ON THE ANALOGS OF BERNOULLI AND EULER NUMBERS, RELATED IDENTITIES AND ZETA AND L-FUNCTIONS

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ABSTRACT. In this paper, by using $q$-deformed bosonic $p$-adic integral, we give $\lambda$-Bernoulli numbers and polynomials, we prove Witt’s type formula of $\lambda$-Bernoulli polynomials and Gauss multiplicative formula for $\lambda$-Bernoulli polynomials. By using derivative operator to the generating functions of $\lambda$-Bernoulli polynomials and generalized $\lambda$-Bernoulli numbers, we give Hurwitz type $\lambda$-zeta functions and Dirichlet’s type $\lambda$-$L$-functions; which are interpolated $\lambda$-Bernoulli polynomials and generalized $\lambda$-Bernoulli numbers, respectively. We give generating function of $\lambda$-Bernoulli numbers with order $r$. By using Mellin transforms to their function, we prove relations between multiply zeta function and $\lambda$-Bernoulli polynomials and ordinary Bernoulli numbers of order $r$ and $\lambda$-Bernoulli numbers, respectively. We also study on $\lambda$-Bernoulli numbers and polynomials in the space of locally constant. Moreover, we define $\lambda$-partial zeta function and interpolation function.

Introduction, definitions and notations

Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be denoted by the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$, (cf. [2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 20, 26]).

When one talks of $q$-extension, $q$ considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, as $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the following

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notations:

\[
[x] = [x : q] = \frac{1 - q^x}{1 - q} \quad \text{(cf. [3, 4, 5, 6, 8, 9, 24, 26, 28])}.
\]

Observe that when \( \lim_{q \to 1} [x] = x \), for any \( x \) with \( |x|^p \leq 1 \) in the present \( p \)-adic case \( [x : a] = \frac{1 - a^x}{1 - a} \).

Let \( d \) be a fixed integer and let \( p \) be a fixed prime number. For any positive integer \( N \), we set

\[
X = \lim_{N \to \infty} (\mathbb{Z}/dp^N\mathbb{Z}),
\]

\[
X^* = \cup_{0 < a < dp, (a, p) = 1} (a + dp\mathbb{Z}_p),
\]

\[
a + dp^N\mathbb{Z}_p = \{ x \in X | x \equiv a \pmod{dp^n} \},
\]

where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp^N \). We assume that \( u \in \mathbb{C}_p \) with \( |1 - u|_p \geq 1 \). (cf. [3, 4, 5, 6, 7, 8, 24, 26]).

For \( x \in \mathbb{Z}_p \), we say that \( f \) is a uniformly differentiable function at point \( a \in \mathbb{Z}_p \), and write \( g \in UD(\mathbb{Z}_p) \), the set of uniformly differentiable functions, if the difference quotients,

\[
F_g(x, y) = \frac{g(y) - g(x)}{y - x},
\]

have a limit \( l = g'(a) \) as \( (x, y) \to (a, a) \). For \( f \in UD(\mathbb{Z}_p) \), the \( q \)-deformed bosonic \( p \)-adic integral was defined as

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x)
\]

\[
(A)
\]

\[
= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \mu_q(x + p^N\mathbb{Z}_p)
\]

\[
= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \frac{q^x}{[p^N]}, \quad \text{(cf. [4, 5, 9])}.
\]

By Eq-(A), we have

\[
\lim_{q \to q} I_q(f) = I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x).
\]

This integral, \( I_{-q}(f) \), give the \( q \)-deformed integral expression of fermioinc. The classical Euler numbers were defined by means of the following generating function:

\[
\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}, \quad |t| < \pi \quad \text{(cf. [6, 7, 20, 21])}.
\]
Let \( u \) be algebraic in complex number field. Then Frobenius-Euler polynomials \([6, 7, 20, 21]\) were defined by

\[
\frac{1 - u}{e^t - u} e^{xt} = e^{H(u,x)t} = \sum_{m=0}^{\infty} H_m(u,x) \frac{t^m}{m!},
\]

where we use technical method’s notation by replacing \( H^m(u,x) \) by \( H_m(u,x) \) symbolically. In case \( x = 0 \), \( H_m(u,0) = H_m(u) \), which is called Frobenius-Euler number. The Frobenius-Euler polynomials of order \( r \), denoted by \( H_n^{(r)}(u,x) \), were defined by

\[
\left( \frac{1 - u}{e^t - u} \right)^r e^{tx} = \sum_{n=0}^{\infty} H_n^{(r)}(u,x) \frac{t^n}{n!}. \quad \text{(cf. [7, 10, 25, 26])}
\]

The values at \( x = 0 \) are called Frobenius-Euler numbers of order \( r \). When \( r = 1 \), these numbers and polynomials are reduced to ordinary Frobenius-Euler numbers and polynomials. In the usual notation, the \( n \)-th Bernoulli polynomial were defined by means of the following generating function:

\[
\left( \frac{t}{e^t - 1} \right) e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
\]

For \( x = 0 \), \( B_n(0) = B_n \) are said to be the \( n \)-th Bernoulli numbers. The Bernoulli polynomials of order \( r \) were defined by

\[
\left( \frac{t}{e^t - 1} \right)^r e^{tx} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}
\]

and \( B_n^{(r)}(0) = B_n^{(r)} \) are called the Bernoulli numbers of order \( r \). Let \( x, w_1, w_2, \ldots, w_r \) be complex numbers with positive real parts. When the generalized Bernoulli numbers and polynomials were defined by means of the following generating function:

\[
\frac{w_1 w_2 \cdots w_r t^r e^{xt}}{(e^{w_1 t} - 1)(e^{w_2 t} - 1)\cdots(e^{w_r t} - 1)} = \sum_{n=0}^{\infty} B_n^{(r)}(x \ | \ w_1, w_2, \ldots, w_r) \frac{t^n}{n!}
\]

and \( B_n^{(r)}(0 \ | \ w_1, w_2, \ldots, w_r) = B_n^{(r)}(w_1, w_2, \ldots, w_r) \) (cf. [13, 15]).

The Hurwitz zeta function is defined by

\[
\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^s},
\]

\( \zeta(s, 1) = \zeta(s) \), which is the Riemann zeta function. The multiple zeta functions \([12, 26]\) were defined by

\[
\zeta_r(s) = \sum_{0 < n_1 < n_2 < \cdots < n_r} \frac{1}{(n_1 + \cdots + n_r)^s}.
\]

We summarize our paper as follows:
In section 1, by using $q$-deformed bosonic $p$-adic integral, the generating functions of $\lambda$-Bernoulli numbers and polynomials are given. From these generating functions, we derive many new interesting identities related to these numbers and polynomials and we prove Gauss multiplicative formula for $\lambda$-Bernoulli numbers. Witt's type formula of $\lambda$-Bernoulli polynomials is given.

In section 2, by using derivative operator $\left( \frac{d}{dt} \right)_t^{k} \bigg|_{t=0}$ to the generating function of the $\lambda$-Bernoulli numbers, we construct Hurwitz' type $\lambda$-zeta function, which interpolates $\lambda$-Bernoulli polynomials at negative integers.

In section 3, by using same method of section 2, we give Dirichlet type $\lambda$-$L$-function which interpolates generalized $\lambda$-Bernoulli numbers.

In section 4, the generating functions of $\lambda$-Bernoulli numbers of order $r$ are obtained. From these generating generating functions, we derive some interesting relations between multiple zeta functions and $\lambda$-Bernoulli numbers of order $r$.

In section 5, we give some important identities related to generalized $\lambda$-Bernoulli numbers of order $r$.

In section 6, we study on $\lambda$-Bernoulli numbers and polynomials in the space of locally constant. In this section, we also define $\lambda$-partial zeta function which interpolates $\lambda$-Bernoulli numbers at negative integers.

In section 7, we give $p$-adic interpolation functions.

1. $\lambda$-Bernoulli numbers

In this section, by using Eq-(A), we give integral equation of bosonic $p$-adic integral. By using this integral equation we define generating function of $\lambda$-Bernoulli polynomials. We give fundamental properties of the $\lambda$-Bernoulli numbers and polynomials. We also give some new identities related to $\lambda$-Bernoulli numbers and polynomials. We prove Gauss multiplicative formula for $\lambda$-Bernoulli numbers as well. Witt's type formula of $\lambda$-Bernoulli polynomials is given.

To give the expression of bosonic $p$-adic integral in Eq-(A), we consider the limit

$$ I_{l}(f) = \lim_{q \to 1} I_{q}(f) = \int_{\mathbb{Z}_{p}} f(x) d\mu_{l}(x) \quad \text{(cf. [16, 17, 18, 21])}, $$

in the sense of bosonic $p$-adic integral on $\mathbb{Z}_{p}$ ($= p$-adic invariant integral on $\mathbb{Z}_{p}$). From this $p$-adic invariant integral on $\mathbb{Z}_{p}$, we derive the following integral equation:

$$ I_{l}(f_{1}) = I_{l}(f) + f'(0) \quad \text{(cf. [17])}, $$

where $f_{1}(x) = f(x + 1)$. Let $C_{p^{n}}$ be the space of primitive $p^{n}$-th root of unity, $C_{p^{n}} = \{ \zeta \mid \zeta^{p^{n}} = 1 \}.$
Then, we denote
\[ T_p = \lim_{n \to \infty} C_{p^n} = \bigcup_{n \geq 0} C_{p^n}. \]

For \( \lambda \in \mathbb{Z}_p \), we take \( f(x) = \lambda^x e^{tx} \), and \( f_1(x) = e^t \lambda f(x) \). Thus we have
\[
(2) \quad f_1(x) - f(x) = (\lambda e^t - 1) f(x).
\]

By substituting (2) into (1), we get
\[
(2a) \quad (\lambda e^t - 1) I_1(f) = f'(0), \quad (\text{cf. [4, 21]}).
\]

Consequently, we have
\[
(3) \quad \frac{\log \lambda + t}{\lambda e^t - 1} := \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}, \quad (\text{cf. [4]}).
\]

By using Eq-(3), we obtain
\[
\lambda(B(\lambda) + 1)^n - B_n(\lambda) = \begin{cases} 
\log \lambda, & \text{if } n = 0 \\
1, & \text{if } n = 1 \\
0, & \text{if } n > 1,
\end{cases}
\]

with the usual convention of replacing \( B_n(\lambda) \) by \( B^n(\lambda) \), (cf. [4, 17, 18, 21]). From this result, we derive the values of some \( B_n(\lambda) \) numbers as follows:
\[
B_0(\lambda) = \frac{\log \lambda}{\lambda - 1}, \quad B_1(\lambda) = \frac{\lambda - 1 - \lambda \log \lambda}{(\lambda - 1)^2}, \ldots, \quad (\text{cf. [4, 17, 21]}).
\]

We note that, if \( \lambda \in T_p \), for some \( n \in \mathbb{N} \), then Eq-(2a) is reduced to the following generating function:
\[
(3a) \quad \frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!} \quad (\text{cf. [4]}).
\]

If \( \lambda = e^{2\pi i/f}, \ f \in \mathbb{N} \) and \( \lambda \in \mathbb{C} \), then Eq-(3) is reduced to (3a). Eq-(3a) is obtained by Kim [3]. Let \( u \in \mathbb{C} \), then by substituting \( x = 0 \) into Eq-(A1), we set
\[
(3b) \quad \frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!} \quad (\text{cf. [4, 17, 18, 21]}).
\]

\( H_n(u) \) is denoted Frobenius-Euler numbers. Relation between \( H_n(u) \) and \( B_n(\lambda) \) is given by the following theorem:

**Theorem 1.** Let \( \lambda \in \mathbb{Z}_p \). Then
\[
B_n(\lambda) = \frac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \frac{n H_{n-1}(\lambda^{-1})}{\lambda - 1},
\]

\[
B_0(\lambda) = \frac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1}).
\]
Proof. By using Eq-(3), we have
\[
\sum_{n=0}^{\infty} \frac{B_n(\lambda) t^n}{n!} = \frac{\log \lambda + t}{\lambda e^t - 1} = \frac{\log \lambda}{\lambda e^t - 1} + \frac{t}{\lambda e^t - 1}
\]
\[
= \frac{1 - \lambda^{-1}}{(1 - \lambda^{-1})\lambda} \cdot \left( \frac{\log \lambda}{e^t - \lambda^{-1}} \right) - \frac{(1 - \lambda^{-1})}{(e^t - \lambda^{-1})} \cdot \frac{t}{\lambda(1 - \lambda^{-1})}
\]
\[
= \frac{\log \lambda}{\lambda - 1} \sum_{n=0}^{\infty} H_n(\lambda^{-1}) \frac{t^n}{n!} + \frac{t}{\lambda - 1} \sum_{n=0}^{\infty} H_n(\lambda^{-1}) \frac{t^n}{n!},
\]
the next to the last step being a consequence of Eq-(3b). After some elementary calculations, we have
\[
\sum_{n=0}^{\infty} \frac{B_n(\lambda) t^n}{n!} = \frac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1})
\]
\[
+ \sum_{n=1}^{\infty} \left( \frac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \frac{n}{\lambda - 1} H_{n-1}(\lambda^{-1}) \right) \frac{t^n}{n!}.
\]
By comparing coefficient \( \frac{t^n}{n!} \) in the above, then we obtain the desired result. \( \square \)

Observe that, if \( \lambda \in T_p \) in Eq-(4), then we have, \( B_0(\lambda) = 0 \) and \( B_n(\lambda) = \frac{nH_{n-1}(\lambda^{-1})}{\lambda - 1}, n \geq 1. \)

By Eq-(3) and Eq-(4), we obtain the following formula:

For \( n \geq 0, \lambda \in \mathbb{Z}_p \)
\[
(4a) \quad \int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = \begin{cases} 
\frac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1}), & n = 0 \\
\frac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \frac{n}{\lambda - 1} H_{n-1}(\lambda^{-1}), & n > 0
\end{cases}
\]
and
\[
(4b) \quad \int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = B_n(\lambda), \quad n \geq 0.
\]

Now, we define \( \lambda \)-Bernoulli polynomials, we use these polynomials to give the sums powers of consecutive. The \( \lambda \)-Bernoulli polynomials are defined by means of the following generating function:
\[
\frac{\log \lambda + t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(\lambda; x) \frac{t^n}{n!}.
\]
By Eq-(3) and Eq-(5), we have
\[
B_n(\lambda; x) = \sum_{k=0}^{n} \binom{n}{k} B_k(\lambda)x^{n-k}.
\]

The Witt’s formula for \( B_n(\lambda; x) \) is given by the following theorem:
Theorem 2. For $k \in \mathbb{N}$ and $\lambda \in \mathbb{Z}_p$, we have

$$B_n(\lambda; x) = \int_{\mathbb{Z}_p} (x + y)^n \lambda^y d\mu_1(y).$$

Proof. By substituting $f(y) = e^{(x+y)\lambda^y}$ into Eq-(1), we have

$$\int_{\mathbb{Z}_p} e^{(x+y)\lambda^y} d\mu_1(y) = \sum_{n=0}^{\infty} B_n(\lambda; x) \frac{t^n}{n!} = \frac{(\log \lambda + t)e^{tx}}{\lambda e^t - 1}.$$

By using Taylor expansion of $e^{tx}$ in the left side of the above equation, after some elementary calculations, we obtain the desired result. \qed

We now give the distribution of the $\lambda$-Bernoulli polynomials.

Theorem 3. Let $n \geq 0$, and let $d \in \mathbb{Z}^+$. Then we have

$$B_n(\lambda; x) = d^{n-1} \sum_{a=0}^{d-1} \lambda^a B_n \left( \lambda^d; \frac{x + a}{d} \right).$$

Proof. By using Eq-(6),

$$B_n(x; \lambda) = \int_{\mathbb{Z}_p} (x + y)^n \lambda^y d\mu_1(y)$$

$$= \lim_{N \to \infty} \frac{1}{dp^N} \sum_{y=0}^{p^N-1} (x + y)^n \lambda^y$$

$$= \lim_{N \to \infty} \frac{1}{dp^N} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (a + dy + x)^n \lambda^{a+dy}$$

$$= d^{n-1} \lim_{N \to \infty} \frac{1}{p^N} \sum_{a=0}^{d-1} \lambda^a \sum_{y=0}^{p^N-1} \left( \frac{a + x}{d} + y \right)^n (\lambda^d)^y$$

$$= d^{n-1} \frac{1}{p^N} \sum_{a=0}^{d-1} \lambda^a \int_{\mathbb{Z}_p} \left( \frac{a + x}{d} + y \right)^n (\lambda^d)^y.$$

Thus, we have the desired result. \qed

By substituting $x = 0$ into Eq-(7), we have the following corollary:

Corollary 1. For $m, n \in \mathbb{N}$, we have

$$mB_n(\lambda) = \sum_{j=0}^{n} \binom{n}{j} B_j(\lambda^m) m^j \sum_{a=0}^{m-1} \lambda^a a^{n-j}.$$  

(Gauss multiplicative formula for $\lambda$-Bernoulli numbers).

By Eq-(8), we have
Theorem 4. For \( m, n \in \mathbb{N} \) and \( \lambda \in \mathbb{Z}_p \), we have
\[
mB_n(\lambda) - m^n[m]_\lambda B_n(\lambda^m) = \sum_{j=0}^{n-1} \binom{n}{j} B_j(\lambda^m) m^j \sum_{k=1}^{m-1} \lambda^k k^{n-j}.
\]

Theorem 5. Let \( k \in \mathbb{Z} \), with \( k > 1 \). Then we have
\[
B_l(\lambda; k) - \lambda^{-k} B_l(\lambda) = \lambda^{-k} \sum_{n=0}^{k-1} \lambda^n n^{l-1} + (\lambda^{-k} \log \lambda) \sum_{n=0}^{k-1} n^l \lambda^n.
\]

Proof. We set
\[
- \sum_{n=0}^{\infty} e^{(n+k)t} \lambda^n + \sum_{n=0}^{\infty} e^{nt} \lambda^{n-k} = \sum_{n=0}^{k-1} e^{nt} \lambda^{n-k}
\]
\[
= \sum_{l=0}^{\infty} (\lambda^{-k} \sum_{n=0}^{k-1} n^l \lambda^n) \frac{t^l}{l!}.
\]
\[
= \sum_{l=1}^{\infty} (\lambda^{-kl} \sum_{n=0}^{k-1} n^{l-1} \lambda^n) \frac{t^{l-1}}{l!}.
\]

Multiplying \((t + \log \lambda)\) both side of Eq-(10a), then by using Eq-(3) and Eq-(5), after some elementary calculations, we have
\[
\sum_{l=0}^{\infty} (B_l(\lambda; k) - \lambda^{-k} B_l(\lambda)) \frac{t^l}{l!}
\]
\[
= \sum_{l=0}^{\infty} (\lambda^{-k} \log \lambda) \sum_{n=0}^{k-1} n^l \lambda^n \frac{t^l}{l!}.
\]

By comparing coefficient \(\frac{t^l}{l!}\) in both sides of Eq-(10b). Thus we arrive at the Eq-(10). Thus we complete the proof of theorem.

Observe that \(\lim_{\lambda \to 1} B_l(\lambda) = B_l\). For \( \lambda \to 1 \), then Eq-(10) reduces the following:
\[
B_l(k) - B_l = l \sum_{n=0}^{k-1} n^{l-1}.
\]

If \( \lambda \in T_p \), then Eq-(10) reduces to the following formula:
\[
B_l(\lambda; k) - \lambda^{-k} B_l(\lambda) = \lambda^{-k} l \sum_{n=0}^{k-1} \lambda^n n^{l-1}.
\]

Remark. Garrett and Hummel [2] proved combinatorial proof of \( q \)-analogue of
\[
\sum_{k=1}^{n} k^3 = \binom{n+1}{k}^2
\]
as follows:
\[
\sum_{k=1}^{n} q^{k-1} [k]_q^{2} \left( \begin{bmatrix} k - 1 \\ 2 \end{bmatrix}_q + \begin{bmatrix} k + 1 \\ 2 \end{bmatrix}_q \right) = \begin{bmatrix} n + 1 \\ 2 \end{bmatrix}_q,
\]
where \([n]_q = \prod_{j=1}^{n} [n + 1 - j]_q\), \(q\)-binomial coefficients. In [12], Kim constructed the following formula
\[
S_{n,q^{k}}(k) = \sum_{l=0}^{k-1} q^l [l]_q^{n}
= \frac{1}{n + 1} \sum_{j=0}^{n} \binom{n + 1}{j} \beta_{j,q} q^{k j} [k]^{n+1-j} - \frac{(1 - q^{(n+1)k}) \beta_{n+1,q}}{n + 1},
\]
where \(\beta_{j,q}\) are the \(q\)-Bernoulli numbers which were defined by
\[
\frac{e^{t/q} - 1}{\log q} - t \sum_{n=0}^{\infty} q^{n+x} e^{(n+x)t} = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!}, \quad |q| < 1, |t| < 1,
\]
\(\beta_{n,q}(0) = \beta_{n,q}\) (cf. [11, 12]).

Schlosser [22] gave for \(n = 1, 2, 3, 4, 5\) the value of \(S_{n,q^{k}}[k]\). In [27], the authors also gave another proof of \(S_{n,q}(k)\) formula.

2. Hurwitz’s type \(\lambda\)-zeta function

In this section, by using generating function of \(\lambda\)-Bernoulli polynomials, we construct Hurwitz’s type \(\lambda\)-zeta function, which is interpolate \(\lambda\)-Bernoulli polynomials at negative integers. By Eq-(5), we get
\[
F_{\lambda}(t; x) = \frac{\log \lambda + t}{\lambda e^t - 1} e^{xt} = - (\log \lambda + t) \sum_{n=0}^{\infty} \lambda^n e^{(n+x)t}
= \sum_{n=0}^{\infty} B_n(\lambda) t^n.
\]
By using \(\frac{d^k}{dt^k}\) derivative operator to the above, we have
\[
B_k(\lambda; x) = \frac{d^k}{dt^k} F_{\lambda}(t; x) \bigg|_{t=0},
B_k(\lambda; x) = - \log \lambda \sum_{n=0}^{\infty} \lambda^n (n + x)^k - k \sum_{n=0}^{\infty} (n + x)^{k-1} \lambda^n.
\]
Thus we arrive at the following theorem:
Theorem 6. For \( k \geq 0 \), we have
\[
-\frac{1}{k} B_k(\lambda; x) = \frac{\log \lambda}{k} \sum_{n=0}^{\infty} \lambda^n(n + x)^k + \sum_{n=0}^{\infty} \lambda^n(n + x)^{k-1}.
\]

Consequently, we define Hurwitz type zeta function as follows:

Definition 1. Let \( s \in \mathbb{C} \). Then we define
\[
\zeta(\lambda, s, x) = \frac{\log \lambda}{1-s} \sum_{n=0}^{\infty} \frac{\lambda^n}{(n + x)^s} + \sum_{n=0}^{\infty} \frac{\lambda^n}{(n + x)^s}.
\]

Note that \( \zeta(\lambda, s, x) \) is analytic continuation, except for \( s = 1 \), in whole complex plane. By Definition 1 and Theorem 6, we have the following:

Theorem 7. Let \( s = 1 - k, k \in \mathbb{N} \). Then
\[
\zeta(1 - k, x) = -\frac{B_k(\lambda, x)}{k}.
\]

3. Generalized \( \lambda \)-Bernoulli numbers associated with Dirichlet type \( \lambda \)-\( L \)-functions

By using Eq-(0), we define
\[
I_1(f_d) = I_1(f) + \sum_{j=0}^{d-1} f'(j),
\]
where \( f_d(x) = f(x + d), \int_X f(x)d\mu(x) = I_1(f) \).

Let \( \chi \) be a Dirichlet character with conductor \( d \in \mathbb{N}^+, \lambda \in \mathbb{Z}_p \).

By substituting \( f(x) = \chi(x)e^{\lambda x} \) into Eq-(12), then we have
\[
\int_X \chi(x) \lambda^x e^{\lambda x} d\mu_1(x) = \sum_{j=0}^{d-1} \frac{\chi(j) \lambda^j e^{\lambda j} (\log \lambda + t)}{\lambda^d e^{dt} - 1}
\]
\[
= \sum_{n=0}^{\infty} B_{n, \chi}(\lambda) \frac{t^n}{n!}.
\]

By Eq-(12a), we easily see that
\[
B_{n, \chi}(\lambda) = \int_X \chi(x) x^n \lambda^x d\mu_1(x).
\]

From Eq-(12a), we define generating function of generalized Bernoulli number by
\[
F_{\lambda, \chi}(t) = \sum_{j=0}^{d-1} \frac{\chi(j) \lambda^j e^{\lambda j} (\log \lambda + t)}{\lambda^d e^{dt} - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}.
\]
Observe that if $\lambda \in T_p$, then the above formula reduces to

$$F_{\lambda, \chi}(t) = \sum_{j=0}^{d-1} \frac{\chi(j) \lambda^j e^{jt}}{\lambda^d e^{dt} - 1} = \sum_{j=0}^{\infty} B_n(\lambda) \frac{e^{tn}}{n!}$$

(for detail see cf. [3, 16, 18, 22, 23, 24]).

From the above, we easily see that

$$F_{\lambda, \chi}(t) = - (\log \lambda + t) \sum_{m=1}^{\infty} \chi(m) \lambda^m e^{tm} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}.$$

By applying $\frac{d^k}{dt^k} \bigg|_{t=0}$ derivative operator both sides of the above equation, we arrive at the following theorem:

**Theorem 8.** Let $k \in \mathbb{Z}^+$, $\lambda \in \mathbb{Z}_p$ and let $\chi$ be a Derichlet character with conductor $d$. Then we have

$$\sum_{m=1}^{\infty} \chi(m) \lambda^m m^{k-1} + \frac{\log \lambda}{k} \sum_{m=1}^{\infty} \lambda^m \chi(m) m^k = - \frac{B_k(\chi(\lambda))}{k}. \quad (13)$$

**Definition 2** (Dirichlet type $\lambda$-$L$ function). For $\lambda, s \in \mathbb{C}$, we define

$$L_\lambda(s, \chi) = \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^s} - \frac{\log \lambda}{s - 1} \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^{s-1}}. \quad (14)$$

Relation between $L_\lambda(s, \chi)$ and $\zeta_\chi(s, y)$ is given by the following theorem:

**Theorem 9.** Let $s \in \mathbb{C}$ and $d \in \mathbb{Z}^+$. Then we have

$$L_\lambda(s, \chi) = d^{-s} \sum_{a=1}^{d} \lambda^a \chi(a) \zeta_\chi^d \left( s, \frac{a}{d} \right).$$

**Proof.** By substituting $m = a + dk$, $a = 1, 2, \ldots, d$, $k = 0, 1, \ldots, \infty$, into Eq. (14), we have

$$L_\lambda(s, \chi) = \sum_{a=1}^{d} \sum_{k=0}^{\infty} \frac{\lambda^a + dk \chi(a + dk)}{(a + dk)^s} - \frac{\log \lambda}{s - 1} \sum_{a=1}^{d} \sum_{k=0}^{\infty} \frac{\lambda^a + dk \chi(a + dk)}{(a + dk)^{s-1}}$$

$$= d^{-s} \sum_{a=1}^{d} (\lambda^a \chi(a)) \left[ \sum_{k=0}^{\infty} \frac{(\lambda^d)^k}{(k + \frac{a}{d})^s} - \frac{\log \lambda^d}{s - 1} \sum_{k=0}^{\infty} \frac{(\lambda^d)^k}{(k + \frac{a}{d})^{s-1}} \right].$$

By using Eq-(11) in the above we obtain the desired result. \qed

**Theorem 10.** For $k \in \mathbb{Z}^+$, we have

$$L_\lambda(1 - k, \chi) = - \frac{1}{k} B_{k, \chi}(\lambda), \quad k > 0.$$

**Proof.** By substituting $s = 1 - k$ in Definition 2 and using Eq-(13), we easily obtain the desired result. \qed
Remark. If $\lambda \in T_p$, then from Definition 2, we have

$$L_{\lambda}(s, \chi) = \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^s}.$$ 

In [21, 18], Kim studied on the $\lambda$-Euler numbers and he gave interesting many relations on $\lambda$-Euler numbers and polynomials. $\lambda$-Bernoulli numbers and polynomials are corresponding to $\lambda$-Euler numbers and polynomials (see [21]). In [17, 18], Kim et al gave $\lambda- (h, q)$ zeta function and $\lambda-(h, q)$ $L$-function. These functions interpolate $\lambda-(h, q)$-Bernoulli numbers at negative integer. Observe that, if we take $s = 1 - k$ in Theorem 9, and then using Eq-(12) in Theorem 7, we get another proof of Theorem 10.

4. $\lambda$-Bernoulli numbers of order $r$ associated with multiple zeta function

In this section, we define generating function of $\lambda$-Bernoulli numbers of order $r$. By using Mellin transforms and Cauchy residue theorem, we obtain multiple zeta function which is given in Eq-(C). We also gave relations between $\lambda$-Bernoulli polynomials of order $r$ and multiple zeta function at negative integers. This relation is important and very interesting. Let $r \in \mathbb{Z}^+$. Generating function of $\lambda$-Bernoulli numbers of order $r$ is defined by

$$F_{\lambda}^{(r)}(t) = \left( \frac{\log \lambda + t}{\lambda e^t - 1} \right)^r = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!}.$$  

Generating function of $\lambda$-Bernoulli polynomials of order $r$ is defined by

$$F_{\lambda}^{(r)}(t, x) = F_{\lambda}^{(r)}(t)e^{tx} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!}.$$ 

Observe that when $r = 1$, Eq-(15) reduces to Eq-(3). By applying Mellin transforms to the Eq-(15) we get

$$\frac{1}{\Gamma(s)} \int_0^\infty \lambda^r e^{-\lambda t} F_{\lambda}^{(r)}(-t)(t - \log \lambda)^{s-r-1} dt$$

$$= \sum_{n_1, \ldots, n_r=0}^{\infty} \frac{1}{(n_1 + n_2 + \cdots + n_r + r)^s}.$$ 

Thus, we get, by (C)

$$\zeta_r(s) = \frac{1}{\Gamma(s)} \int_0^\infty \lambda^r e^{-\lambda t} F_{\lambda}^{(r)}(-t)(t - \log \lambda)^{s-r-1} dt.$$ 

By using the above relation, we obtain the following theorem:

**Theorem 11.** Let $r, m \in \mathbb{Z}^+$. Then we have

$$(D1) \quad \zeta_r(-m) = (-\lambda)^r m! \sum_{j=0}^{\infty} \left( \begin{array}{c} -m - r - 1 \\ j \end{array} \right) \left( \log \lambda \right)^j \frac{B_{m+r+j}(\lambda; \tau)}{(m + r + j)!}.$$
Remark. If \( \lambda \to 1 \), the above theorem reduces to

\[
(D2) \quad \zeta_r(-m) = (-1)^r m! \frac{B_{m+r}(1; r)}{(m + r)!}
\]

which is given Theorem 6 in [13].

By \((D1)\) and \((D2)\), we obtain relation between \( \lambda \)-Bernoulli polynomials of order \( r \) and ordinary Bernoulli polynomials of order \( r \) as follows:

\[
B_{m+r}(r) = \lambda^r \sum_{j=0}^{\infty} \binom{-m - r - 1}{j} \frac{B_{m+r+j}(\lambda; r)}{(m + r + j)!} (m + r)!,
\]

where \( m, r \in \mathbb{Z}^+ \).

We now give relations between \( B_3^{(r)}(\lambda) \) and \( H^{(r)}_n(\lambda^{-1}) \) as follows:

If \( \lambda \in T_r \), then Eq-(15) reduces to the following equation

\[
\frac{t^r}{(\lambda e^t - 1)^r} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!}.
\]

Thus by the above equation, we easily see that

\[
t^r = (\lambda e^t - 1)^r e^{B^{(r)}(\lambda)t}
\]

\[
= \sum_{l=0}^{r} \lambda^l (-1)^{r-l} e^{(B^{(r)}(\lambda)+l)t}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{r} \lambda^l (-1)^{r-l} (B^{(r)}(\lambda) + l)^n \right) \frac{t^n}{n!}.
\]

Consequently we have

\[
\sum_{l=0}^{r} \lambda^l (-1)^{r-l} (B^{(r)}(\lambda) + l)^n = \begin{cases} 0 & \text{if } n \neq r \\ 1 & \text{if } n = r. \end{cases}
\]

By Eq-(15) we obtain

\[
\sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!} = \frac{t^r}{(\lambda - 1)^r} \sum_{n=0}^{\infty} H_n^{(r)}(\lambda^{-1}) \frac{t^n}{n!}.
\]

By comparing coefficient \( \frac{t^n}{n!} \) in the both sides of the above equation, we have

\[
B_{n+r}(r) = \frac{\Gamma(n + r + 1)}{\Gamma(n + 1)} \frac{1}{(\lambda - 1)^r} H_n^{(r)}(\lambda^{-1}).
\]

Observe that, if we take \( r = 1 \), then the above identity reduce to Eq-(4), that is

\[
B_{n+1}(1) = \frac{(n + 1)}{\lambda - 1} H_n(\lambda^{-1}).
\]
5. $\lambda$-Bernoulli numbers and polynomials associated with multivariate $p$-adic invariant integral

In this section, we give generalized $\lambda$-Bernoulli numbers of order $r$. Consider the multivariate $p$-adic invariant integral on $\mathbb{Z}_p$ to define $\lambda$-Bernoulli numbers and polynomials.

\[
\int_{\mathbb{Z}_p} \ldots \int_{\mathbb{Z}_p} \lambda^{w_1 x_1 + \cdots + w_r x_r} e^{(w_1 x_1 + \cdots + w_r x_r)t} \, d\mu_1(x_1) \cdots d\mu_1(x_r)
\]

\[
= \frac{(w_1 \log \lambda + w_1 t) \cdots (w_r \log \lambda + w_r t)}{(\lambda^{w_1} e^{w_1 t} - 1) \cdots (\lambda^{w_r} e^{w_r t} - 1)}
\]

\[
= \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; w_1, w_2, \ldots, w_r) \frac{t^n}{n!},
\]

where we called $B_n^{(r)}(\lambda; w_1, w_2, \ldots, w_r)$ $\lambda$-extension of Bernoulli numbers. Substituting $\lambda = 1$ into Eq-(16), $\lambda$-extension of Bernoulli numbers reduce to Barnes Bernoulli numbers as follows:

\[
\frac{(w_1 t) \cdots (w_r t)}{(e^{w_1 t} - 1) \cdots (e^{w_r t} - 1)} = \sum_{n=0}^{\infty} B_n^{(r)}(w_1, \ldots, w_r) \frac{t^n}{n!},
\]

where $B_n^{(r)}(w_1, \ldots, w_r)$ are denoted Barnes Bernoulli numbers and $w_1, \ldots, w_r$ complex numbers with positive real parts [1, 7, 26]. Observe that when $w_1 = w_2 = \cdots = w_r = 1$ in Eq-(16), we obtain the $\lambda$-Bernoulli numbers of higher order as follows:

\[
\left( \frac{\log \lambda + t}{\lambda e^t - 1} \right)^r = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!}.
\]

We note that $B_n^{(r)}(\lambda; 1, 1, \ldots, 1) = B_n^{(r)}(\lambda)$. Consider

\[
\left( \frac{\log \lambda + t}{\lambda e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x) \frac{t^n}{n!}.
\]

Observe that

\[
\sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x) \frac{t^n}{n!} = \left( \frac{\log \lambda + t}{\lambda e^t - 1} \right)^r e^{(\log \lambda + t)x} \lambda^{-x} = \frac{1}{\lambda x} \sum_{m=0}^{\infty} B_m^{(r)}(\lambda; x) \frac{(t + \log \lambda)^m}{m!}
\]
\[
= \frac{1}{\lambda^x} \sum_{m=0}^{\infty} \frac{B_m^{(r)}(\lambda; x)}{m!} \sum_{l=0}^{m} \binom{m}{l} (\log \lambda)^m t^{m-l} \\
= \sum_{n=0}^{\infty} \left( \frac{1}{\lambda^x} \sum_{l=0}^{\infty} \frac{B_{n+l}^{(r)}(\lambda; x)}{l!} (\log \lambda)^l \right) \frac{t^n}{n!}.
\]

Now, comparing coefficient \( \frac{t^n}{n!} \) both sides of the above equation, we easily arrive at the following theorem:

**Theorem 12.** For \( n, r \in \mathbb{N} \) and \( \lambda \in \mathbb{Z}_p \), we have

\[
B_n^{(r)}(\lambda; x) = \frac{1}{\lambda^x} \sum_{l=0}^{\infty} B_{n+l}^{(r)}(\lambda; x) \frac{(\log \lambda)^l}{l!},
\]

where \( 0^l = \begin{cases} 
1 & \text{if } l = 0 \\
0 & \text{if } l \neq 0.
\end{cases} \)

**Remark.** In Theorem 12, we see that

\[
\lim_{\lambda \to 1} B_n^{(r)}(\lambda; x) = \begin{cases} 
B_n^{(r)}(x) & \text{if } l = 0, \\
0 & \text{if } l \neq 0.
\end{cases}
\]

6. \( \lambda \)-Bernoulli numbers and polynomials in the space of locally constant

In this section, we construct partial \( \lambda \)-zeta functions, we need this function in the following section. We need this function in the following section. By Eq-(3b), Frobenius-Euler polynomials are defined by means of the following generating function:

\[
\left( \frac{1 - u}{e^t - u} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(u, x) \frac{t^n}{n!}.
\]

As well known, we note that the Frobenius-Euler polynomials of order \( r \) were defined by

\[
\left( \frac{1 - u}{e^t - u} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(u, x) \frac{t^n}{n!}.
\]

The case \( x = 0 \), \( H_n^{(r)}(u, 0) = H_n^{(r)}(u) \), which are called Frobenius-Euler numbers of order \( r \).

If \( \lambda \in T_p \), then \( \lambda \)-Bernoulli polynomials of order \( r \) are given by

\[
\frac{t^r}{(\lambda e^t - 1)^r} e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x) \frac{t^n}{n!}.
\]
Hurwitz type $\lambda$-zeta function is given by

$$\zeta_\lambda(s, x) = \sum_{n=0}^{\infty}\frac{\lambda^n}{(n + x)^s}, \quad \lambda \in T_p.$$  

Thus, from Theorem 7, we have

$$\zeta_\lambda(1 - k, x) = -\frac{1}{k}B(\lambda; x), \quad k \in \mathbb{Z}^+.\tag{17a}$$

We now define $\lambda$-partial zeta function as follows

$$H_\lambda(s, a|F) = \sum_{m \equiv a \pmod{F}}\frac{\lambda^m}{m^s}.\tag{17b}$$

From (17), we have

$$H_\lambda(s, a|F) = \frac{\lambda^a}{F^s}\zeta_{\lambda^F}\left(s, \frac{a}{F}\right),\tag{17c}$$

where $\zeta_{\lambda^F}\left(s, \frac{a}{F}\right)$ is given by Eq-(17). By Eq-(17a) we have

$$H_\lambda(1 - n, a|F) = -\frac{F^{n-1}\lambda^aB_n(\lambda^F; \frac{a}{F})}{n}, \quad n \in \mathbb{Z}^+.\tag{18}$$

If $\lambda \in T_p$, then by Eq-(14), we have

$$L_\lambda(s, \chi) = \sum_{n=1}^{\infty}\frac{\lambda^n\chi(n)}{n^s},$$

where $s \in \mathbb{C}$, $\chi$ be the primitive Dirichlet character with conductor $f \in \mathbb{Z}^+$. By Theorem 9, Eq-(17c) and Eq-(18) we easily see that

$$L_\lambda(s, \chi) = \sum_{a=1}^{F}\chi(a)H_\lambda\left(s, \frac{a}{F}\right),$$

and

$$L_\lambda(1 - k, \chi) = -\frac{B_{k, \chi}(\lambda)}{k}, \quad k \in \mathbb{Z}^+,\tag{19}$$

where $B_{k, \chi}(\lambda)$ is defined by

$$\sum_{a=0}^{F-1}t^a\chi(a)e^{at}\frac{\lambda^F e^{Ft} - 1}{\lambda^F e^{Ft}} = \sum_{a=0}^{\infty}B_{n, \chi}(\lambda)\frac{t^n}{n!}, \quad \lambda \in T_p$$

and $F$ is multiple of $f$.

**Remark.**

$$\frac{B_m(\lambda)}{m} = \frac{1}{\lambda - 1}H_{n-1}(\lambda^{-1}), \quad \lambda \in T_p.$$
7. \( p \)-adic interpolation function

In this section we give \( p \)-adic \( \lambda \)-L function. Let \( w \) be the Teichimuller character and let \( \langle x \rangle = \frac{x}{w(x)} \).

When \( F \) is multiple of \( p \) and \( f \) and \( (a, p) = 1 \), we define

\[
H_{p, \lambda}(s, a|F) = \frac{1}{s - 1} \lambda^a \langle a \rangle^{1 - s} \sum_{j=0}^{\infty} \binom{1 - s}{j} \left( \frac{F}{a} \right)^j B_j(\lambda^F).
\]

From this we note that

\[
H_{p, \lambda}(1 - n, a|F) = -\frac{1}{n} \lambda^a \langle a \rangle^n \sum_{j=0}^{n} \binom{n}{j} \left( \frac{F}{a} \right)^j B_j(\lambda^F)
\]

\[
= -\frac{1}{n} F^{n-1} \lambda^a w^{-n}(a) B_n(\lambda^F; \frac{a}{F})
\]

\[
= w^{-n}(a) H_{\lambda}(1 - n; \frac{a}{F})
\]

since by Theorem 3 for \( \lambda \in T_p \), Eq-(18).

By using this formula, we can consider \( p \)-adic \( \lambda \)-L-function for \( \lambda \)-Bernoulli numbers as follows:

\[
L_{p, \lambda}(s, \chi) = \sum_{\langle a, p \rangle = 1}^{\sigma} \chi(a) H_{p, \lambda} \left(s, \frac{a}{F}\right).
\]

By using the above definition, we have

\[
L_{p, \lambda}(1 - n, \chi) = \sum_{\langle a, p \rangle = 1}^{\sigma} \chi(a) H_{p, \lambda} \left(1 - n, \frac{a}{F}\right)
\]

\[
= -\frac{1}{n} \left(B_n, \chi_{w^{-n}}(\lambda) - p^{n-1} \chi w^{-n}(p) B_n, \chi_{w^{-n}}(\lambda^F)\right).
\]

Thus, we define the formula

\[
L_{p, \lambda}(s, \chi) = \frac{1}{F} \frac{1}{s - 1} \sum_{n=1}^{\sigma} \chi(a) \langle a \rangle^{1 - s} \sum_{j=0}^{\infty} \binom{1 - s}{j} B_j(\lambda^F)
\]

for \( s \in \mathbb{Z}_p \).

References


ON THE ANALOGS OF BERNOULLI AND EULER NUMBERS

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