CERTAIN GENERALIZATIONS OF G-SEQUENCES AND THEIR EXACTNESS

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Abstract. In this paper, we generalize the Gottlieb groups and the related G-sequence of those groups, and present some sufficient conditions to ensure the exactness or non-exactness of G-sequences at some terms. We also give some applications of the exactness or non-exactness of G-sequences. Especially, we show that the non-exactness of G-sequences implies the non-triviality of homotopy groups of some function spaces.

1. Introduction

Some subgroups of homotopy groups were introduced in [3] and [4], and were named later as Gottlieb groups. Their properties and structures turned out to be an interesting and active topic because of the connection with other mathematical fields such as the fixed point theory ([2] and [6]), the rational homotopy theory ([10]) and transformation groups ([15]). Unfortunately, concrete calculations of Gottlieb groups are very difficult, even more than those of homotopy groups. Although they are all homotopy type invariant, Gottlieb groups have no functoriality. Many traditional and effective methods in studying homotopy groups can not be applied to the Gottlieb groups.

After a series works in [9], [13], and [14], the Gottlieb groups, together with other “Gottlieb-like” groups, are successfully arranged in some sequences namely, the G-sequences. It gives us a useful tool to access the Gottlieb groups. Since the homomorphisms in G-sequences are the restrictions of those in homotopy sequences, any G-sequence is half exact, i.e., compositions of consecutive homomorphisms are trivial. Some conditions are known under which the G-sequence is exact, but in general it is not.

This paper presents some results in this direction. We generalize Gottlieb groups and related G-sequences. We present some conditions under which the G-sequence is exact or non-exact at the given terms, rather than the exactness of the whole sequence. Furthermore, the relations amongst the exactness of generalized G-sequences and other topological invariants are addressed.

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This paper is organized as follows. We give the definitions of generalized Gottlieb groups and related \(G\)-sequence in Section 2. We bring focus into the dependence of the generalized Gottlieb groups on the choice of base points in the pairs of spaces involved in Section 3. In Section 4, we present some sufficient conditions to ensure the exactness or non-exactness of \(G\)-sequence at some terms. Some general applications of our conditions for exactness and non-exactness will be given. In the meanwhile, we get a piece of information of homotopy groups of some function spaces.

In this paper, all spaces are finite CW-complexes, all topological pairs are CW-pairs and all subspaces mentioned contain the same base point as their total spaces. We denote the \(n\)-dimensional cube by \(I^n\), its boundary by \(\partial I^n\) and the union of all \(n - 1\) faces of \(I^n\) except for the initial face by \(J^{n-1}\).

2. Generalized \(G\)-sequences

In this section, we shall generalize the original Gottlieb groups in [4] and the \(G\)-sequence introduced in [14].

Let \((X, A)\) and \((Y, B)\) be two pairs of non-empty spaces. We write \(X^Y\) for the space consisting of continuous maps from \(Y\) to \(X\) with compact open topology, and \((X, A)^{(Y,B)}\) for the subspace of \(X^Y\) consisting of relative maps from \((Y, B)\) to \((X, A)\). We then have a natural pair of spaces \((X^Y, (X, A)^{(Y,B)})\). Pick a point \(b_0 \in B\), \(g_0 \in (X, A)^{(Y,B)}\) and set \(g_0(b_0) = a_0\). We choose \(b_0\), \(g_0\) and \(a_0\) as base points of \((Y, B)\), \((X^Y, (X, A)^{(Y,B)})\) and \((X, A)\) respectively. Through the whole of this paper, we always choose base points in this manner. Then there is a natural based map:

\[
\omega^{b_0} : (X^Y, (X, A)^{(Y,B)}, g_0) \rightarrow (X, A, a_0)
\]

which is given by \(\omega^{b_0}(g) = g(b_0)\) for any \(g \in X^Y\), and is said to be the evaluation map at \(b_0\). Thus, we have the following commutative diagram: [Diagram 1]

\[
\begin{array}{ccc}
\pi_n((X, A)^{(Y,B)}, g_0) & \xrightarrow{i_*} & \pi_n(X^Y, g_0) & \xrightarrow{j_*} & \pi_n((X, A)^{(Y,B)}, g_0) \\
\omega^{b_0}_{*} & \downarrow & \omega^{b_0}_{*} & \downarrow & \omega^{b_0}_{*} \\
\pi_n(A, a_0) & \xrightarrow{i_*} & \pi_n(X, a_0) & \xrightarrow{j_*} & \pi_n((X, A), a_0)
\end{array}
\]

where the two horizontal sequences are homotopy exact sequences of the based pairs of spaces \((X^Y, (X, A)^{(Y,B)}, g_0)\) and \((X, A, a_0)\) respectively. If we consider the images of these evaluation homomorphisms \(\omega^{b_0}_{*}\), we get a half exact sequence: [Sequence 2]

\[
\rightarrow \omega^{b_0}_{*}\pi_n((X, A)^{(Y,B)}, g_0) \rightarrow \omega^{b_0}_{*}\pi_n(X^Y, g_0) \rightarrow \omega^{b_0}_{*}\pi_n((X, A)^{(Y,B)}, g_0) \rightarrow
\]

We call this sequence as \(G\)-sequence for \((X^Y, (X, A)^{(Y,B)}, g_0)\) at \(b_0\).

The first \(G\)-sequence defined in [14], which is said to be the \(G\)-sequence for the triple \((X, A, a_0)\), is just the \(G\)-sequence for \((X^A, (X, A)^{(A,A)}, i_A) = (X^Y, (X, A)^{(Y,B)}, g_0)\).
\((X^A, A^A, i_A)\) at \(a_0\), where \(i_A\) is the inclusion from \(A\) to \(X\). Moreover, we have 
\[G_n(X, A, a_0) = \omega^n_0 \pi_n(X^A, i_A), \quad G^n_{\text{rel}}(X, A, a_0) = \omega^n_0 \pi_n(X^A, (X, A)^{(A,A)}, i_A).\] 

The \(G\)-sequence for a map \(f: A \to X\) was introduced in [12]. Let \(Z_f\) denote the mapping cylinder of the map \(f\), i.e., 
\[Z_f = (A \times I \cup X) / \{(a,0) \sim f(a)\}.\] 

The space \(A\) is regarded as a subspace according to the embedding \(i^A_{Z_f}: A \to Z_f\) given by \(i^A_{Z_f}(a) = (a,1)\). Since \((Z_f^A, (Z_f, A \times 1)^{(A,A)}, i^A_{Z_f}) = (Z_f^A, (A \times 1)^A, i^A_{Z_f})\), the \(G\)-sequence for \((Z_f^A, (Z_f, A \times 1)^{(A,A)}, i^A_{Z_f})\) at \(a_0 \in A\) is the same as the \(G\)-sequence for the map \(f: A \to X\) (see [12, Theorem 3.3 and Corollary 3.4]).

Finally, we illustrate a property of the relative term in \(G\)-sequence. For any subset \(Y'\) of \(Y\), there is a natural map \(r^{Y,Y'}: X^Y \to X^{Y'}\), which is just the map restriction. Moreover, for a triple \((Y, Y', B)\), we have a relative map 
\[r^{Y,Y'}: (X^Y, (X, A)^{(Y,B)}) \to (X^{Y'}, (X, A)^{(Y',B)}).\] 

Clearly, \(r^{Y,Y'}\) is not surjective in general. But we have

**Proposition 2.1.** Let \((X, A)\) and \((Y, B)\) be finite CW-pairs, and let \(Y'\) be a sub-complex of \(Y\) with \(B \subset Y' \subset Y\). Then, for any \(f: (Y, B) \to (X, A)\) as a base point of the pair \((X^Y, (X, A)^{(Y,B)})\), the homomorphism 
\[r^*_n: \pi_n(X^Y, (X, A)^{(Y,B)}, f) \to \pi_n(X^{Y'}, (X, A)^{(Y',B)}, f|_{Y'})\] 
induced by the natural map \(r^{Y,Y'}\) is surjective for each positive integer \(n\).

**Proof.** Pick an arbitrary element in \(\pi_n(X^{Y'}, (X, A)^{(Y',B)}, f|_{Y'})\), which is represented by a map \(\alpha: (I^n, \partial I^n, J^{n-1}) \to (X^Y, (X, A)^{(Y,B)}, f|_{Y'})\). Then \(\alpha\) gives rise to a map \(H: (Y' \times I^n, B \times \partial I^n) \to (X, A)\), which is defined by \(H(y', s) = \alpha(s)(y')\) for all \(y' \in Y'\) and \(s \in I^n\). Note that \(H(y', s) = f(y')\) for all \(y' \in Y'\) and \(s \in J^{n-1}\).

Now, we can find an extension \(H': (Y' \times I^n) \cup (Y \times J^{n-1}) \to X\) of \(H\) by defining \(H(y, s) = f(y)\) for all \(y \in Y - Y'\) and \(s \in J^{n-1}\). Since \(Y' \times I^n \cup Y \times J^{n-1} = Y \times I^n \cup Y \times J^{n-1} \cap (Y' \times I^n \cup Y \times \partial I^n) \times I\) and \(Y' \times I^n \cup Y \times \partial I^n\) is a sub-complex of \(X \times I^{n-1}\), there is an extension \(\tilde{H}: (Y \times I^n, B \times \partial I^n) \to (X, A)\) of \(H'\) by the absolute homotopy extension property.

Define a map \(\beta: I^n \to X^Y\) by \(\beta(s) = H(\cdot, s)\) for all \(s \in I^n\). It is easy to check that \(\beta\) maps \((I^n, \partial I^n, J^{n-1})\) into \((X^Y, (X, A)^{(Y,B)}, f)\), and that \(\alpha = r^{Y,Y'} \circ \beta\). Thus the proof is complete. \(\Box\)

From the following commutative diagram
\[
\begin{array}{ccc}
(X^Y, (X, A)^{(Y,B)}, f) & \xrightarrow{r^{Y,Y'}} & (X^{Y'}, (X, A)^{(Y',B)}, f|_{Y'}) \\
\omega^n_{b_0} \downarrow & & \omega^n_{b_0} \downarrow \\
(X, A, a_0) & = & (X, A, a_0)
\end{array}
\]
where \(b_0 \in B\) and \(f(b_0) = a_0\), we have the following corollary.
Corollary 2.2. Let \((X, A), (Y, B)\) and \(Y'\) be the same as in Proposition 2.1. Then for each \(f \in (X, A)^{(Y,B)}\) and each \(b_0 \in B\),
\[
\omega^b_0(\pi_n(X^Y, (X, A)^{(Y,B)}, f)) = \omega^b_0(\pi_n(X^{Y'}, (X, A)^{(Y',B)}, f|_{Y'})).
\]

In [14], a relative Gottlieb group \(G^\text{Rel}_n(X, A)\) is equal to
\[
\omega^g_0\pi_n(X^X, (X, A)^{(X,A)}, id_X)
\]
([14, Theorem 2]). Applying Corollary 2.2 to the case \((X, A) = (Y, B)\) and \(Y' = B\), we have
\[
G^\text{Rel}_n(X, A) = \omega^g_0\pi_n(X^A, (X, A)^{(A,A)}, id_A) = \omega^g_0\pi_n(X^A, A^A, id_A).
\]
So, Proposition 2.1 is a generalization of [9, Lemma 3.1].

3. Dependence on base points

In this section, we shall discuss the dependence of the groups on the base points \(b_0\) and \(g_0\) in the \(G\)-sequence defined in Section 2 when the base point \(a_0\) of the pair \((X, A)\) is fixed. It sounds an elementary topic, but it is in fact related to the root theory. Such a connection seems to be interesting.

As the remark in the beginning of section 2, we may assume that \(g_0^{-1}(a_0)\) is always non-empty. When we choose base points, the condition may not be strong, since we have the following proposition.

Proposition 3.1. Let \(g_0 \in (X, A)^{(Y,B)}\). If the image \(g_0(B)\) of \(B\) under \(g_0\) meets the path component of \(A\) containing \(a_0\), then there is a map \(g'_0\) in the path-component of \((X, A)^{(Y,B)}\) containing \(g_0\) and a point \(b_0 \in B\) such that \(g'_0(b_0) = a_0\).

Proof. Since the image \(g_0(B)\) of \(B\) under \(g_0\) meets the path component \(A_0\) of \(A\) containing \(a_0\), there is a point \(a_1 \in g_0(B) \cap A_0\). Hence, there is a point \(b_0 \in B\) with \(g_0(b_0) = a_1\) and a path \(\alpha\) from \(a_0\) to \(a_1\) in \(A_0\). We then obtain a map \(H : (B \times \{0\}) \cup \{b_0\} \times I \to A\) given by \(H(b, 0) = g_0(b)\) and \(H(b_0, t) = a_1(1 - t)\) for all \(b \in B\) and \(t \in I\). Using the absolute homotopy extension property twice, we can find an extension \(H' : (Y \times I, B \times I) \to (X, A)\) of \(H\). The map \(H'(\cdot, 1)\) is the desired map \(g'_0\).

First, we prove the following simple fact.

Proposition 3.2. The final two \(\omega^b_0\)'s in Diagram 1 induced by the evaluation map at \(b_0\) are all surjective, i.e.,
\[
\omega^b_0(\pi_0((X, A)^{(Y,B)}, g_0)) = \pi_0(A, a_0)
\]
and
\[
\omega^b_0(\pi_0(X^Y, g_0)) = \pi_0(X, a_0).
\]
Proof. Consider the last part of Diagram 1:

\[
\begin{array}{c}
\ldots \xrightarrow{\beta} \pi_0((X, A)^{(Y, B)}, g_0) \xrightarrow{i_*} \pi_0(X^Y, g_0) \\
\omega_*^{b_0} \downarrow \quad \omega_*^{b_0} \downarrow \\
\ldots \xrightarrow{\alpha} \pi_0(A, a_0) \xrightarrow{i_*} \pi_0(X, a_0)
\end{array}
\]

By definition, \( \pi_0(A, a_0) \) is a based set consisting of all components of \( A \) with base element the component containing \( a_0 \). Pick an arbitrary component \( A_m \) and a constant map \( c_{a_m} \) from \( Y \) to \( X \) with value \( a_m \) in \( A_m \). Then \( c_{a_m} \) is an element in \( (X, A)^{(Y, B)} \). It is obvious that \( \omega_*^{b_0}(c_{a_m}) = a_m \). It follows that as an element of \( \pi_0(A, a_0) \), \( A_m = \omega_*^{b_0}(c_{a_m}) \), where \( c_{a_m} \) is regarded as an element of \( \pi_0((X, A)^{(Y, B)}, g_0) \). Thus, we obtain that \( \omega_*^{b_0}(\pi_0((X, A)^{(Y, B)}, g_0)) = \pi_0(A, a_0) \).

By using the same argument, we can prove that \( \omega_*^{b_0}\pi_0(X^Y, g_0) = \pi_0(X, a_0) \).

By Proposition 3.2, the final two terms in \( G \)-sequence, which are based sets, not groups, are independent on the choice of \( g_0 \) and \( b_0 \). In general, the groups in the \( G \)-sequence (Sequence 2) are dependent on the choice of \( g_0 \) and \( b_0 \).

Example. Let \( X = Y = S^1 \vee S^1 \) be a wedge of two circles, and \( A = B = \{a_0\} \) be the wedge point. Write \( c_{a_0} : X \to A \) for the constant map at \( a_0 \). Then we have that \( \omega_*^{a_0}(\pi_1(X^Y, c_{a_0})) = \pi_1(X, a_0) \) (see [16]). Since \( \omega_*^{a_0}(\pi_1(X^Y, id_X)) \) is contained in the center of \( \pi_1(X, a_0) \) and \( \pi_1(X, a_0) \) is a free group \( F_2 \) with two generators, it follows that \( \omega_*^{a_0}(\pi_1(X^Y, id_X)) \) is a trivial group.

From this example, we know that the groups in the \( G \)-sequence (Sequence 2) maybe quite different if the base points \( b_0 \) and \( g_0 \) come from different components.

Proposition 3.3. Let \( g_0 \) and \( g_1 \) be in the same component of \( (X, A)^{(Y, B)} \). If \( b_0 \) and \( b_1 \) are in the same path component of \( B \) and \( g_0(b_0) = g_1(b_1) \), then we obtain following three types of isomorphisms for each positive integer \( n \):

\[
\omega_*^{b_0}(\pi_n((X, A)^{(Y, B)}, g_0)) \cong \omega_*^{b_1}(\pi_n((X, A)^{(Y, B)}, g_1)),
\omega_*^{b_0}(\pi_n(X^Y, g_0)) \cong \omega_*^{b_1}(\pi_n(X^Y, g_1)),
\omega_*^{b_0}(\pi_n(X^Y, (X, A)^{(Y, B)}, g_0)) \cong \omega_*^{b_1}(\pi_n(X^Y, (X, A)^{(Y, B)}, g_1)).
\]

Proof. Since \( g_0 \) and \( g_1 \) are in the same component of \( (X, A)^{(Y, B)} \), \( g_0, g_1 : (Y, B) \to (X, A) \) are relatively homotopic. Thus there is a relative homotopy \( H : (Y \times I, B \times I) \to (X, A) \) from \( g_0 \) to \( g_1 \). As a path in \( (X, A)^{(Y, B)} \), \( H(\cdot, t) \) induces a classical isomorphism \( \theta_{H(\cdot, t)} : \pi_n(X^Y, g_0) \to \pi_n(X^Y, g_1) \). Pick a path \( \alpha : I \to B \) from \( b_0 \) to \( b_1 \). Then \( \alpha \) determines a homotopy \( \omega^{\alpha(t)} : X^Y \to X \) from \( \omega^{b_0} \) and \( \omega^{b_1} \). Note that \( \omega^{\alpha(t)}(H(\cdot, 1)) = H(\alpha(t), 1) \). Let \( a_0 = g_0(b_0) = g_1(b_1) \).
Then we obtain a commutative diagram:

\[
\begin{array}{c}
\pi_n(X^Y, g_0) \\ \downarrow \omega_*^{b_0} \\
\pi_n(X, a_0) = g_0(b_0)
\end{array}
\xrightarrow{\theta_{\{H(b_0, t)\}}} 
\begin{array}{c}
\pi_n(X^Y, g_1) \\ \downarrow \omega_*^{b_1} \\
\pi_n(X, a_0 = g_1(b_1))
\end{array}
\xrightarrow{\theta_{\{H(a(t), 1)\}}} 
\begin{array}{c}
\pi_n(X^Y, g_1) \\ \downarrow \omega_*^{b_1} \\
\pi_n(X, a_0 = g_1(b_1))
\end{array}
\]

The commutativity of the left square comes from the functoriality of the isomorphism \(\theta_{\{\}}\). The commutativity of the right square is just the relation between the homomorphisms induced from homotopic maps. Since the path product \(\{H(b_0, t)\} \cdot \{H(a(t), 1)\}\) and \(\{H(a(t), t)\}\) are homotopic keeping end points fixed, \(\theta_{\{H(b_0, t)\}} \theta_{\{H(a(t), 1)\}} = \theta_{\{H(a(t), t)\}}\). Thus we have the following commutative diagram

\[
\begin{array}{c}
\pi_n(X^Y, g_0) \\ \downarrow \omega_*^{b_0} \\
\pi_n(X, a_0)
\end{array}
\xrightarrow{\theta_{\{H(a(t), 1)\}}} 
\begin{array}{c}
\pi_n(X^Y, g_1) \\ \downarrow \omega_*^{b_1} \\
\pi_n(X, a_0)
\end{array}
\]

It follows that \(\omega_*^{b_0}(\pi_n(X^Y, g_0))\) and \(\omega_*^{b_1}(\pi_n(X^Y, g_1))\) are isomorphic subgroups in \(\pi_n(X, a_0)\).

Similarly, we can prove that

\[
\omega_*^{b_0}(\pi_n((X, A)^{(Y, B)}, g_0)) \cong \omega_*^{b_1}(\pi_n((X, A)^{(Y, B)}, g_1))
\]

by using the commutative diagram:

\[
\begin{array}{c}
\pi_n((X, A)^{(Y, B)}, g_0) \\ \downarrow \omega_*^{b_0} \\
\pi_n(A, a_0)
\end{array}
\xrightarrow{\theta_{\{H(a(t), 1)\}}} 
\begin{array}{c}
\pi_n((X, A)^{(Y, B)}, g_1) \\ \downarrow \omega_*^{b_1} \\
\pi_n(A, a_0)
\end{array}
\]

and that

\[
\omega_*^{b_0}(\pi_n(X^Y, (X, A)^{(Y, B)}, g_0)) \cong \omega_*^{b_1}(\pi_n(X^Y, (X, A)^{(Y, B)}, g_1))
\]

by using the commutative diagram:

\[
\begin{array}{c}
\pi_n(X^Y, (X, A)^{(Y, B)}, g_0) \\ \downarrow \omega_*^{b_0} \\
\pi_n(X, A, a_0)
\end{array}
\xrightarrow{\theta_{\{H(a(t), 1)\}}} 
\begin{array}{c}
\pi_n(X^Y, (X, A)^{(Y, B)}, g_1) \\ \downarrow \omega_*^{b_1} \\
\pi_n(X, A, a_0)
\end{array}
\]

\[
\square
\]

By Proposition 3.2, the property \(g(B)\) meets a component of \(A\) is preserved by elements in a path-component of \((X, A)^{(Y, B)}\). Thus, by Proposition 3.3, we obtain

**Theorem 3.4.** The isomorphism types of groups in the G-sequence do not depend on the choice of base points on the path-components of \((X, A)^{(Y, B)}\) and the path components of \(B\).
Here we recall some basic concepts in root theory (see [1] or [8] for more details). Given a map \( g : Y \to X \) and a point \( a_0 \in X \), a point \( y \in Y \) is said to be a root of \( g \) at \( a_0 \) if \( g(y) = a_0 \). Let \( K \) be a normal subgroup of \( \pi_1(X, a_0) \). All the roots of \( g \) at \( a_0 \) are split into equivalence classes, namely mod \( K \) root classes, in the sense: two roots \( y' \) and \( y'' \) are in the same mod \( K \) root class if there is a path \( \beta \) in \( Y \) from \( y' \) to \( y'' \) such that \( g \circ \beta \), a loop in \( X \) at \( a_0 \), represents an element in \( K \). There is a well-defined correspondence between the sets of root classes of homotopic maps. Let \( H : Y \times I \to X \) be a homotopy from \( g_0 \) to \( g_1 \). We called a mod \( K \) root class of containing a root \( y_0 \) of \( g_0 \) at \( a_0 \) and a mod \( K \) root class containing a root \( y_1 \) of \( g_1 \) at \( a_0 \) are \( H \)-related or \( y_0 \) and \( y_1 \) are \( H \)-related mod \( K \) if there is a path \( \gamma \) in \( Y \) from \( y_0 \) to \( y_1 \) such that \( \{ H(\gamma(t), t) \} \) is an element in \( K \). In the case that \( K \) is trivial group, we call just \( y_0 \) and \( y_1 \) are \( H \)-related. The elements of \( \pi_1(X, a_0) \) operate on \( \pi_n(X, a_0) \) as a group of automorphisms in a standard way. It is well-known fact that the set of all elements of \( \pi_1(X, a_0) \) which operate trivially on \( \pi_n(X, a_0) \) for all positive integer \( n \) is a subgroup, which will be denoted as \( P_1(X, a_0) \), and it is contained in the center of \( \pi_1(X, a_0) \).

**Lemma 3.5.** Let \( g_0, g_1 : Y \to X \) be homotopic maps by a homotopy \( H : Y \times I \to X \) and \( b_0 \) and \( b_1 \) be two roots of \( g_0 \) and \( g_1 \) at \( a_0 \) respectively. If \( b_0 \) and \( b_1 \) are \( H \)-related mod \( P_1(X, a_0) \), then \( \omega_n^{b_0} (\pi_n(X^Y, g_0)) = \omega_n^{b_1} (\pi_n(X^Y, g_1)) \).

**Proof.** Since the mod \( P_1(X, a_0) \) root classes containing \( b_0 \) and \( b_1 \) are \( H \)-related by the hypothesis, there is a path \( \alpha : I \to Y \) from \( b_0 \) to \( b_1 \) such that the diagonal path \( H(\alpha(\cdot), \cdot) \) of \( \alpha \) under \( H \) determines an element in \( P_1(X, a_0) \). Consider the following commutative diagram (see Proposition 3.3):

\[
\begin{array}{ccc}
\pi_n(X^Y, g_0) & \xrightarrow{\theta_{H(\cdot, \cdot)}} & \pi_n(X^Y, g_1) \\
\omega_n^{b_0} \downarrow & & \omega_n^{b_1} \downarrow \\
\pi_n(X, a_0) & \xrightarrow{\theta_{H(\alpha(\cdot), \cdot)}} & \pi_n(X, a_0).
\end{array}
\]

Since \( H(\alpha(0), 0) = g_0(\alpha(0)) = g_0(b_0) = a_0 \), and \( H(\alpha(1), 1) = g_1(\alpha(1)) = g_1(b_1) = a_0 \), the path \( H(\alpha(\cdot), \cdot) \) is a loop at \( a_0 \). The isomorphism \( \theta_{H(\alpha(\cdot), \cdot)} \) is just the action of the element \( \{ H(\alpha(\cdot), \cdot) \} \) in \( \pi_1(X, a_0) \) on \( \pi_n(X, a_0) \). Since \( \{ H(\alpha(\cdot), \cdot) \} \) is an element of \( P_1(X, a_0) \), \( \theta_{H(\alpha(\cdot), \cdot)} \) is the identity by the definition of \( P_1(X, a_0) \). \( \square \)

**Lemma 3.6.** Let \( g_0, g_1 : (Y, B) \to (X, A) \) be relatively homotopic maps by a homotopy \( H : (Y \times I, B \times I) \to (X, A) \), and \( b_0 \) and \( b_1 \) two roots of \( g_0|_B \) and \( g_1|_B \) at \( a_0 \) respectively. If \( b_0 \) and \( b_1 \) are \( H_{|B \times I} \)-related, then

\[
\begin{align*}
\omega_n^{b_0} (\pi_n((X, A)^{(Y, B)}, g_0)) &= \omega_n^{b_1} (\pi_n((X, A)^{(Y, B)}, g_1)), \\
\omega_n^{b_0} (\pi_n(X^Y, g_0)) &= \omega_n^{b_1} (\pi_n(X^Y, g_1)), \\
\omega_n^{b_0} (\pi_n(X^Y, (X, A)^{(Y, B)}, g_0)) &= \omega_n^{b_1} (\pi_n(X^Y, (X, A)^{(Y, B)}, g_1)).
\end{align*}
\]
Proof. Since $b_0$ and $b_1$ are $H|_{B \times I}$-related, the diagonal path $H(\alpha(\cdot), \cdot)$ is homotopic to constant path in $A$. Recall the proof of Proposition 3.3. Since the isomorphism $\theta_{H(\alpha(t), t)}$ is the identity, the proof is complete. 

**Theorem 3.7.** Given a path component $B_0$ of $B$ and a component $M$ of $(X, A)^{(Y, B)}$, the groups in the $G$-sequence for $(X^Y, (X, A)^{(Y, B)}, g_0)$ at $b_0$, as subgroups of those in homotopy exact sequence of $(X, A, a_0)$, are independent of the choice of $b_0 \in B_0$ and $g_0 \in M$ provided one of the following conditions is hold:

1. the based spaces $(X, a_0)$ and $(A, a_0)$, and the based pair $(X, A, a_0)$ are all simple;
2. one map, hence for all maps, in $M$ induces a surjective homomorphism from $\pi_1(X, b_0)$ to $\pi_1(X, a_0)$ and a surjective homomorphism from $\pi_1(B_0, b_0)$ to $\pi_1(A, a_0)$.

Proof. Suppose (1) holds. By the definition of simple spaces, the action of fundamental group $\pi_1(X, a_0)$ on high dimensional homotopy groups $\pi_n(X, a_0)$ ($n > 0$), and the action of fundamental group $\pi_1(A, a_0)$ on high dimensional homotopy groups $\pi_n(A, a_0)$ and $\pi_n(X, A, a_0)$ ($n > 0$) are all trivial. By Proposition 3.3, we have the result.

Suppose (2) holds. Recall from [8, p.133] that a root class of $g_0$ corresponds to the coset of $\text{im}g_0\ast$ in $\pi_1(X, a_0)$. Since $g_0\ast$ is onto and hence there is one coset of $\text{im}g_0\ast$ in $\pi_1(X, a_0)$, $g_0$ as well as any other maps in $M$ has only one root class. Similarly, $g_0|_{B_0}$ has only one root class. By Lemma 3.6, we have the conclusion. 

4. **Criteria for the exactness and applications**

Our purpose in this section is to give some conditions under which the $G$-sequence is exact at some terms, instead of those for the exactness of the whole $G$-sequence and present some applications.

By Proposition 3.1, the final two terms in the $G$-sequence are nothing but those in the homotopy exact sequence for the pair $(X, A)$. Thus we have

**Proposition 4.1.** The $G$-sequence for $(X^Y, (X, A)^{(Y, B)}, g_0)$ at $b_0$ is exact at final term.

About the exactness of the term prior to the final term, we have

**Proposition 4.2.** Let $B_0$ be the component of $B$ containing $b_0$, $X_0$ the component of $X$ containing $a_0$. Then the $G$-sequence for $(X^Y, (X, A)^{(Y, B)}, g_0)$ at $b_0$ is exact at the term $\omega_{b_0}^*\pi_0((X, A)^{(Y, B)}, g_0))$ if and only if for each component $A_k$ of $A$ lying in $X_0$, $g_0|_{B_0} : B_0 \to X$ is homotopic to a map from $B_0$ to $A_k$.

Proof. Let us prove "if" part first. From Proposition 3.1, we know

$$\omega_{b_0}^*\pi_0((X, A)^{(Y, B)}, g_0)) = \pi_0(A, a_0).$$
By definition, \( \pi_0(A, a_0) \) is a based set consisting of all components of \( A \) with the component containing \( a_0 \) as the base element, and \( \pi_0(X, a_0) \) is a based set consisting of all components of \( X \) with base element \( X_0 \). If an element of \( \pi_0(A, a_0) \) is sent to the base element \( X_0 \) by \( i_*: \pi_0(A, a_0) \rightarrow \pi_0(X, a_0) \), it is a component, say \( A_k \), lying in \( X_0 \). By the assumption, there is a homotopy \( H': B_0 \times I \rightarrow X \) from \( g_0|_B_0 \) to a map in \( A_k^{B_0} \). We can extend \( H' \) to \( H'' : B \times I \rightarrow X \) by defining \( H''(b, t) = g_0(b) \) for all \( b \in B - B_0 \). By the homotopy extension property, we obtain an extension \( H : Y \times I \rightarrow X \) of \( H'' \) such that \( H|_{Y \times 0} = g_0 \). It is evident that \( H(b, 1) \in A \) for all \( b \in B \). Thus, its adjoint \( \tilde{H} \) can be regarded as an element in that \( \pi_1(X^Y, (X, A)^{(Y,B)}, g_0) \). Since \( \omega^b_*(\tilde{H})(t) = H(b_0, t) \) is a path from \( g_0(b_0) = a_0 \) to \( H(b_0, 1) \) which is a point in \( A_k \), we have that \( \partial_* (\omega^b_*(\tilde{H})) = H(b_0, 1) \), i.e., the element in \( \pi_0(A, a_0) \) represented by \( A_k \). Hence, \( A_k \) lies in the image of the homomorphism
\[
\partial_* : \omega^b_* (\pi_1(X^Y, (X, A)^{(Y,B)}, g_0)) \rightarrow \pi_0(A, a_0).
\]

Let us show the “only if” part. Pick an arbitrary component \( A_k \) of \( A \) which is contained in \( X_0 \). Then it is an element of the \( \ker(i_* : \pi_0(A, a_0) \rightarrow \pi_0(X, a_0)) \). Since the \( G \)-sequence is exact at
\[
\omega^b_* (\pi_0((X, A)^{(Y,B)}), g_0)) = \pi_0(A, a_0),
\]
\( A_k \) as an element of \( \pi_0(A, a_0) \), lies in the
\[
\text{Im}(\partial_* : \omega^b_* (\pi_1(X^Y, (X, A)^{(Y,B)}, g_0)) \rightarrow \pi_0(A, a_0)),
\]
i.e., there is a map \( H : Y \times I \rightarrow X \) with \( H(y, 0) = g_0(y) \) for all \( y \in Y \) and \( H(b, 1) \in A \) for all \( b \in B \) such that \( \partial_* (\omega^b_*(H)) = \partial_* (\{H(b_0, t)\}) = H(b_0, 1) \in A_k \). Thus, \( H|_{B_0 \times I} \) is a desired homotopy. \( \Box \)

**Corollary 4.3.** The \( G \)-sequence is exact at \( \omega^b_* (\pi_0((X, A)^{(Y,B)}), g_0)) \), if the component of \( X \) containing \( a_0 \) meets only one component of \( A \), especially, if \( A \) has only one path component.

Here we need a lemma in algebra.

**Lemma 4.4.** Consider the following commutative diagram of groups and homomorphisms:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{i} & A_2 & \xrightarrow{j} & A_3 \\
\phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow \\
A_1' & \xrightarrow{i'} & A_2' & \xrightarrow{j'} & A_3'
\end{array}
\]

where \( \phi_1, \phi_2 \) and \( \phi_3 \) are surjective, and the top row is exact at \( A_2 \), i.e., \( \text{Im}i = \text{Ker}j \). Then we have:

1. the bottom row is exact at \( A_2' \) if \( \phi_3 \) is injective on \( \text{Im}j \);
2. the bottom row is not exact at \( A_2' \) if \( \phi_3 \) is non-injective on \( \text{Im}j \) and if \( \phi_2 \) is injective.
Proof. (1) Pick an element $a'_2$ in $\ker j'$. Since $\phi_2$ is surjective, we may assume that $a'_2 = \phi_2(a_2)$ for $a_2 \in A_2$. By the commutativity of the diagram, we have $\phi_3(j(a_2)) = j'(\phi_2(a_2)) = j'(a'_2) = 0$. Because $\phi_3$ is injective on $\Im j$, $j(a_2) = 0$. The exactness of the top row implies that $a_2 \in \Im i$, i.e., $a_2 = i(a_1)$ for some $a_1 \in A_1$. It follows that $a'_2 = \phi_2(a_2) = \phi_2(i(a_1)) = i'(\phi_1(a_1))$. We then obtain that $\ker j' \subseteq \Im i'$. By the commutativity of the diagram, it is easy to check that $\ker j' \supseteq \Im i'$.

(2) Suppose that $\phi_3$ is not injective on $\Im j$. Then there is an element $a_2$ in $A_2$ such that $j(a_2) \neq 0$, but $\phi_3(j(a_2)) = 0$. Consider the element $\phi_2(a_2) \in A'_2$, which lies in $\ker j'$ because $j'(\phi_2(a_2)) = \phi_3(j(a_2)) = 0$. We claim that $\phi_2(a_2) \notin \Im i'$.

Assume that $\phi_2(a_2) = i'(a'_1)$ for some $a'_1 \in A'_1$. We would have $\phi_2(a_2) = i'(\phi_1(a_1))$ for some $a_1 \in A_1$ because $\phi_1$ is surjective. Hence, $\phi_2(a_2) = \phi_2(i(a_1))$. As $\phi_2$ is injective, $a_2 = i(a_1) \in \Im i$. The exactness of top row would implies that $j(a_2) = j(i(a_1)) = 0$. This is a contradiction. It follows that the bottom row is non-exact at $A'_2$. □

By Lemma 4.4, we obtain immediately that

Theorem 4.5. If all evaluation homomorphisms are injective on the image of $i_*$, or $j_*$ or $\partial_*$ depending on the domain of $\omega_*$ in Diagram 1, then the G-sequence for $(X^Y, (X,A)^{(Y,B)}, g_0)$ at $b_0$ is exact.

Corollary 4.6. If all homomorphisms induced on homotopy by the evaluation maps concerned are injective, then the G-sequence for $(X^Y, (X,A)^{(Y,B)}, g_0)$ at $b_0$ is exact.

On the other hand, we have

Theorem 4.7. If an evaluation homomorphism $\omega_*^{b_0}$ in Diagram 1 is surjective, then the G-sequence for $(X^Y, (X,A)^{(Y,B)}, g_0)$ at $b_0$ is exact at the term next to that containing the image of this $\omega_*^{b_0}$.

Proof. There are three types of evaluation homomorphism in Diagram 1. We give a proof for one type, the proofs for other two types are similar.

Consider Diagram 1. Suppose

$$\omega_*^{b_0} : \pi_n((X,A)^{(Y,B)}, g_0) \rightarrow \pi_n(A, a_0)$$

is surjective. We shall prove the G-sequence for $(X^Y, (X,A)^{(Y,B)}, g_0)$ at $b_0$ is exact at $\omega_*^{b_0}(\pi_n(Y,g_0))$. Since the G-sequence is already half exact, it is sufficient to show that the kernel of

$$j_* : \omega_*^{b_0}(\pi_n(X^Y,g_0)) \rightarrow \omega_*^{b_0}(\pi_n(X,Y,A)^{(Y,B)}, g_0))$$

is contained in the image of

$$i_* : \omega_*^{b_0}(\pi_n((X,A)^{(Y,B)}, g_0)) \rightarrow \omega_*^{b_0}(\pi_n(X^Y,g_0)).$$
By assumption, the latter is just $i_*(\pi_n(A, a_0))$ and by the exactness of the bottom sequence in Diagram 1, we have
\[
\text{Ker} j_* |_{\omega^b_*(\pi_n(X^n, y_0))} \subset \text{Ker} j_* \subset \text{Im} i_*. 
\]

Recall that a space $X$ is said to be a Gottlieb space if $\omega^b_*(\pi_n(X^n, id_X)) = \pi_n(X, x_0)$ for all positive integer $n$. Note that
\[
\omega^b_*(\pi_n(X^n, id_X)) \subset \omega^b_*(\pi_n(X^A, i_A))
\]
for any $A \subset X$. By Theorem 4.7, we obtain a necessary condition for a space to be a Gottlieb space.

**Corollary 4.8.** If $X$ is a Gottlieb space, then the $G$-sequence for the pair $(X, A)$ is exact at $G_n^{\text{Rel}}(X, A)$ for each subspace $A$ of $X$ and each positive integer $n$.

Especially, we have

**Corollary 4.9.** If $X$ is a Jiang space, i.e., $\omega^b_*(\pi_1(X^n, id_X)) = \pi_1(X, x_0)$ then the $G$-sequence for the pair $(X, A)$ is exact on $G_1^{\text{Rel}}(X, A)$ for each subspace $A$ of $X$.

**Theorem 4.10.** If $X$ is aspherical, Then the $G$-sequence for pair $(X, A)$ for pair $(X, A)$ exact at all terms except for three: $G_1(X, A)$, $G_1^{\text{Rel}}(X, A)$ and $G_0(A)$.

**Proof.** In fact, the exactness at the terms before $G_2(X, A)$ was already proved in [9]. Consider the following commutative diagram:

\[
\begin{array}{cccccc}
\pi_2(X^A, i_A) & \xrightarrow{j_*} & \pi_2(X^A, A^A, id_A) & \xrightarrow{\partial_*} & \pi_1(A^A, id_A) & \xrightarrow{i_*} & \pi_1(X^A, i_A) \\
\omega^a_0 \downarrow & & \omega^a_0 \downarrow & & \omega^a_0 \downarrow & & \omega^a_0 \downarrow \\
G_2(X, A) & \xrightarrow{j_*} & G_2^{\text{Rel}}(X, A) & \xrightarrow{\partial_*} & G_1(A) & \xrightarrow{i_*} & G_1(X, A) \\
\cap \downarrow & & \cap \downarrow & & \cap \downarrow & & \cap \downarrow \\
\pi_2(X, a_0) & \xrightarrow{j_*} & \pi_2(X, A, a_0) & \xrightarrow{\partial_*} & \pi_1(A, a_0) & \xrightarrow{i_*} & \pi_1(X, a_0) \\
\end{array}
\]

Since $\pi_2(X) = 0$, the exactness at $G_2(X, A) = 0$ is obvious. By the exactness of the bottom sequence, we have that $\partial_* : \pi_2(X, A, a_0) \rightarrow \pi_1(A, a_0)$ is injective. Hence, the $G$-sequence in the middle is exact at $G_2^{\text{Rel}}(X, A)$. By [5, Lemma 2], we know $\omega^a_0 : \pi_1(X^A, i_A) \rightarrow \pi_1(X, a_0)$ is injective. From Lemma 4.4, the $G$-sequence is exact on $G_1(A)$. The exactness at final two terms was proved in general in Proposition 4.1.

**Theorem 4.11.** If there is a map $f : A \rightarrow X$ such that the $G$-sequence of $f$ is non-exact at $G_n^{\text{Rel}}(Z_f, A)$, then $\pi_{n-1}(A^A, id_A)$ is non-trivial, where $Z_f$ is the mapping cylinder of $f$. 

Proof. Pick a base point $a_0$ and consider the following commutative diagram
\[
\begin{array}{c}
\pi_n(Z_f, i^A_{Z_f}) \xrightarrow{id} \pi_n(Z_f^A, A^A, i^A_{Z_f}) \xrightarrow{\partial_*} \pi_{n-1}(A^A, id_A) \\
\omega_*^{i^A_0} \downarrow \quad \omega_*^{i^A_0} \downarrow \quad \omega_*^{i^A_0} \downarrow \\
G_n(Z_f, A) \xrightarrow{id} G_n^{Rel}(Z_f, A) \xrightarrow{\partial_*} G_{n-1}(A) \\
\cap \downarrow \quad \cap \downarrow \quad \cap \downarrow \\
\pi_n(Z_f, a_0) \xrightarrow{id} \pi_n(Z_f, A, a_0) \xrightarrow{\partial_*} \pi_{n-1}(A, a_0)
\end{array}
\]

The top and bottom rows are parts of the homotopy exact sequence of $(Z_f^A, A^A)$ and $(Z_f, A)$ respectively, and the middle one is just a part of $G$-sequence of $f$. If $\pi_{n-1}(A^A, id_A)$ is trivial, then
\[
\omega_*^{i^A_0} : \pi_{n-1}(A^A, id_A) \to G_{n-1}(A) = \omega_*^{i^A_0}(\pi_{n-1}(A^A, id_A))
\]
is trivial homomorphism between trivial groups, and hence is injective. By Lemma 4.4, the $G$-sequence in the middle is exact at $G_n^{Rel}(Z_f, A)$. \hfill \square

Theorem 4.11 shows that the non-exactness of $G$-sequence provides information of the homotopy groups of function spaces. For example, the $G$-sequence of the Hopf map $p : S^7 \to S^4$ is not exact at $G_n^{Rel}(Z_p, S^7)$ (see [12, Corollary 4.7]). It follows that $\pi_3((S^7)(S^7)) \neq 0$.

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