*^{b_0}(\pi_n((X,A)^{(Y,B)},g_0)) \cong \omega_*^{b_1}(\pi_n((X,A)^{(Y,B)},g_1)),$$

$$\omega_*^{b_0}(\pi_n(X^Y,g_0)) \cong \omega_*^{b_1}(\pi_n(X^Y,g_1)),$$

$$\omega_*^{b_0}(\pi_n(X^Y,(X,A)^{(Y,B)},g_0)) \cong \omega_*^{b_1}(\pi_n(X^Y,(X,A)^{(Y,B)},g_1)).$$

Proof. Since g_0 and g_1 are in the same component of $(X,A)^{(Y,B)}$, $g_0, g_1 : (Y,B) \to (X,A)$ are relatively homotopic. Thus there is a relative homotopy $H : (Y \times I, B \times I) \to (X,A)$ from g_0 to g_1. As a path in $(X,A)^{(Y,B)}$, $H(\cdot,t)$ induces a classical isomorphism $\theta_{\{H(\cdot,t)\}} : \pi_n(X^Y,g_0) \to \pi_n(X^Y,g_1)$. Pick a path $\alpha : I \to B$ from b_0 to b_1. Then α determines a homotopy $\omega^{\alpha(t)} : X^Y \to X$ from ω^{b_0} and ω^{b_1}. Note that $\omega^{\alpha(t)}(H(\cdot,1)) = H(\alpha(t),1)$. Let $a_0 = g_0(b_0) = g_1(b_1)$.

Then we obtain a commutative diagram:

$$\begin{array}{ccccc}
\pi_n(X^Y, g_0) & \xrightarrow{\theta_{\{H(\cdot,t)\}}} & \pi_n(X^Y, g_1) & \xrightarrow{=} & \pi_n(X^Y, g_1) \\
\omega_*^{b_0} \downarrow & & \omega_*^{b_0} \downarrow & & \omega_*^{b_1} \downarrow \\
\pi_n(X, a_0 = g_0(b_0)) & \xrightarrow{\theta_{\{H(b_0,t)\}}} & \pi_n(X, g_1(b_0)) & \xrightarrow{\theta_{\{H(\alpha(t),1)\}}} & \pi_n(X, a_0 = g_1(b_1))
\end{array}$$

The commutativity of the left square comes from the functoriality of the isomorphism $\theta_{\{\}}$. The commutativity of the right square is just the relation between the homomorphisms induced from homotopic maps. Since the path product $\{H(b_0,t)\} \cdot \{H(\alpha(t),1)\}$ and $\{H(\alpha(t),t)\}$ are homotopic keeping end points fixed, $\theta_{\{H(b_0,t)\}}\theta_{\{H(\alpha(t),1)\}} = \theta_{\{H(\alpha(t),t)\}}$. Thus we have the following commutative diagram

$$\begin{array}{ccc}
\pi_n(X^Y, g_0) & \xrightarrow{\theta_{\{H(\cdot,t)\}}} & \pi_n(X^Y, g_1) \\
\omega_*^{b_0} \downarrow & & \omega_*^{b_1} \downarrow \\
\pi_n(X, a_0) & \xrightarrow{\theta_{\{H(\alpha(t),t)\}}} & \pi_n(X, a_0)
\end{array}$$

It follows that $\omega_*^{b_0}(\pi_n(X^Y, g_0))$ and $\omega_*^{b_1}(\pi_n(X^Y, g_1))$ are isomorphic subgroups in $\pi_n(X, a_0)$.

Similarly, we can prove that

$$\omega_*^{b_0}(\pi_n((X,A)^{(Y,B)}, g_0)) \cong \omega_*^{b_1}(\pi_n((X,A)^{(Y,B)}, g_1))$$

by using the commutative diagram:

$$\begin{array}{ccc}
\pi_n((X,A)^{(Y,B)}, g_0) & \xrightarrow{\theta_{\{H(\cdot,t)\}}} & \pi_n((X,A)^{(Y,B)}, g_1) \\
\omega_*^{b_0} \downarrow & & \omega_*^{b_1} \downarrow \\
\pi_n(A, a_0) & \xrightarrow{\theta_{\{H(\alpha(t),t)\}}} & \pi_n(A, a_0)
\end{array}$$

and that

$$\omega_*^{b_0}(\pi_n(X^Y, (X,A)^{(Y,B)}, g_0)) \cong \omega_*^{b_1}(\pi_n(X^Y, (X,A)^{(Y,B)}, g_1))$$

by using the commutative diagram:

$$\begin{array}{ccc}
\pi_n(X^Y, (X,A)^{(Y,B)}, g_0) & \xrightarrow{\theta_{\{H(\cdot,t)\}}} & \pi_n(X^Y, (X,A)^{(Y,B)}, g_1) \\
\omega_*^{b_0} \downarrow & & \omega_*^{b_1} \downarrow \\
\pi_n(X, A, a_0) & \xrightarrow{\theta_{\{H(\alpha(t),t)\}}} & \pi_n(X, A, a_0)
\end{array}$$

□

By Proposition 3.2, the property $g(B)$ meets a component of A is preserved by elements in a path-component of $(X,A)^{(Y,B)}$. Thus, by Proposition 3.3, we obtain

Theorem 3.4. *The isomorphism types of groups in the G-sequence do not depend on the choice of base points on the path-components of $(X,A)^{(Y,B)}$ and the path components of B.*

Here we recall some basic concepts in root theory (see [1] or [8] for more details). Given a map $g : Y \to X$ and a point $a_0 \in X$, a point $y \in Y$ is said to be *a root of* g *at* a_0 if $g(y) = a_0$. Let K be a normal subgroup of $\pi_1(X, a_0)$. All the roots of g at a_0 are split into equivalence classes, namely mod K root classes, in the sense: two roots y' and y'' are in the same mod K root class if there is a path β in Y from y' to y'' such that $g \circ \beta$, a loop in X at a_0, represents an element in K. There is a well-defined correspondence between the sets of root classes of homotopic maps. Let $H : Y \times I \to X$ be a homotopy from g_0 to g_1. We called a mod K root class of containing a root y_0 of g_0 at a_0 and a mod K root class containing a root y_1 of g_1 at a_0 are *H-related* or y_0 and y_1 are *H-related mod K* if there is a path γ in Y from y_0 to y_1 such that $\{H(\gamma(t), t)\}$ is an element in K. In the case that K is trivial group, we call just y_0 and y_1 are *H-related.* The elements of $\pi_1(X, a_0)$ operate on $\pi_n(X, a_0)$ as a group of automorphisms in a standard way. It is well-known fact that the set of all elements of $\pi_1(X, a_0)$ which operate trivially on $\pi_n(X, a_0)$ for all positive integer n is a subgroup ,which will be denoted as $P_1(X, a_0)$, and it is contained in the center of $\pi_1(X, a_0)$.

Lemma 3.5. *Let $g_0, g_1 : Y \to X$ be homotopic maps by a homotopy H : $Y \times I \to X$ and b_0 and b_1 be two roots of g_0 and g_1 at a_0 respectively. If b_0 and b_1 are H-related mod $P_1(X, a_0)$, then $\omega_*^{b_0}(\pi_n(X^Y, g_0)) = \omega_*^{b_1}(\pi_n(X^Y, g_1))$.*

Proof. Since the mod $P_1(X, a_0)$ root classes containing b_0 and b_1 are H-related by the hypothesis, there is a path $\alpha : I \to Y$ from b_0 to b_1 such that the diagonal path $H(\alpha(\cdot), \cdot)$ of α under H determines an element in $P_1(X, a_0)$. Consider the following commutative diagram (see Proposition 3.3):

$$\begin{array}{ccc}
\pi_n(X^Y, g_0) & \xrightarrow{\theta_{\{H(\cdot,t)\}}} & \pi_n(X^Y, g_1) \\
\omega_*^{b_0} \downarrow & & \omega_*^{b_1} \downarrow \\
\pi_n(X, a_0) & \xrightarrow{\theta_{\{H(\alpha(t),t)\}}} & \pi_n(X, a_0).
\end{array}$$

Since $H(\alpha(0), 0) = g_0(\alpha(0)) = g_0(b_0) = a_0$, and $H(\alpha(1), 1) = g_1(\alpha(1)) = g_1(b_1) = a_0$, the path $H(\alpha(\cdot), \cdot)$ is a loop at a_0. The isomorphism $\theta_{\{H(\alpha(\cdot),\cdot)\}}$ is just the action of the element $\{H(\alpha(\cdot), \cdot)\}$ in $\pi_1(X, a_0)$ on $\pi_n(X, a_0)$. Since $\{H(\alpha(\cdot), \cdot)\}$ is an element of $P_1(X, a_0)$, $\theta_{\{H(\alpha(\cdot),\cdot)\}}$ is the identity by the definition of $P_1(X, a_0)$. $\square$

Lemma 3.6. *Let $g_0, g_1 : (Y, B) \to (X, A)$ be relatively homotopic maps by a homotopy $H : (Y \times I, B \times I) \to (X, A)$, and b_0 and b_1 two roots of $g_0|_B$ and $g_1|_B$ at a_0 respectively. If b_0 and b_1 are $H|_{B\times I}$-related, then*

$$\omega_*^{b_0}(\pi_n((X, A)^{(Y,B)}, g_0)) = \omega_*^{b_1}(\pi_n((X, A)^{(Y,B)}, g_1)),$$
$$\omega_*^{b_0}(\pi_n(X^Y, g_0)) = \omega_*^{b_1}(\pi_n(X^Y, g_1)),$$
$$\omega_*^{b_0}(\pi_n(X^Y, (X, A)^{(Y,B)}, g_0)) = \omega_*^{b_1}(\pi_n(X^Y, (X, A)^{(Y,B)}, g_1)).$$

Proof. Since b_0 and b_1 are $H|_{B\times I}$-related, the diagonal path $H(\alpha(\cdot),\cdot)$ is homotopic to constant path in A. Recall the proof of Proposition 3.3. Since the isomorphism $\theta_{\{H(\alpha(t),t)\}}$ is the identity, the proof is complete. □

Theorem 3.7. *Given a path component B_0 of B and a component M of $(X,A)^{(Y,B)}$, the groups in the G-sequence for $(X^Y,(X,A)^{(Y,B)},g_0)$ at b_0, as subgroups of those in homotopy exact sequence of (X,A,a_0), are independent of the choice of $b_0 \in B_0$ and $g_0 \in M$ provided one of the following conditions is hold:*

(1) *the based spaces (X,a_0) and (A,a_0), and the based pair (X,A,a_0) are all simple;*

(2) *one map, hence for all maps, in M induces a surjective homomorphism from $\pi_1(Y,b_0)$ to $\pi_1(X,a_0)$ and a surjective homomorphism from $\pi_1(B_0,b_0)$ to $\pi_1(A,a_0)$.*

Proof. Suppose (1) holds. By the definition of simple spaces, the action of fundamental group $\pi_1(X,a_0)$ on high dimensional homotopy groups $\pi_n(X,a_0)$ $(n>0)$, and the action of fundamental group $\pi_1(A,a_0)$ on high dimensional homotopy groups $\pi_n(A,a_0)$ and $\pi_n(X,A,a_0)$ $(n>0)$ are all trivial. By Proposition 3.3, we have the result.

Suppose (2) holds. Recall from [8, p.133] that a root class of g_0 corresponds to the coset of $\mathrm{im}g_{0*}$ in $\pi_1(X,a_0)$. Since g_{0*} is onto and hence there is one coset of $\mathrm{im}g_{0*}$ in $\pi_1(X,a_0)$, g_0 as well as any other maps in M has only one root class. Similarly, $g_0|_{B_0}$ has only one root class. By Lemma 3.6, we have the conclusion. □

4. Criterions for the exactness and applications

Our purpose in this section is to give some conditions under which the G-sequence is exact at some terms, instead of those for the exactness of the whole G-sequence and present some applications.

By Proposition 3.1, the final two terms in the G-sequence are nothing but those in the homotopy exact sequence for the pair (X,A). Thus we have

Proposition 4.1. *The G-sequence for $(X^Y,(X,A)^{(Y,B)},g_0)$ at b_0 is exact at final term.*

About the exactness of the term priori to the final term, we have

Proposition 4.2. *Let B_0 be the component of B containing b_0, X_0 the component of X containing a_0. Then the G-sequence for $(X^Y,(X,A)^{(Y,B)},g_0)$ at b_0 is exact at the term $\omega_*^{b_0}(\pi_0((X,A)^{(Y,B)},g_0))$ if and only if for each component A_k of A lying in X_0, $g_0|_{B_0}: B_0 \to X$ is homotopic to a map from B_0 to A_k.*

Proof. Let us prove "if" part first. From Proposition 3.1, we know

$$\omega_*^{b_0}(\pi_0((X,A)^{(Y,B)},g_0)) = \pi_0(A,a_0).$$

By definition, $\pi_0(A, a_0)$ is a based set consisting of all components of A with the component containing a_0 as the base element, and $\pi_0(X, a_0)$ is a based set consisting of all components of X with base element X_0. If an element of $\pi_0(A, a_0)$ is sent to the base element X_0 by $i_* : \pi_0(A, a_0) \to \pi_0(X, a_0)$, it is a component, say A_k, lying in X_0. By the assumption, there is a homotopy $H' : B_0 \times I \to X$ from $g_0|_{B_0}$ to a map in $A_k^{B_0}$. We can extend H' to $H'' : B \times I \to X$ by defining $H''(b,t) = g_0(b)$ for all $b \in B - B_0$. By the homotopy extension property, we obtain an extension $H : Y \times I \to X$ of H'' such that $H|_{Y\times 0} = g_0$. It is evident that $H(b, 1) \in A$ for all $b \in B$. Thus, its adjoint $\widetilde{H}$ can be regarded as an element in that $\pi_1(X^Y, (X, A)^{(Y,B)}, g_0))$. Since $\omega_*^{b_0}(\widetilde{H})(t) = H(b_0, t)$ is a path from $g_0(b_0) = a_0$ to $H(b_0, 1)$ which is a point in A_k, we have that $\partial_*(\omega_*^{b_0}(\widetilde{H})) = H(b_0, 1)$, i.e., the element in $\pi_0(A, a_0)$ represented by A_k. Hence, A_k lies in the image of the homomorphism

$$\partial_* : \omega_*^{b_0}(\pi_1(X^Y, (X, A)^{(Y,B)}, g_0)) \to \pi_0(A, a_0).$$

Let us show the "only if" part. Pick an arbitrary component A_k of A which is contained in X_0. Then it is an element of the $\ker(i_* : \pi_0(A, a_0) \to \pi_0(X, a_0))$. Since the G-sequence is exact at

$$\omega_*^{b_0}(\pi_0((X, A)^{(Y,B)}, g_0)) = \pi_0(A, a_0),$$

A_k as an element of $\pi_0(A, a_0)$, lies in the

$$\mathrm{Im}(\partial_* : \omega_*^{b_0}(\pi_1(X^Y, (X, A)^{(Y,B)}, g_0)) \to \pi_0(A, a_0)),$$

i.e., there is a map $H : Y \times I \to X$ with $H(y, 0) = g_0(y)$ for all $y \in Y$ and $H(b, 1) \in A$ for all $b \in B$ such that $\partial_*(\omega^{b_0}(H)) = \partial_*(\{H(b_0, t)\}) = H(b_0, 1) \in A_k$. Thus, $H|_{B_0 \times I}$ is a desired homotopy. □

Corollary 4.3. *The G-sequence is exact at $\omega_*^{b_0}(\pi_0((X, A)^{(Y,B)}, g_0))$, if the component of X containing a_0 meets only one component of A, especially, if A has only one path component.*

Here we need a lemma in algebra.

Lemma 4.4. *Consider the following commutative diagram of groups and homomorphisms:*

$$\begin{array}{ccccc} A_1 & \xrightarrow{i} & A_2 & \xrightarrow{j} & A_3 \\ \phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow \\ A_1' & \xrightarrow{i'} & A_2' & \xrightarrow{j'} & A_3' \end{array}$$

where ϕ_1, ϕ_2 and ϕ_3 are surjective, and the top row is exact at A_2, i.e., $\mathrm{Im} i = \mathrm{Ker} j$. *Then we have:*

(1) *the bottom row is exact at A_2' if ϕ_3 is injective on* $\mathrm{Im} j$;

(2) *the bottom row is not exact at A_2' if ϕ_3 is non-injective on* $\mathrm{Im} j$ *and if ϕ_2 is injective.*

Proof. (1) Pick an element a_2' in $\ker j'$. Since ϕ_2 is surjective, we may assume that $a_2' = \phi_2(a_2)$ for $a_2 \in A_2$. By the commutativity of the diagram, we have $\phi_3(j(a_2)) = j'(\phi_2(a_2)) = j'(a_2') = 0$. Because ϕ_3 is injective on $\mathrm{Im} j$, $j(a_2) = 0$. The exactness of the top row implies that $a_2 \in \mathrm{Im} i$, i.e., $a_2 = i(a_1)$ for some $a_1 \in A_1$. It follows that $a_2' = \phi_2(a_2) = \phi_2(i(a_1)) = i'(\phi_1(a_1))$. We then obtain that $\ker j' \subset \mathrm{Im} i'$. By the commutativity of the diagram, it is easy to check that $\ker j' \supset \mathrm{Im} i'$.

(2) Suppose that ϕ_3 is not injective on $\mathrm{Im} j$. Then there is an element a_2 in A_2 such that $j(a_2) \neq 0$, but $\phi_3 j(a_2)) = 0$. Consider the element $\phi_2(a_2) \in A_2'$, which lies in $\ker j'$ because $j'(\phi_2(a_2)) = \phi_3(j(a_2)) = 0$. We claim that $\phi_2(a_2) \notin \mathrm{Im} i'$.

Assume that $\phi_2(a_2) = i'(a_1')$ for some $a_1' \in A_1'$. We would have $\phi_2(a_2) = i'(\phi_1(a_1))$ for some $a_1 \in A_1$ because ϕ_1 is surjective. Hence, $\phi_2(a_2) = \phi_2(i(a_1))$. As ϕ_2 is injective, $a_2 = i(a_1) \in \mathrm{Im} i$. The exactness of top row would implies that $j(a_2) = j(i(a_1)) = 0$. This is a contradiction. It follows that the bottom row is non-exact at A_2'. □

By Lemma 4.4, we obtain immediately that

Theorem 4.5. *If all evaluation homomorphisms are injective on the image of i_*, or j_* or ∂_* depending on the domain of ω_* in Diagram 1, then the G-sequence for $(X^Y, (X, A)^{(Y,B)}, g_0)$ at b_0 is exact.*

Corollary 4.6. *If all homomorphisms induced on homotopy by the evaluation maps concerned are injective, then the G-sequence for $(X^Y, (X, A)^{(Y,B)}, g_0)$ at b_0 is exact.*

On the other hand, we have

Theorem 4.7. *If an evaluation homomorphism $\omega_*^{b_0}$ in Diagram 1 is surjective, then the G-sequence for $(X^Y, (X, A)^{(Y,B)}, g_0)$ at b_0 is exact at the term next to that containing the image of this $\omega_*^{b_0}$.*

Proof. There are three types of evaluation homomorphism in Diagram 1. We give a proof for one type, the proofs for other two types are similar.

Consider Diagram 1. Suppose

$$\omega_*^{b_0} : \pi_n((X, A)^{(Y,B)}, g_0) \to \pi_n(A, a_0)$$

is surjective. We shall prove the G-sequence for $(X^Y, (X, A)^{(Y,B)}, g_0)$ at b_0 is exact at $\omega_*^{b_0}(\pi_n(X^Y, g_0))$. Since the G-sequence is already half exact, it is sufficient to show that the kernel of

$$j_* : \omega_*^{b_0}(\pi_n(X^Y, g_0)) \to \omega_*^{b_0}(\pi_n(X^Y, (X, A)^{(Y,B)}, g_0))$$

is contained in the image of

$$i_* : \omega_*^{b_0}(\pi_n((X, A)^{(Y,B)}, g_0)) \to \omega_*^{b_0}(\pi_n(X^Y, g_0)).$$

By assumption, the latter is just $i_*(\pi_n(A, a_0))$ and by the exactness of the bottom sequence in Diagram 1, we have

$$\text{Ker}j_*|_{\omega_*^{b_0}(\pi_n(X^Y, g_0))} \subset \text{Ker}j_* \subset \text{Im}i_*.$$

□

Recall that a space X is said to be a *Gottlieb space* if $\omega_*^{b_0}(\pi_n(X^X, id_X)) = \pi_n(X, x_0)$ for all positive integer n. Note that

$$\omega_*^{b_0}(\pi_n(X^X, id_X)) \subset \omega_*^{b_0}(\pi_n(X^A, i_A))$$

for any $A \subset X$. By Theorem 4.7, we obtain a necessary condition for a space to be a Gottlieb space.

Corollary 4.8. *If X is a Gottlieb space, then the G-sequence for the pair (X, A) is exact at $G_n^{Rel}(X, A)$ for each subspace A of X and each positive integer n.*

Especially, we have

Corollary 4.9. *If X is a Jiang space, i.e., $\omega_*^{b_0}(\pi_1(X^X, id_X)) = \pi_1(X, x_0)$ then the G-sequence for the pair (X, A) is exact on $G_1^{Rel}(X, A)$ for each subspace A of X.*

Theorem 4.10. *If X is aspherical, Then the G-sequence for pair (X, A) for pair(X, A) exact at all terms except for three: $G_1(X, A)$, $G_1^{Rel}(X, A)$ and $G_0(A)$.*

Proof. In fact, the exactness at the terms before $G_2(X, A)$ was already proved in [9]. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\pi_2(X^A, i_A) & \xrightarrow{j_*} & \pi_2(X^A, A^A, id_A) & \xrightarrow{\partial_*} & \pi_1(A^A, id_A) & \xrightarrow{i_*} & \pi_1(X^A, i_A) \\
\omega_*^{a_0} \downarrow & & \omega_*^{a_0} \downarrow & & \omega_*^{a_0} \downarrow & & \omega_*^{a_0} \downarrow \\
G_2(X, A,) & \xrightarrow{j_*} & G_2^{Rel}(X, A) & \xrightarrow{\partial_*} & G_1(A) & \xrightarrow{i_*} & G_1(X, A) \\
\cap \downarrow & & \cap \downarrow & & \cap \downarrow & & \cap \downarrow \\
\pi_2(X, a_0) & \xrightarrow{j_*} & \pi_2(X, A, a_0) & \xrightarrow{\partial_*} & \pi_1(A, a_0) & \xrightarrow{i_*} & \pi_1(X, a_0)
\end{array}$$

Since $\pi_2(X) = 0$, the exactness at $G_2(X, A) = 0$ is obvious. By the exactness of the bottom sequence, we have that $\partial_* : \pi_2(X, A, a_0) \to \pi_1(A, a_0)$ is injective. Hence, the G-sequence in the middle is exact at $G_2^{Rel}(X, A)$. By [5, Lemma 2], we know $\omega_*^{a_0} : \pi_1(X^A, i_A) \to \pi_1(X, a_0)$ is injective. From Lemma 4.4, the G-sequence is exact on $G_1(A)$. The exactness at final two terms was proved in general in Proposition 4.1. □

Theorem 4.11. *If there is a map $f : A \to X$ such that the G-sequence of f is non-exact at $G_n^{Rel}(Z_f, A)$, then $\pi_{n-1}(A^A, id_A)$ is non-trivial, where Z_f is the mapping cylinder of f.*

Proof. Pick a base point a_0 and consider the following commutative diagram

$$\begin{array}{ccccc}
\pi_n(Z_f^A, i_{Z_f}^A) & \xrightarrow{j_*} & \pi_n(Z_f^A, A^A, i_{Z_f}^A) & \xrightarrow{\partial_*} & \pi_{n-1}(A^A, id_A) \\
\omega_*^{a_0}\downarrow & & \omega_*^{a_0}\downarrow & & \omega_*^{a_0}\downarrow \\
G_n(Z_f, A) & \xrightarrow{j_*} & G_n^{Rel}(Z_f, A) & \xrightarrow{\partial_*} & G_{n-1}(A) \\
\cap\downarrow & & \cap\downarrow & & \cap\downarrow \\
\pi_n(Z_f, a_0) & \xrightarrow{j_*} & \pi_n(Z_f, A, a_0) & \xrightarrow{\partial_*} & \pi_{n-1}(A, a_0)
\end{array}$$

The top and bottom rows are parts of the homotopy exact sequence of (Z_f^A, A^A) and (Z_f, A) respectively, and the middle one is just a part of G-sequence of f. If $\pi_{n-1}(A^A, id_A)$ is trivial, then

$$\omega_*^{a_0} : \pi_{n-1}(A^A, id_A) \to G_{n-1}(A) = \omega_*^{a_0}(\pi_{n-1}(A^A, id_A))$$

is trivial homomorphism between trivial groups, and hence is injective. By Lemma 4.4, the G-sequence in the middle is exact at $G_n^{Rel}(Z_f, A)$. $\square$

Theorem 4.11 shows that the non-exactness of G-sequence provides information of the homotopy groups of function spaces. For example, the G-sequence of the Hopf map $p : S^7 \to S^4$ is not exact at $G_4^{Rel}(Z_p, S^7)$ (see [12, Corollary 4.7]). It follows that $\pi_3((S^7)^{(S^7)}) \neq 0$.

References

[1] R. Brooks, *Certain subgroups of the fundamental group and the number of roots of $f(x) = a$*, Amer. J. Math. **95** (1973), 720–728.

[2] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Co., Glenview, Ill.-London 1971.

[3] D. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840–856.

[4] ———, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729–756.

[5] ———, *Covering transformations and universal fibrations*, Illinois J. Math. **13** (1969), 432–437.

[6] B. Jiang, *Estimation of the Nielsen numbers*, Acta Math. Sinica **14** (1964), 304–312.

[7] ———, *Lectures on Nielsen Fixed Point Theory*, Contemporary Mathematics, 14. American Mathematical Society, Providence, R.I., 1983.

[8] T. H. Kiang, *The Theory of Fixed Point Classes*, Translated from the second Chinese edition. Springer-Verlag, Berlin; Science Press, Beijing, 1989.

[9] K. Lee and M. Woo, *The G-sequence and the ω-homology of CW-pair*, Topology Appl. **52** (1993), no. 3, 221–236.

[10] G. Lupton and S. Smith, *Rationalized evaluation subgroups of a map. I. Sullivan models, derivations and G-sequences*, J. Pure Appl. Algebra **209** (2007), no. 1, 159–171.

[11] J. Pak and M. Woo, *A remark on G-sequences*, Math. Japon. **46** (1997), no. 3, 427–432.

[12] J. Pan, X. Shen, and M. Woo, *The G-sequence of a map and its exactness*, J. Korean Math. Soc. **35** (1998), no. 2, 281–294.

[13] M. Woo and J. Kim, *Certain subgroups of homotopy groups*, J. Korean Math. Soc. **21** (1984), no. 2, 109–120.

[14] M. Woo and K. Lee, *On the relative evaluation subgroups of a CW-pair*, J. Korean Math. Soc. **25** (1988), no. 1, 149–160.

[15] M. Woo and Y. Yoon, *Certain subgroups of homotopy groups of a transformation group*, J. Korean Math. Soc. **20** (1983), no. 2, 223–233.

[16] X. Zhao, *Realization of fixed point sets*, Acta Math. Sinica (N.S.) **12** (1996), no. 1, 71–76.

Kee Young Lee
Department of Information and Mathematics
Korea University
Chochiwon 339-701, Korea
E-mail address: keyolee@korea.ac.kr

Moo Ha Woo
Department of Mathematics
Korea University
Seoul 136-701, Korea
E-mail address: woomh@korea.ac.kr

Xuezhi Zhao
Department of Mathematics
Capital Normal University
Beijing 100037, China
E-mail address: zhaoxve@mail.cnu.edu.cn