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On the fuzzy convergence of sequences in a fuzzy normed linear space

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Abstract

In this paper, we introduce the notions of a fuzzy convergence of sequences, fuzzy Cauchy sequence and the related fuzzy completeness on a fuzzy normed linear space. And we investigate some properties relative to fuzzy normed linear spaces. In particular, we prove an equivalent conditions that a fuzzy norm defined on a ordinary normed linear space is fuzzy complete.

Key Words : fuzzy norm, fuzzy convergence of sequences, fuzzy Cauchy sequence , fuzzy complete fuzzy normed linear space.

1. Introduction

The notions of fuzzy vector spaces and fuzzy topological vector spaces were introduced in Katsaras and Liu [3]. These ideas were modified by Katsaras [1] and in [2] Katsaras defined the fuzzy norm on a vector space. In [4] Krishna and Sarma discussed the generation of a fuzzy vector topology from an ordinary vector topology on a vector space. Also Krishna and Sarma [5] observed the convergence of sequence of fuzzy points. Rhie et al[8] introduced the notion of fuzzy α-Cauchy sequence of fuzzy points and fuzzy completeness.

Since the concept of the completeness is essential to describe the aspects of normed linear spaces relative to the closedness of a space, there may be rich applications for fuzzyfying Banach spaces if the concept of a new type of the fuzzy completeness is introduced in a fuzzy normed linear space. In this paper, we introduce the notions of a fuzzy convergence of sequences, fuzzy Cauchy sequences on a fuzzy normed space and the related fuzzy completeness, as a generalization of those in ordinary normed linear spaces. And we investigate some related properties on fuzzy normed linear spaces. In particular, we prove an equivalent condition that a fuzzy norm defined on a ordinary normed linear space is fuzzy complete.

2. Preliminaries

Throughout this paper, X is a vector space over the field K(R or C).

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Fuzzy subsets of X are denoted by Greek letters in general. χ_A denotes the characteristic function of the set A.

Definition 2.1[3]. For two fuzzy subsets μ_1 and μ_2 of X, the fuzzy subset $\mu_1+\mu_2$ is defined by

$$(\mu_1+\mu_2)(x) = sup_{\ x_1+x_2=x} min\{\mu_1(x_1), \mu_2(x_2)\}$$

And for a scalar t of K and a fuzzy subset μ of X, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu(\frac{x}{t}) & \text{if } t \neq 0 \\ 0 & \text{if } t=0 \text{ and } x \neq 0 \\ sup_{y \in X}\, \mu(y) & \text{if } t=0 \text{ and } x=0 \end{cases}$$

Definition 2.2 [1]. $\mu \in I^X$ is said to be

i) convex if $t\mu+(1-t)\mu \le \mu$ for each $t \in [0,1]$
ii) balanced if $t\mu \le \mu$ for each $t \in K$ with $|t| \le 1$
iii) absorbing is $sup_{\ t>0} t\mu(x) = 1$ for all $x \in X$

Definition2.3[1]. Let (X,τ) be *a* topological space and $\omega(\tau) = \{f : (X,\tau) \to [0,1] \quad f\, is\ lower\, semicontinuous\}$. Then $\omega(\tau)$ is a fuzzy topology on X. This topology is called the fuzzy topology generated by τ on X. The fuzzy usual topology on K means the fuzzy topology generated by the usual topology of K.

Definition 2.4 [1]. A fuzzy linear topology on a vector space X over K is a fuzzy topology on X such that the two mappings

$$+ : X \times X \to X,\ (x,y) \to x+y$$
$$\cdot : K \times X \to X,\ (t,x) \to tx$$

are continuous when K has the fuzzy usual topology. A linear space with a fuzzy linear topology is called a fuzzy topological linear space or a fuzzy topological vector space.

Definition 2.5[1]. Let now X be a fuzzy topological space and $x \in X$. A fuzzy set μ in X is called a neighborhood of x if there exists an open fuzzy set ρ with $\rho \leq \mu$ and $\rho(x) = \mu(x) > 0$. Warren has proved in [9] that a fuzzy set μ in X is open iff μ is a neighborhood of x for each $x \in X$ with $\mu(x) > 0$.

Theorem 2.6[1]. Let μ be a neighborhood of $z_0 = x_0 + y_0$ in a fuzzy topological vector space X. Then, for each real number θ with $0 < \theta < \mu(z_0)$ there exist open neighborhoods μ_1, μ_2 of the points x_0, y_0 respectively, such that $\mu_1 + \mu_2 \leq \mu$ and $\mu_1(x_0) \wedge \mu_2(y_0) > \theta$. In case $x_0 = y_0 = 0$, there exists an open neighborhood μ_3 of zero with $\mu_3(0) > \theta$. $\mu_3 \leq \mu$ and $\mu_3 + \mu_3 \leq \mu$.

Definition 2.7[1]. Let x be a point in a fuzzy topological space X. A family F of neighborhoods of x is called a base for the system of all neighborhoods of x if for each neighborhood μ of x and each $0 < \theta < \mu(x)$ there exists $\mu_1 \in F$ with $\mu_1 \leq \mu$ and $\mu_1(x) > \theta$.

Definition 2.8[2]. A fuzzy seminorm on X is a fuzzy set ρ in X which is convex, balanced and absorbing. If in addition $\wedge_{t>0} t\rho(x) = 0$ for every nonzero x, then ρ is called a fuzzy norm.

Theorem 2.9[2]. If ρ is a fuzzy seminorm on X, then the family

$B_\rho = \{\theta \wedge t\rho | 0 < \theta \leq 1, t > 0\}$ is a base for a fuzzy linear topology τ_ρ, where $\theta \wedge t\rho$ is the function $X \to [0,1]$ such that

$$\theta \wedge t\rho(x) = \theta \wedge \rho\left(\frac{x}{t}\right).$$

Definition 2.10[2]. Let ρ be a seminorm on a linear space. The fuzzy topology τ_ρ in Theorem 2.9 is called the fuzzy topology induced by the fuzzy seminorm ρ. And a linear space equipped with a fuzzy seminorm(resp.fuzzy norm) is called a fuzzy seminormed(resp.fuzzy normed) linear space.

3. Fuzzy convergence and fuzzy completeness

In this section, we introduce the notions of a fuzzy convergence of sequences, fuzzy Cauchy sequences and fuzzy completeness on a fuzzy normed linear space. And we investigate related properties on fuzzy normed linear spaces.

Definition 3.1. Let (X, ρ) be a fuzzy normed linear space. A sequence $< x_n > \subset X$ is said to fuzzy converge to a point $x \in X$ if and only if for every $\epsilon > 0$ and for every neighborhood of zero μ with $\mu(0) > 1 - \varepsilon$, there exists a positive integer N such that $n \geq N$ implies

$\mu(x_n - x) > 1 - \epsilon$ (denoted by $x_n \to x$) and x is said to be a fuzzy limit of $< x_n >$.

Theorem 3.2. Let (X,ρ) be a fuzzy normed linear space. Then

(a) If $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$.

(b) If $t \in K$ and $x_n \to x$, then $tx_n \to tx$.

Proof. (a) Let $\epsilon > 0$ and μ be a neighborhood of zero with $\mu(0) > 1 - \epsilon$. Then there exists an open neighborhood μ_1 such that $\mu_1 \leq \mu$, $\mu_1 + \mu_1 \leq \mu$ and $\mu(0) > 1 - \epsilon$ by Theorem 2.6. Since $x_n \to x$ and $y_n \to y$, there exist two positive integers N_1 , N_2 such that

$n \geq N_1$, implies $\mu_1(x_n - x) > 1 - \epsilon$ and
$n \geq N_2$ implies $\mu_1(y_n - y) > 1 - \epsilon$.

Let $N = \max\{N_1, N_2\}$ and $n \geq N$. Then

$$\begin{aligned}
&\mu((x_n + y_n) - (x + y)) \\
&\geq (\mu_1 + \mu_1)((x_n + y_n) - (x + y)) \\
&= (\mu_1 + \mu_1)((x_n - x) + (y_n - y)) \\
&\geq \mu_1(x_n - x) \wedge \mu_1(y_n - y) > 1 - \epsilon
\end{aligned}$$

Therefore $x_n + y_n$ converges to $x + y$.

(b) If $t = 0$, then it is clear. Let $t \neq 0$. Since for every $\epsilon > 0$ and for every neighborhood of zero μ with $\mu(0) > 1 - \epsilon$, $\frac{1}{t}\mu$ is also a neighborhood of zero with $\frac{1}{t}\mu(0) = \mu(0) > 1 - \epsilon$, and $\langle x_n \rangle$ fuzzy converges to x, there exists a positive integer N such that $n \geq N$ implies $\mu(tx_n - tx) = \frac{1}{t}\mu(x_n - x) > 1 - \epsilon$. Therefore $< tx_n >$ fuzzy converges to tx. This completes the proof.

Now, we will prove that the fuzzy limit is unique. For the proof, we begin with following two lemmas.

Lemma 3.3. Let (X, ρ) be a fuzzy normed linear space, $x \in X$ and $\alpha \in (0,1)$. If for every $t > 0$, $t\rho(x) > \alpha$, then x is the zero vector of the space X.

Proof. Suppose that x is not zero vector. Since for every $t > 0$, $t\rho(x) > \alpha$, $\wedge_{t>0} t\rho(x) \geq \alpha > 0$. This contradicts to the fact that ρ is a fuzzy norm on X. Hence x is the zero vector of X.

Lemma 3.4. Let (X,ρ) be a fuzzy normed linear space and $x \in X$. If for each neighborhood of zero μ with $\mu(0) > \alpha$, $\mu(x) > \alpha$ then x is the zero vector of X.

Proof. Fix a $\theta > \alpha$. Since for every $t>0$, $\theta \wedge t\rho$ is a neighborhood of zero and $\theta \wedge t\rho(0) = \theta \wedge \rho(0) = \theta \wedge 1 > \alpha$, $\theta \wedge t\rho(x) > \alpha$ for all $t>0$. This implies that for every $t>0$, $t\rho(x) > \alpha$. By the above lemma, x is the zero vector of X.

Theorem 3.5. The fuzzy limit of a sequence $<x_n>$ is unique.

Proof. Suppose that $<x_n>$ fuzzy converges to x and x'. If $\epsilon > 0$ and μ is a neighborhood of zero with $\mu(0) > 1-\epsilon$, then there exist two positive integers N_1 and N_2 such that

$$n \geq N_1 \text{ implies } \mu(x_n - x) > 1-\epsilon \text{ and}$$
$$n \geq N_2 \text{ implies } \mu(x_n - x') > 1-\epsilon.$$

Since μ is a neighborhood of zero and $\mu(0) > 1-\epsilon$, there exists a neighborhood of zero μ_1 such that $\mu_1(0) > 1-\epsilon$, $\mu_1 \leq \mu$ and $\mu_1 + \mu_1 \leq \mu$ by Theorem 2.6. Now, we have

$$\begin{aligned}\mu(x-x') &\geq (\mu_1+\mu_1)(x-x') \\ &= (\mu_1+\mu_1)((x-x_n)+(x_n-x')) \\ &\geq \min\{\mu_1(x_n-x),\mu_1(x_n-x')\} > 1-\epsilon \quad \text{for all}\end{aligned}$$
$n \geq N_1 \vee N_2$

By the above lemma, we get $x - x' = 0$ equivalently $x = x'$. This completes the proof.

Definition 3.6. Let (X,ρ) be a fuzzy normed linear space. A sequence $<x_n>$ is said to be a fuzzy Cauchy sequence if and only if for every $\epsilon > 0$ and for every neighborhood of zero μ with $\mu(0) > 1-\epsilon$, there exists a positive integer N such that $n, m \geq N$ implies $\mu(x_n - x_m) > 1-\epsilon$.

Theorem 3.7. Let (X,ρ) be a fuzzy normed linear space. Then every fuzzy convergent sequence in (X,ρ) is a fuzzy Cauchy sequence.

Proof. Let $<x_n>$ fuzzy converge to a point $x \in X$. Then for every $\epsilon > 0$ and for every neighborhood of zero μ with $\mu(0) > 1-\epsilon$, there exists a positive integer N such that $n \geq N$ implies $\mu(x_n - x) > 1-\epsilon$. Let a neighborhood of zero μ be given and $\mu(0) > 1-\epsilon$. Then there exists a neighborhood of zero μ_1 such that $\mu_1 \leq \mu$, $\mu_1 + \mu_1 \leq \mu$ and $\mu_1(0) > 1-\epsilon$ by Theoorem 2.6. Since μ_1 is a neighborhood of zero and $\mu_1(x) > 1-\epsilon$, there exists a positive integer N_1 such that

$$n \geq N_1 \text{ implies } \mu_1(x_n - x) > 1-\epsilon.$$

Now, we have

$$\begin{aligned}\mu(x_n - x_m) &\geq (\mu_1+\mu_1)(x_n - x_m) \\ &= (\mu_1+\mu_1)((x_n - x)+(x-x_m)) \\ &\geq \min\{\mu_1(x_n - x),\ \mu_1(x-x_m)\} > 1-\epsilon \quad \text{for all}\end{aligned}$$
$n, m \geq N_1$.

Therefore $<x_n>$ is a fuzzy Cauchy sequence. This completes the proof.

Now , we consider some relations between the fuzzy completeness and ordinary completeness on a linear space.

Definition 3.8. A fuzzy normed linear space (X,ρ) is said to be complete if and only if every fuzzy Cauchy sequence fuzzy converges to a point $x \in X$.

Lemma 3.9. Let$(X, \| \cdot \|)$ be a normed linear space and B is the closed unit ball of X. Then every fuzzy Cauchy sequence on the fuzzy normed linear space (X,χ_B) is a Cauchy sequence with respect to the ordinary norm.

Proof. Let $\eta > 0$ be given. Since $\theta \wedge \frac{\eta}{2}\chi_B(0) > 1-\epsilon$ for every $\epsilon > 0$, if $\theta > 1-\epsilon$, $\theta \wedge \frac{\eta}{2}\chi_B$ is a neighborhood of zero with $\theta \wedge \frac{\eta}{2}\chi_B(0) > 1-\epsilon$. Hence there exists a positive integer N such that $n, m \geq N$ implies

$$\begin{aligned}&\theta \wedge \frac{\eta}{2}\chi_B(x_n - x_m) > 1-\epsilon \\ &\Rightarrow \frac{\eta}{2}\chi_B(x_n - x_m) > 1-\epsilon \\ &\Rightarrow \chi_B(\frac{2}{\eta}(x_n - x_m)) > 1-\epsilon \\ &\Rightarrow \chi_B(\frac{2}{\eta}(x_n - x_m)) = 1 \\ &\Rightarrow \| x_n - x_m \| \leq \frac{\eta}{2} < \eta.\end{aligned}$$

Therefore $<x_n>$ is a Cauchy sequence in $(X, \| \cdot \|)$.

Theorem 3.10. Let $(X, \| \cdot \|)$ be a Banach space. Then the fuzzy normed linear space (X, χ_B) is fuzzy complete where B is the closed unit ball of X

Proof. Let $<x_n>$ be a fuzzy Cauchy sequence in (X, χ_B). Then it is a Cauchy sequence with respect to the ordinary norm $\| \cdot \|$ by the above lemma. Since $(X, \| \cdot \|)$ is complete, there exists an $x \in X$ such that $\| x_n - x \| \to 0$.

Now, we show that x_n converges to this x in (X, χ_B). Let $\epsilon > 0$ and μ be a neighborhood of zero with $\mu(0) > 1-\epsilon$. Then there exist $1-\epsilon < \theta \leq 1$, $\eta > 0$ such that $\theta \wedge \eta\chi_B \leq \mu$ because that $\{\theta \wedge \eta\chi_B\colon \eta > 0, 0 < \theta \leq 1\}$ is a base at zero. For this $\eta > 0$, there exists a positive integer N such that

$n \geq N$ implies $\| x_n - x \| < \eta$
$\Rightarrow$ $n \geq N$ implies $\theta \wedge \eta\chi_B(x_n - x) > 1 - \epsilon$
$\Rightarrow$ $n \geq N$ implies $\mu(x_n - x) > 1 - \epsilon$.

That is $<x_n>$ fuzzy converges to x, therefore(X, χ_B) is fuzzy complete. This completes the proof.

Corollary 3.11. The field $K(R \text{ or } C)$with the fuzzy topology generated by the usual topology on K is a complete fuzzy normed linear space.

Definition 3.12 [2]. Two fuzzy seminorms ρ_1, ρ_2 on X are said to be equivalent if $\tau_{\rho_1} = \tau_{\rho_2}$.

Proposition 3.13[8]. Let $(X, \| \cdot \|)$ be a normed linear space. If ρ is a lower semicontinuous fuzzy norm on X, and have the bounded support : $\{x \in X \mid \rho(x) > 0\}$ is bounded, then ρ is equivalent to the fuzzy norm χ_B where B is the closed unit ball of X.

By Theoorem 3.10 and the above proposition, we get the following theoorem.

Theorem 3.14. If X is a Banach space and ρ is a lower semicontinuous fuzzy norm having the bounded support, then the fuzzy normed linear space(X, ρ) is fuzzy complete.

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