CONVERGENCE OF APPROXIMATING PATHS TO
SOLUTIONS OF VARIATIONAL INEQUALITIES INVOLVING
NON-LIPSCHITZIAN MAPPINGS

JONG SOO JUNG† AND DAYA RAM SAHU ‡

ABSTRACT. Let $X$ be a real reflexive Banach space with a uniformly
Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $X, T :$
$C \to X$ a continuous pseudocontractive mapping, and $A : C \to C$ a
continuous strongly pseudocontractive mapping. We show the existence
of a path $\{x_t\}$ satisfying $x_t = tAx_t + (1-t)Tx_t, \ t \in (0,1)$ and prove that
$\{x_t\}$ converges strongly to a fixed point of $T$, which solves the variational
inequality involving the mapping $A$. As an application, we give strong
convergence of the path $\{x_t\}$ defined by $x_t = tAx_t + (1-t)(2I-T)x_t$
to a fixed point of firmly pseudocontractive mapping $T$.

1. Introduction

Let $X$ be a real Banach space with dual $X^*$ and $T$ be a mapping with domain
$D(T)$ and range $R(T)$ in $X$. Following Morales [12], the mapping $T$ is called
strongly pseudocontractive if for some constant $k < 1$ and for all $x,y \in D(T),$
\begin{equation}
(\lambda - k)||x - y|| \leq ||(\lambda I - T)(x) - (\lambda I - T)(y)||
\end{equation}
for all $\lambda > k$; while $T$ is called a pseudocontraction if (1) holds for $k = 1$. The
mapping $T$ is called Lipschitzian if there exists $L \geq 0$ such that
\[||Tx - Ty|| \leq L||x - y|| \text{ for all } x, y \in D(T).
\]
Otherwise, the mapping is called non-Lipschitzian. The Lipschitzian mapping
$T$ is called nonexpansive if $L = 1$ and is called a contraction if $L < 1$. Every
nonexpansive mapping is a pseudocontractive. The converse is not true. The
example, $Tx = (1 - x^{2/3})x^{1/3}, x \in [0,1]$ is a continuous pseudocontraction which is

Received July 24, 2006.
2000 Mathematics Subject Classification. Primary 47H10, 47J20.
Keywords and phrases. pseudocontractive mapping, strongly pseudocontractive map-
ing, firmly pseudocontractive mapping, nonexpansive mapping, fixed points, uniformly
Gâteaux differentiable norm, variational inequality.
† This paper was supported by Dong-A University Research Fund in 2006.
‡ The second author wishes to acknowledge the financial support of Department of Science

©2008 The Korean Mathematical Society

377
not nonexpansive. Indeed,
\[
\left| T\left(\frac{1}{4^3}\right) - T\left(\frac{1}{2^3}\right) \right| = \left| \left(\frac{15}{16}\right)^{\frac{3}{2}} - \left(\frac{3}{4}\right)^{\frac{3}{2}} \right| = \frac{|(15)^{\frac{3}{2}} - (12)^{\frac{3}{2}}|}{64} > \frac{7}{64} = \left| \frac{1}{4^3} - \frac{1}{2^3} \right|.
\]

A mapping \(T\) with domain \(D(T)\) and range \(R(T)\) in \(X\) is called firmly pseudocontractive if for all \(x, y \in D(T)\),
\[
\|x - y\| \leq \| (1 - \lambda)(x - y) + \lambda (Tx - Ty) \|
\]
for all \(\lambda > 0\). Following Kato [10], we are able to find an equivalent definition for firmly pseudocontractive operators. An operator \(T : D(T) \to R(T)\) is firmly pseudocontractive if and only if for every \(x, y \in D(T)\), there exists \(j(x - y) \in J(x - y)\) such that
\[
\langle Tx - Ty, j(x - y) \rangle \geq \|x - y\|^2,
\]
where \(J : X \to 2^{X^*}\) is the normalized duality mapping which is defined by
\[
J(u) = \{ j \in X^* : \langle u, j \rangle = \| u \|^2, \| j \| = \| u \| \}
\]
(see Browder [2] and Kato [10]). It is an immediate consequence of the Hahn-Banach theorem that \(J(u)\) is nonempty for each \(u \in X\).

The firmly pseudocontractive mappings are characterized by the fact that a mapping \(T\) is firmly pseudocontractive if and only if the mapping \(f = T - I\) is accretive (see Lemma 5).

The concept of firmly pseudocontractive mapping was introduced by Sharma and Sahu [20]. The mapping \(T : D(T) \to R(T)\) is firmly pseudocontractive if and only if \(2I - T\) is pseudocontractive (see Lemma 5).

In [15], Moudafi proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping which is a unique solution of a variational inequality in a Hilbert space. He proved the following theorem:

**Theorem M** (Theorem 2.1, Moudafi [15]). Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\). Let \(T : C \to C\) be a nonexpansive mapping and \(f : C \to C\) a contraction mapping. Let \(\{x_n\}\) be the sequence defined by the scheme
\[
x_n = \frac{1}{1 + \varepsilon_n} Tx_n + \frac{\varepsilon_n}{1 + \varepsilon_n} f x_n,
\]
where \(\varepsilon_n\) is a sequence \((0, 1)\) with \(\varepsilon_n \to 0\). Then \(\{x_n\}\) converges strongly to the unique solution of the variational inequality:
\[
\langle (I - f)\bar{x}, \bar{x} - x \rangle \leq 0 \text{ for all } x \in F(T).
\]

In other word, \(\bar{x}\) is the unique fixed point of \(P_{F(T)}f\).

**Theorem X** (Theorem 4.1, Xu [22]). Let $C$ be a nonempty closed convex subset of a uniformly smooth Banach space $X$, $f \in \Pi_C$ the set of all contractions on $C$ and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Then the path $\{x_t\}$ defined by

$$x_t = tfx_t + (1-t)Tx_t, \quad t \in (0,1)$$

converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by

$$Q(f) = \lim_{t \to 0^+} x_t, \quad f \in \Pi_C,$$

then $Q(f)$ solves the variational inequality:

$$\langle (I-f)Q(f), J(Q(f) - v) \rangle \leq 0, \quad f \in \Pi_C \text{ and } v \in F(T).$$

It is well known that for certain applications the Lipschitzian assumption of mapping becomes a rather strong condition. In view of this the following natural question arises:

**Question.** Is it possible to replace contraction mapping $f$ involving in variational inequality (2) by a non-Lipschitzian mapping $A$?

Motivated and inspired by the above question, we will consider a more general situation. In this paper our purpose is to prove that in reflexive Banach space $X$, for pseudocontractive mapping $T$, the path $\{x_t\}$ defined by

$$x_t = tAx_t + (1-t)Tx_t$$

converges strongly to a fixed point of $T$, which solves the certain variational inequality involving non-Lipschitzian mapping $A$. Using our results, we derive strong convergence theorems for firmly pseudocontractive mappings. Our results generalize and improve the results of Jung and Kim [9], Morales [13], Morales and Jung [14], Moudafi [15], O’Hara, Pillay, and Xu [16], Reich [18], Schu [19], Sharma and Sahu [20], and Xu [21, 22].

## 2. Preliminaries and lemmas

Recall that a Banach space $X$ is said to be smooth if the limit

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x$ and $y$ in $S = \{x \in X : \|x\| = 1\}$. In this case, the norm of $X$ is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in S$, this limit is attained uniformly for $x \in S$. It is well known that every uniformly smooth space (e.g., $L_p$ space, $1 < p < \infty$) has uniformly Gâteaux differentiable norm (see e.g., [3]).

When $\{x_n\}$ is a sequence in $X$, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \rightharpoonup x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to $x$. 
Suppose that the duality mapping $J$ is single valued. Then $J$ is said to be weakly sequentially continuous if, for each $\{x_n\} \in X$ with $x_n \rightharpoonup x$, $J(x_n) \rightharpoonup J(x)$.

A Banach space $X$ is said to satisfy Opial’s condition (see for example [17]) if for each sequence $\{x_n\}$ in $X$ which converges weakly to a point $x \in X$ we have

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \quad \text{for all } y \in X.$$  

It is well-known that, if $X$ admits a weakly sequentially continuous duality mapping, then $X$ satisfies Opial’s condition.

Let $X$ be a Banach space and let $T$ be a mapping with domain $D(T)$ and range $R(T)$ in $X$. The mapping $T$ is said to be demiclosed at a point $p \in D(T)$ if whenever $\{x_n\}$ is a sequence in $D(T)$ which converges weakly to a point $z \in D(T)$ and $\{Tx_n\}$ converges strongly to $p$, then $Tz = p$. The mapping $T$ is said to be demicontinuous if, whenever a sequence $\{x_n\}$ in $C$ converges strongly to $x \in C$, then $\{Tx_n\}$ converges weakly to $Tx$. The set of fixed point of $T$ will be denoted by $F(T)$.

Let $C$ be a convex subset of $X$, $D$ a nonempty subset of $C$, and $P$ a retraction from $C$ onto $D$, that is, $Px = x$ for each $x \in D$. A retraction $P$ is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If the sunny retraction $P$ is also nonexpansive, then $D$ is said to be a sunny nonexpansive retract of $C$.

Let $C$ be a nonempty closed convex subset of a Banach space $X$. For $x \in C$, let

$$I_C(x) = \{y \in X : y = x + \lambda(z - x), z \in C \text{ and } \lambda \geq 0\}.$$  

$I_C(x)$ is called the inward set of $x \in C$ with respect to $C$ (see, for example [5]). $I_C(x)$ is a convex set containing $C$. A mapping $T : C \to X$ is said to be satisfying the inward condition if $Tx \in I_C(x)$ for all $x \in C$, $T$ is also said to be satisfying the weakly inward condition if for each $x \in C, Tx \in I_C(x)$ ($I_C(x)$ is the closure of $I_C(x)$). It is well-known (Lemma 18.1, Deimling [5]) that $T : C \to X$ is weakly inward if and only if $\lim_{\lambda \to 0^+} \lambda^{-1}d((1 - \lambda)x + \lambda Tx, C) = 0$ for all $x \in C$, where $d$ denotes the distance to $C$.

Recall that a Banach limit $LIM$ is a bounded linear functional on $l^\infty$ such that

$$||LIM|| = 1, \liminf_{n \to \infty} t_n \leq LIM_n t_n \leq \limsup_{n \to \infty} t_n,$$

and $LIM_n t_n = LIM_n t_{n+1}$ for all $t_n \in l^\infty$.

In what follows, we shall make use of the following lemmas.

**Lemma 1** (Corollary 5.1, Cioranescu [3]). If $X$ is a smooth Banach space, then any duality mapping on $X$ is norm to weak* continuous.

**Lemma 2** (Lemma 13.1, Goebel and Reich [6]). Let $C$ be a convex subset of a smooth Banach space $X$, $D$ a non-empty subset of $C$ and $P$ a retraction from $C$ onto $D$. Then the following are equivalent:

(a) $P$ is a sunny and nonexpansive;
(b) $\langle x - Px, J(z - Px) \rangle \leq 0$ for all $x \in C, z \in D$;
(c) $\langle x - y, J(Px - Py) \rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.

**Lemma 3** (Lemma 1, Ha and Jung [8]). Let $X$ be a Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $X$ and $\{x_n\}$ a bounded sequence in $X$. Let $\text{LIM}$ be a Banach limit and $y \in C$. Then

$$\text{LIM}_n \|x_n - y\|^2 = \min_{z \in C} \text{LIM}_n \|x_n - z\|^2$$

if and only if

$$\text{LIM}_n \langle x - y, J(x_n - y) \rangle \leq 0 \text{ for all } x \in C.$$  

**Lemma 4** (Theorem 10.3, Goebel and Kirk [7]). Let $X$ be a reflexive Banach space which satisfies Opial condition, $C$ a nonempty closed convex subset of $X$ and $T : C \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demi-closed on $C$, where $I$ is the identity mapping.

**Lemma 5** (Lemma 2.2, Sharma and Sahu [20]). Let $X$ be a Banach space and $T$ a mapping with domain and range in $X$. Then following are equivalent:

(a) $T$ is firmly pseudocontractive;
(b) $2I - T$ is pseudocontractive;
(c) $T - I$ is accretive.

**Lemma 6** (Corollary 1, Deimling [4]). Let $C$ be a nonempty closed subset of a Banach space $X$ and $T : C \to X$ a continuous strongly pseudocontractive mapping with constant $k \in [0, 1)$ satisfying

$$\lim_{\lambda \to 0^+} \lambda^{-1}d((1 - \lambda)x + \lambda Tx, C) = 0 \text{ for all } x \in C,$$

where $d$ denotes the distance to $C$ (equivalently, the weakly inward condition under additional assumption that $C$ is convex). Then $T$ has a unique fixed point.

**Lemma 7.** Let $C$ be a nonempty closed convex subset of a smooth Banach space $X$. Let $A : C \to C$ be a continuous strongly pseudocontractive with constant $k \in [0, 1)$. Then variational inequality problem $\text{VIP}(I - A, C)$:

$$\text{to find } u \in C \text{ such that } \langle (I - A)u, J(u - x) \rangle \leq 0 \text{ for all } x \in C$$

has at most one solution.

**Proof.** Let $x^*$ and $y^*$ be two distinct solutions of $\text{VIP}(I - A, C)$. Then

$$\langle x^* - Ax^*, J(x^* - y^*) \rangle \leq 0 \text{ and } \langle y^* - Ay^*, J(y^* - x^*) \rangle \leq 0.$$ 

Adding these inequalities, we get

$$\langle x^* - y^* - (Ax^* - Ay^*), J(x^* - y^*) \rangle \leq 0,$$

which implies that

$$\|x^* - y^*\|^2 \leq \langle Ax^* - Ay^*, J(x^* - y^*) \rangle \leq k\|x^* - y^*\|^2,$$

a contradiction. Therefore, $x^* = y^*$. 

3. Main results

Before proving main results we need the following propositions:

**Proposition 1.** Let $C$ be a nonempty closed convex subset of a normed space $X$. Let $A : C \to C$ be a mapping and $T : C \to X$ another mapping satisfying the weakly inward condition. Then for each $\lambda \in (0, 1)$, the mapping $T^A_\lambda : C \to X$ defined by
\[
T^A_\lambda x = (1 - \lambda)Ax + \lambda Tx, \quad x \in C
\]
satisfies the weakly inward condition.

**Proof.** Let $x \in C$ and $\varepsilon > 0$. Since $T$ is weakly inward, there exists $y \in I_C(x)$ such that $\|y - Tx\| \leq \varepsilon$, and since $C$ is convex, there exists $t_0$ such that $z_t := (1 - t)x + ty \in C$ for $0 < t \leq t_0$. For these $t$ we have
\[
d((1 - t)x + tTx, C) \leq \|(1 - t)x + tTx - z_t\| \leq t\varepsilon.
\]
Moreover, since $C$ is convex,
\[
w_t = \frac{(1 - t + \lambda t)x + (1 - \lambda)tx + \lambda z_t}{1 + \lambda} \in C
\]
for all $\lambda \in (0, 1)$ whenever $t \in (0, 1)$. Set $\alpha := \frac{t}{1 + \lambda}$ and let $t \in (0, 1)$. Then we have
\[
d((1 - \alpha)x + \alpha T^A_\lambda x, C)
\]
\[
\leq \|(1 - \alpha)x + \alpha T^A_\lambda x - w_t\|
\]
\[
= \|(1 + \lambda - t)x + \lambda T^A_\lambda x - (1 + \lambda)w_t\|/(1 + \lambda)
\]
\[
= \|(1 + \lambda - t)x + \lambda [(1 - \lambda)Ax + \lambda Tx] - (1 + \lambda)w_t\|/(1 + \lambda)
\]
\[
= \frac{\lambda}{1 + \lambda} \|(1 - t)x + tTx - z_t\| \leq \frac{t}{1 + \lambda} \varepsilon,
\]
and hence $\lim_{\alpha \to 0^+} \alpha^{-1}d((1 - \alpha)x + \alpha T^A_\lambda x, C) = 0$. By (Lemma 18.1, Deimling [5]), $T^A_\lambda$ satisfies the weakly inward condition. \hfill \Box

**Proposition 2.** Let $C$ be a nonempty closed convex subset of a Banach space $X$. Let $A : C \to C$ be a continuous strongly pseudocontractive with constant $k \in [0, 1)$ and $T : C \to X$ a continuous pseudocontractive mapping satisfying the weakly inward condition. Then

(a) for each $t \in (0, 1)$, there exists unique solution $x_t \in C$ of equation
(3)
\[
x = tAx + (1 - t)Tx,
\]

(b) Moreover, if $v$ is a fixed point of $T$, then for each $t \in (0, 1)$, there exists $j(x_t - v) \in J(x_t - v)$ such that
\[
\langle x_t - Ax_t, j(x_t - v) \rangle \leq 0,
\]

(c) $\{x_t\}$ is bounded.
Proof. (a) For each \( t \in (0, 1) \), the mapping \( T^A_t : C \to X \) defined by
\[
T^A_t x = tAx + (1 - t)Tx, \quad x \in C
\]
is continuous strongly pseudocontractive with constant \( 1 - t(1 - k) \in (0, 1) \).
Indeed, for \( x, y \in C \), there exists \( j(x - y) \in J(x - y) \) such that
\[
\langle T^A_t x - T^A_t y, j(x - y) \rangle = t\langle Ax - Ay, j(x - y) \rangle \\
+ (1 - t)\langle Tx - Ty, j(x - y) \rangle \\
\leq tk\|x - y\|^2 + (1 - t)\|Tx - Ty\|\|x - y\| \\
\leq (1 - t(1 - k))\|x - y\|^2.
\]
From Proposition 1, \( T^A_t \) satisfies the weakly inward condition. Thus, by Lemma 6, there exists a unique fixed point \( x_t \in C \) of \( T^A_t \) such that
\[
(4) \quad x_t = tAx_t + (1 - t)Tx_t.
\]
(b) Suppose that \( v \) is a fixed point of \( T \). Since \( T \) is pseudocontractive, for \( j(x_t - v) \in J(x_t - v) \), we have
\[
\langle x_t - Tx_t, j(x_t - v) \rangle = \langle x_t - v + Tv - Tx_t, j(x_t - v) \rangle \\
= \|x_t - v\|^2 - \langle Tx_t -Tv, j(x_t - v) \rangle \geq 0.
\]
Hence from (4) we have
\[
\langle x_t - Ax_t, j(x_t - v) \rangle = (1 - t)\langle Tx_t - Ax_t, j(x_t - v) \rangle \\
\leq (1 - t)\langle Tx_t - x_t + x_t - Ax_t, j(x_t - v) \rangle,
\]
which implies that
\[
\langle x_t - Ax_t, j(x_t - v) \rangle \leq 0.
\]
(c) By strong pseudocontractivity of \( A \), there exists \( j(x_t - v) \in J(x_t - v) \) such that
\[
\langle Ax_t - Av, j(x_t - v) \rangle \leq k\|x_t - v\|^2.
\]
Using Proposition 2(b), we obtain
\[
\|x_t - v\|^2 = \langle x_t - v, j(x_t - v) \rangle \\
= \langle x_t - Ax_t, j(x_t - v) \rangle + \langle Ax_t - Av, j(x_t - v) \rangle \\
+ \langle Av - v, j(x_t - v) \rangle \\
\leq k\|x_t - v\|^2 + \langle Av - v, j(x_t - v) \rangle.
\]
Thus,
\[
\|x_t - v\|^2 \leq \frac{1}{1 - k}\langle Av - v, j(x_t - v) \rangle,
\]
which yields
\[
\|x_t - v\| \leq \frac{1}{1 - k}\|Av - v\|.
\]
Therefore, \( \{x_t\} \) is bounded. \( \Box \)
Theorem 1. Let $X$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $X$, $A : C \to C$ a continuous strongly pseudocontractive mapping with constant $k \in [0,1)$ and $T : C \to X$ a continuous pseudocontractive mapping satisfying the weakly inward condition. Suppose that every closed convex bounded subset of $C$ has fixed point property for nonexpansive self-mappings. Suppose also that the set

$$E = \{ x \in C : Tx = \lambda x + (1 - \lambda)Ax \text{ for some } \lambda > 1 \}$$

is bounded. For $t \in (0,1)$, let $\{x_t\}$ be the path defined by (4). Then we have the following:

(a) $\lim_{t \to 0^+} x_t = \tilde{x}$ exists,

(b) $\tilde{x}$ is a fixed point of $T$ and it is the unique solution of the variational inequality:

$$( (I - A)\tilde{x}, J(\tilde{x} - v) ) \leq 0 \text{ for all } v \in F(T).$$

Proof. (a) It follows from Theorem 6 of [11] that the mapping $2I - T$ has a nonexpansive inverse, denoted by $g$, which maps $C$ into itself with $F(T) = F(g)$. By Proposition 2(c), $\{x_t\}$ is bounded and hence, the sets $\{Tx_t : t \in (0,1)\}$ and $\{Ax_t : t \in (0,1)\}$ are also bounded. By (4), we have

$$||x_t - Tx_t|| = t||Ax_t - Tx_t|| \to 0 \text{ as } t \to 0^+,$$

which implies that

$$x_t - gx_t \to 0 \text{ as } t \to 0^+. \tag{6}$$

Since $X$ is reflexive, there exists a weakly convergent subsequence $\{x_{t_n}\} \subseteq \{x_t\}$ such that $x_{t_n} \rightharpoonup z$, where $\{t_n\}$ is a sequence in $(0,1)$ such that $t_n \to 0$ as $n \to \infty$.

Now define the function $\varphi : C \to \mathbb{R}$ by

$$\varphi(x) := LIM_n||x_n - x||^2, \quad x \in C.$$

Since $X$ is reflexive, $\varphi(x) \to \infty$ as $||x|| \to \infty$, and $\varphi$ is continuous convex function, by Theorem 1.2 of [1, p. 79] we have that the set

$$M := \{ y \in C : \varphi(y) = \inf_{x \in C} \varphi(x) \} \tag{7}$$

is nonempty. $M$ is also closed convex and bounded. Moreover, $M$ is invariant under $g$. In fact, we have for each $y \in M$,

$$\varphi(gy) = LIM_n||x_n - gy||^2$$
$$= LIM_n||gx_n - gy||^2$$
$$\leq LIM_n||x_n - y||^2 = \varphi(y).$$

So, by the hypothesis, there exists a fixed point $u$ of $g$ in $M$. By Lemma 3, we have

$$LIM_n(z, J(x_n - u)) \leq 0 \text{ for all } z \in C.$$
In particular,

\[(8) \quad LIM_n \langle Au - u, J(x_n - u) \rangle \leq 0.\]

Observe that

\[\|x_n - u\|^2 = \langle x_n - Ax_n, J(x_n - u) \rangle + \langle Ax_n - Au, J(x_n - u) \rangle + \langle Au - u, J(x_n - u) \rangle.\]

By pseudocontractivity of \(T\),

\[(1 - k)\|x_n - u\|^2 \leq \langle x_n - Ax_n, J(x_n - u) \rangle + \langle Au - u, J(x_n - u) \rangle.\]

From (8) and Proposition 2(b), we obtain

\[LIM_n \|x_n - u\|^2 \leq 0.\]

Therefore, there exists a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) such that \(x_{n_i} \to u\). Assume that there is another subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) such that \(x_{n_j} \to \tilde{u}\). Since \(x_n - gx_n \to 0\), it follows that \(\tilde{u} \in F(g)\). Using Proposition 2(b), we have that

\[(9) \quad \langle x_t - Ax_t, J(x_t - v) \rangle \leq 0 \text{ for all } v \in F(T).\]

By norm to weak* uniform continuity of \(J\), we obtain

\[\langle u - Au, J(u - \tilde{u}) \rangle \leq 0 \quad \text{and} \quad \langle \tilde{u} - A\tilde{u}, J(\tilde{u} - u) \rangle \leq 0.\]

Adding these two inequalities yields that

\[\langle u - \tilde{u} + A\tilde{u} - Au, J(u - \tilde{u}) \rangle \leq 0.\]

This implies that

\[\|u - \tilde{u}\|^2 \leq k\|u - \tilde{u}\|^2.\]

Since \(k \in [0, 1]\), it follows that \(u = \tilde{u}\). Thus, \(\{x_n\}\) converges strongly to \(u\).

We finally prove that the entire net \(\{x_t\}\) converges strongly. To this end, we assume that \(\{t_n'\}\) is another subsequence in \((0, 1)\) such that \(x_{t_{n'}} \to u'\) as \(t_{n'} \to 0\). By (6), we obtain \(u' \in F(T)\). From (9), we have that

\[\langle u - Au, J(u - u') \rangle \leq 0 \quad \text{and} \quad \langle u' - Au', J(u' - u) \rangle \leq 0.\]

We must have \(u = u'\). Therefore, \(\{x_t\}\) converges strongly to \(u \in F(T)\).

(b) Since \(x_t \to u \in F(T)\), it follows from Proposition 2(b) and Lemma 7 that \(u\) is a unique point satisfying

\[\langle u - Au, J(u - v) \rangle \leq 0 \text{ for all } v \in F(T).\]

\[\square\]

**Corollary 1.** Let \(X\) be a reflexive Banach space with a uniformly Gâteaux differentiable norm, \(C\) a nonempty closed convex subset of \(X\), \(A : C \to C\) a continuous strongly pseudocontractive mapping with constant \(k \in [0, 1]\) and \(T : C \to C\) a continuous pseudocontractive mapping. Suppose that every closed convex bounded subset of \(C\) has fixed point property for nonexpansive self-mappings. Suppose also that the set

\[E = \{x \in C : Tx = \lambda x + (1 - \lambda)Ax \text{ for some } \lambda > 1\}\]
is bounded. For \( t \in (0, 1) \), let \( \{x_t\} \) be the path defined by (4). Then we have the following:

(a) \( \lim_{t \to 0^+} x_t = \hat{x} \) exists,

(b) \( \hat{x} \) is a fixed point of \( T \) and it is the unique solution of the variational inequality:

\[
\langle (I - A)\hat{x}, J(\hat{x} - v) \rangle \leq 0 \text{ for all } v \in F(T).
\]

**Corollary 2** (Theorem 1, Morales and Jung [14]). Let \( X \) be a reflexive Banach space with a uniformly Gâteaux differentiable norm, \( C \) nonempty closed convex subset of \( X \) and \( T : C \to X \) a continuous pseudocontractive mapping satisfying the weakly inward condition. Suppose every closed convex bounded subset of \( C \) has fixed point property for nonexpansive self mappings. If there exists \( u_0 \in C \) such that the set

\[
E = \{ x \in C : Tx = \lambda x + (1 - \lambda)u_0 \text{ for some } \lambda > 1 \}
\]

is bounded, then the path \( \{x_t : t \in (0, 1)\} \) defined by

\[
x_t = tu_0 + (1 - t)Tx_t
\]

converges strongly to a fixed point of \( T \).

**Proof.** In this case the mapping \( A : C \to C \) defined by \( Ax = u_0 \) for all \( x \in C \) is continuous strongly pseudocontractive with constant 0. The proof follows from Theorem 1. \( \square \)

**Remark 1.** (1) Theorem 1 is also an extension of Theorem 5 of Morales [13] in terms of the space itself and the viscosity type method.

(2) Corollary 1 generalizes the corresponding results in Ha and Jung [8], Moudafi [15], Reich [18], and Xu [22] to ones for pseudocontractive mappings.

(3) Corollary 2 improves Theorem 1 of Xu [21], which is done for nonexpansive mapping and the inwardness condition, as well as Theorem 1 of Jung and Kim [9] for nonexpansive mappings under the additional assumption that \( C \) is a sunny nonexpansive retract of \( X \).

(4) In Theorem 1 and Corollary 1, boundedness of the set \( E \) can be replaced by the assumption that \( F(T) \neq \emptyset \).

We now replace the fixed point property assumption, mentioned in Theorem 1 by imposing certain conditions on the space \( X \) or on the mapping \( T \).

**Theorem 2.** Let \( X \) be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, \( C \) a nonempty closed convex subset of \( X \), \( A : C \to C \) a continuous strongly pseudocontractive mapping with constant \( k \in [0, 1) \) and \( T : C \to X \) a continuous pseudocontractive mapping satisfying the weakly inward condition. If \( T \) has a fixed point in \( C \), then the path \( \{x_t\} \) defined by (4) converges strongly to a fixed point of \( T \), which is a unique solution of variational inequality:

\[
\langle (I - A)\hat{x}, J(\hat{x} - v) \rangle \leq 0 \text{ for all } v \in F(T).
\]
Proof. To be able to use the argument of the proof of Theorem 1, we just need to show that the set \( M \) defined by (7) has a fixed point of \( g \). Since \( F(T) = F(g) \neq \emptyset \), let \( v \in F(g) \). Then the set \( M_0 \) defined by
\[
M_0 = \{ u \in M : ||u - v|| = \inf_{x \in M} ||x - v|| \}
\]
is singleton since \( X \) is strictly convex. Let \( M_0 = \{ u_0 \} \) for some \( u_0 \in M \). Observe that
\[
||gu_0 - v|| = ||gu_0 - gv|| \leq ||u_0 - v|| = \inf_{x \in M} ||x - v||.
\]
Therefore \( gu_0 = u_0 \). We now follow the proof of Theorem 1. \( \Box \)

Next we obtain a convergence of path described by (4) in which continuity assumption of operator \( T \) is weaken and convexity of \( C \) is dispensed.

**Theorem 3.** Let \( X \) be a reflexive Banach space with a weakly continuous duality mapping \( J : X \to X^* \). Let \( C \) be a nonempty closed subset of \( X \), \( A : C \to C \) a continuous strongly pseudocontractive mapping with constant \( k \in [0,1) \) and \( T : C \to X \) a demicontinuous pseudocontractive mapping such that the equation
\[
x = tAx + (1 - t)Tx
\]
has a solution \( x_t \) in \( C \) for each \( t \in [0,1) \). Suppose the path \( \{ x_t \} \) is bounded. Then we have the following:

(a) \( \lim_{t \to 0^+} x_t = \hat{x} \) exists,

(b) \( \hat{x} \) is a fixed point of \( T \) and it is the unique solution of the variational inequality:
\[
\langle (I - A)\hat{x}, J(\hat{x} - v) \rangle \leq 0 \text{ for all } v \in F(T).
\]

**Proof.** (a) Since \( \{ x_t \} \) is bounded, it follows from reflexivity of \( X \) that there exists a subsequence \( \{ x_{t_n} \} \subseteq \{ x_t \} \) such that \( x_{t_n} \rightharpoonup z \in C \) as \( t_n \to 0 \), where \( \{ t_n \} \) is a sequence in \( (0,1) \) such that \( \lim_{n \to \infty} t_n = 0 \). Set \( x_n := x_{t_n} \). As in Theorem 1, \( g : C \to C \) a nonexpansive with \( F(T) = F(g) \). Also \( x_n - gx_n \to 0 \) as \( n \to \infty \). Since \( J \) is weakly continuous, it follows from Lemma 4 that \( z \in F(g) \). By (5), we get
\[
||x_n - z||^2 \leq \frac{1}{1 - k} \langle Az - z, J(x_n - z) \rangle.
\]
Since \( J \) is weakly continuous duality mapping, it follows that \( x_n \rightharpoonup z \) as \( n \to \infty \).

We have already proved that there exists a subsequence \( \{ x_{t_{n'}} \} \) of \( \{ x_t : t \in (0,1) \} \) that converges strongly to a point \( z \in F(T) \). Now it remains to prove that the entire net \( \{ x_t \} \) converges strongly to \( z \). Suppose, for contradiction, that there exists another sequence \( \{ x_{t_{n''}} \} \subseteq \{ x_t \} \) such that \( x_{t_{n''}} \rightharpoonup z' \neq z \) as \( t_{n''} \to 0 \). Then, we have \( z' \in F(T) \). From (9), we have
\[
\langle z - Az, J(z - z') \rangle \leq 0 \text{ and } \langle z'-Az', J(z - z') \rangle \leq 0.
\]
This gives that \( z = z' \). Therefore, \( \lim_{t \to 0^+} x_t \) exists and \( \lim_{t \to 0^+} x_t = z \in F(T) \).
(b) Since \( \lim_{t \to 0^+} x_t = z \), it follows Proposition 2(b) and Lemma 7 that \( z \) is a unique point satisfying
\[
(\langle (I - A)z, J(z - v) \rangle) \leq 0 \quad \text{for all } v \in F(T).
\]

\[\square\]

**Corollary 3** (Theorem 1.2, Schu [19]). Let \( X \) be a reflexive Banach space with a weakly continuous duality mapping \( J: X \to X^* \). Let \( C \) be a nonempty closed convex bounded subset of \( X \), \( u \in C \) and \( T: C \to C \) a continuous pseudocontractive mapping. Let \( \{\lambda_n\} \) be a sequence in \((0,1)\) with \( \lim_{n \to \infty} \lambda_n = 1 \). Then

(a) for each \( n \in \mathbb{N} \), there is exactly one \( x_n \in C \) such that
\[
x_n = (1 - \lambda_n)u + \lambda_nTx_n,
\]

(b) \( \{x_n\} \) converges strongly to a fixed point of \( T \).

**Remark 2.** By putting \( Ax = u \) for all \( x \in C \) in Theorem 2 and Theorem 3, we can also obtain Theorem 2 and Theorem 3 of Morales and Jung [14] as Corollary 2.

### 4. Applications

In 1980, Reich [18] proved the following theorem.

**Theorem R** (Reich [18]). Let \( X \) be a uniformly smooth Banach space and \( C \) a nonempty closed convex subset of \( X \). Let \( T: C \to C \) be a nonexpansive mapping with a fixed point and let \( z \in C \). For each \( t \in (0,1) \), let \( x_t \) be given by \( x_t = tz + (1-t)Tx_t \). Then \( \{x_t\}_{t < 1} \) converges to a fixed point of \( T \) as \( t \to 0^+ \). Thus,
\[
Q(z) := s - \lim_{t \to 0^+} z_t
\]
defines the unique sunny nonexpansive retraction form \( C \) onto \( F(T) \).

O'Hara, Pillay and Xu [16] introduced the Reich's property.

**Definition 1.** A Banach space \( X \) is said to have **Reich property** if for any closed and convex subset \( C \) of \( X \), any nonexpansive mapping \( T: C \to C \) with a fixed point and any \( z \in C \), \( \{x_t\} \) defined by \( x_t = tz + (1-t)Tx_t \) converges strongly to a fixed point of \( T \) as \( t \to 0^+ \).

Thus, every uniformly smooth Banach space has Reich’s property. Let \( C \) be a nonempty closed convex subset of a Banach space \( X \) and \( T: C \to C \) a pseudocontractive mapping. Let \( \Sigma_C \) denote the set of all strongly pseudocontractive mappings \( A: C \to C \) with constant \( k \in [0,1) \). We now introduce the following property:

**Definition 2.** We say that a Banach space \( X \) has **property (S)** if for any closed convex subset \( C \) of \( X \), any pseudocontractive mapping \( T: C \to C \) with \( F(T) \neq \emptyset \) and any \( A \in \Sigma_C \), the path \( \{x_t\} \) defined by (4) converges strongly to a fixed point of \( T \) as \( t \to 0^+ \).
The following theorem shows that property (S) plays a key role in the existence of sunny nonexpansive retraction.

**Theorem 4.** Let $X$ be a smooth Banach space with property (S). Let $C$ be a nonempty closed convex subset of $X$ and $T : C \to C$ a pseudocontractive mapping with $F(T) \neq \emptyset$. If we define $Q : \Sigma_C \to F(T)$ by

$$Q(A) := \lim_{t \to 0^+} x_t, \quad A \in \Sigma_C,$$

then $(AQ(A) - BQ(B), J(Q(A) - Q(B))) \geq \|Q(A) - Q(B)\|^2$ for all $A, B \in \Sigma_C$. In particular, if $A = u \in C$ is a constant, then $Q$ is the sunny nonexpansive retraction from $C$ onto $F(T)$.

**Proof.** For any $A \in \Sigma_C$ and $t \in (0, 1)$, let $x_t$ be the unique point in $C$ such that $x_t = tAx_t + (1 - t)Tx_t$. By Property (S), $\lim_{t \to 0} x_t$ exists; hence $Q(A) = \lim_{t \to 0} x_t$. By Proposition 2(b), we have

$$\langle x_t - Ax_t, J(x_t - v) \rangle \leq 0 \text{ for all } v \in F(T).$$

Taking the limit as $t \to 0^+$ and using Lemma 1, we obtain

$$\langle Q(A) - AQ(A), J(Q(A) - v) \rangle \leq 0.$$

Thus, for $A, B \in \Sigma_C$, we have

$$\langle Q(A) - AQ(A), J(Q(A) - Q(B)) \rangle \leq 0$$

and

$$\langle Q(B) - BQ(B), J(Q(B) - Q(A)) \rangle \leq 0.$$

Adding these two inequalities, we get

$$\langle Q(A) - AQ(A) + BQ(B) - Q(B), J(Q(A) - Q(B)) \rangle \leq 0.$$

Therefore,

$$\|Q(A) - Q(B)\|^2 \leq \langle AQ(A) - BQ(B), J(Q(A) - Q(B)) \rangle.$$

If $A = u$ and $B = v$ then

$$\langle u - v, J(Qu - Qv) \rangle \geq \|Qu - Qv\|^2.$$

By Lemma 2(c), $Q$ is a sunny nonexpansive retraction from $C$ onto $F(T)$. \(\square\)

The following theorem extends Theorem R to one for pseudocontractive mapping. This also improves Theorem 5 of Morales [13].

**Theorem 5.** Let $X$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $X$ and $T : C \to X$ a continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Suppose that every closed convex bounded subset of $C$ has fixed point property for nonexpansive self-mappings. If $T$ satisfies the weakly inward condition, then there exists a unique sunny nonexpansive retraction $Q : C \to F(T)$. 
Proof. For any \( u \in C \) and \( t \in (0,1) \), let \( x_t \) be the unique point in \( C \) such that \( x_t = tu + (1-t)T x_t \). By Theorem 1, \( X \) has property (S) and hence by Theorem 4, there exists a unique sunny nonexpansive retraction form \( C \) onto \( F(T) \) which is given by \( Q(u) = \lim_{t \to 0^+} x_t \).

We now generalize Theorem 3.10 of O’Hara, Pillay and Xu [16] to pseudocontractive one.

**Theorem 6.** Let \( X \) be a reflexive Banach space with a weakly continuous duality mapping \( J : X \to X^* \). \( C \) a nonempty closed convex subset of \( X \) and \( T : C \to X \) a continuous pseudocontractive mapping with \( F(T) \neq \emptyset \). If \( T \) satisfies the weakly inward condition, then there exists a unique sunny nonexpansive retraction \( Q : C \to F(T) \).

**Proof.** The definition of the weak continuity of duality mapping \( J \) implies that \( X \) is smooth. For any \( u \in C \) and \( t \in (0,1) \), let \( x_t \) be the unique point in \( C \) such that \( x_t = tu + (1-t)T x_t \). By Corollary 4, \( X \) has property (S) and hence by Theorem 4, there exists a unique sunny nonexpansive retraction form \( C \) onto \( F(T) \) which is given by \( Q(u) = \lim_{t \to 0^+} x_t \).

Finally, using Lemma 5, Theorem 1 and Theorem 3, we derive strong convergence theorems for firmly pseudocontractive mappings.

**Theorem 7.** Let \( X \) be a reflexive Banach space with a uniformly Gâteaux differentiable norm, \( A : X \to X \) a continuous strongly pseudocontractive mapping with constant \( k \in [0,1) \) and \( T : X \to X \) continuous firmly pseudocontractive mapping. Suppose that every closed convex bounded subset of \( X \) has fixed point property for nonexpansive self mappings. Suppose also that the set 

\[
E' = \{ x \in X : T x = (2 - \lambda)x + (\lambda - 1)A x \text{ for some } \lambda > 1 \}
\]

is bounded. Then we have the following:

(a) For each \( t \in (0,1) \), there is a path \( \{x_t\} \) in \( X \) defined by 

\[
x_t = tA x_t + (1-t)(2I - T)x_t
\]

such that \( \lim_{t \to 0^+} x_t = \bar{x} \) exists,

(b) \( \bar{x} \) is a fixed point of \( T \) and it is the unique solution of variational inequality:

\[
\langle (I - A)\bar{x}, J(\bar{x} - v) \rangle \text{ for all } v \in F(T).
\]

**Theorem 8.** Let \( X \) be a reflexive Banach space with a weakly continuous duality mapping \( J : X \to X^* \). Let \( A : X \to X \) be a continuous strongly pseudocontractive mapping with constant \( k \in [0,1) \) and \( T : X \to X \) a demicontinuous firmly pseudocontractive mapping such that the equation

\[
x = tA x + (1-t)(2I - T)x
\]
has a solution $x_t$ in $C$ for each $t \in [0, 1)$. Suppose the path $\{x_t\}$ is bounded. Then we have the following:

(a) $\lim_{t \to 0^+} x_t = \bar{x}$ exists,

(b) $\bar{x}$ is a fixed point of $T$ and it is the unique solution of the variational inequality:

$$\langle (I - A)\bar{x}, J(\bar{x} - v) \rangle \leq 0 \text{ for all } v \in F(T).$$

References


JONG SOO JUNG  
DEPARTMENT OF MATHEMATICS  
DONG-A UNIVERSITY  
BUSAN 604-714, KOREA  
E-mail address: jungjs@donga.ac.kr

DAYA RAM SAHU  
DEPARTMENT OF APPLIED MATHEMATICS  
SHRI SHANKARACHARYA COLLEGE OF ENGINEERING AND TECHNOLOGY  
JUNWANI, BHILAI - 490020, INDIA  
CURRENT ADDRESS  
DEPARTMENT OF MATHEMATICS  
BANARAS HINDU UNIVERSITY  
VARANASI - 221005, INDIA  
E-mail address: sahuadr@yahoo.com