J. Korean Math. Soc. **45** (2008), No. 3, pp. 597–609

GROUP-FREENESS AND CERTAIN AMALGAMATED FREENESS

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Abstract. In this paper, we will consider certain amalgamated free product structure in crossed product algebras. Let M be a von Neumann algebra acting on a Hilbert space H and G, a group and let $\alpha : G \to \text{Aut}M$ be an action of G on M, where $\text{Aut}M$ is the group of all automorphisms on M. Then the crossed product $\mathbb{M} = M \times_\alpha G$ of M and G with respect to α is a von Neumann algebra acting on $H \otimes l^2(G)$, generated by M and $\{u_g\}_{g\in G}$, where u_g is the unitary representation of g on $l^2(G)$. We show that $M \times_\alpha (G_1 * G_2) = (M \times_\alpha G_1) *_M (M \times_\alpha G_2)$. We compute moments and cumulants of operators in $\mathbb{M}$. By doing that, we can verify that there is a close relation between Group Freeness and Amalgamated Freeness under the crossed product. As an application, we can show that if F_N is the free group with N-generators, then the crossed product algebra $L_M(F_n) \equiv M \times_\alpha F_n$ satisfies that

$$L_M(F_n) = L_M(F_{k_1}) *_M L_M(F_{k_2}),$$

whenever $n = k_1 + k_2$ for $n, k_1, k_2 \in \mathbb{N}$.

In this paper, we will consider a relation between a free product of groups and a certain free product of von Neumann algebras with amalgamation over a fixed von Neumann subalgebra. In particular, we observe such relation when we have crossed product algebras. Crossed product algebras have been studied by various mathematicians. Let M be a von Neumann algebra acting on a Hilbert space H and G, a group, and let $\mathbb{M} = M \times_\alpha G$ be the crossed product of M and G via an action $\alpha : G \to \text{Aut}M$ of G on M, where $\text{Aut}M$ is the automorphism group of M. This new von Neumann algebra $\mathbb{M}$ acts on the Hilbert space $H \otimes l^2(G)$, where $l^2(G)$ is the group Hilbert space. Each element x in $\mathbb{M}$ has its Fourier expansion

$$x = \sum_{g\in G} m_g u_g \quad \text{for } m_g \in M,$$

where u_g is the (left regular) unitary representation of $g \in G$ on $l^2(G)$.

On $\mathbb{M}$, we have the following basic computations;

Received March 28, 2006.

2000 *Mathematics Subject Classification.* 46L54.

Key words and phrases. crossed products of von Neumann algebras and groups, free product of algebras, moments and cumulants.

(0.1) If u_h is the unitary representation of $h \in G$, as an element in $\mathbb{M}$, then

$$u_{g_1} u_{g_2} = u_{g_1 g_2} \text{ and } u_g^* = u_{g^{-1}} \text{ for all } g, g_1, g_2 \in G.$$

(0.2) If m_1, $m_2 \in M$ and g_1, $g_2 \in G$, then

$$\begin{aligned}(m_1 u_{g_1})(m_2 u_{g_2}) &= m_1 u_{g_1} m_2 (u_{g_1}^{-1} u_{g_1}) u_{g_2} \\ &= (m_1 (\alpha_{g_1}(m_2))) u_{g_1 g_2}.\end{aligned}$$

(0.3) If $m u_g \in \mathbb{M}$, then

$$\begin{aligned}(m u_g)^* &= u_g^* m^* = u_{g^{-1}} m^* \left(u_g u_{g^{-1}}\right) \\ &= \left(\alpha_{g^{-1}}(m^*)\right) u_{g^{-1}} = \left(\alpha_{g^{-1}}(m^*) u_g^*\right).\end{aligned}$$

(0.4) If $m \in M$ and $g \in G$, then

$$u_g m = u_g m u_{g^{-1}} u_g = \alpha_g(m) u_g \quad \text{and} \quad m u_g = u_g u_{g^{-1}} m u_g = u_g \cdot \alpha_{g^{-1}}(m).$$

The element $u_g m$ is of course contained in $\mathbb{M}$, since it can be regarded as $u_g m u_{e_G}$, where e_G is the group identity of G, for $m \in M$ and $g \in G$.

Free probability has been researched from mid 1980's. There are two approaches to study it; the Voiculescu's original analytic approach and the Speicher's combinatorial approach. We will use the Speicher's approach. Let M be a von Neumann algebra and N, a W^*-subalgebra and assume that there is a conditional expectation $E : M \to N$ satisfying that (i) E is a continuous $\mathbb{C}$-linear map, (ii) $E(n) = n$ for all $n \in N$, (iii) $E(n_1 \, m \, n_2) = n_1 \, E(m) \, n_2$, for all $m \in M$ and n_1, $n_2 \in N$, and (iv) $E(m^*) = E(m)^*$ for all $m \in M$. If $N = \mathbb{C}$, then E is a continuous linear functional on M, satisfying that $E(m^*) = \overline{E(m)}$, for all $m \in M$. The algebraic pair (M, E) is called an N-valued W^*-probability space. All operators in (M, E) are said to be N-valued random variables. Let $x_1, \ldots, x_s \in (M, E)$ be N-valued random variables for $s \in \mathbb{N}$. Then $x_1, \ldots, x_s$ contain the following free distributional data.

- $(i_1, \ldots, i_n)$-th joint $*$-moment : $E\left(x_{i_1}^{u_{i_1}} \cdots x_{i_n}^{u_{i_n}}\right)$
- $(j_1, \ldots, j_m)$-th joint $*$-cumulant : $k_m\left(x_{j_1}^{u_{j_1}}, \ldots, x_{j_m}^{u_{j_m}}\right)$ such that

$$k_m\left(x_{j_1}^{u_{j_1}}, \ldots, x_{j_m}^{u_{j_m}}\right) \overset{def}{=} \sum_{\pi \in NC(m)} E_\pi\left(x_{j_1}^{u_{j_1}}, \ldots, x_{j_m}^{u_{j_m}}\right) \mu(\pi, 1_m)$$

for $(i_1, \ldots, i_n) \in \{1, \ldots, s\}^n$, $(j_1, \ldots, j_m) \in \{1, \ldots, s\}^m$ for $n, m \in \mathbb{N}$, and u_{i_k}, $u_{j_i} \in \{1, *\}$, and where $NC(m)$ is the lattice of all noncrossing partitions over $\{1, \ldots, m\}$ with its minimal element $0_m = \{(1), \ldots, (m)\}$ and its maximal element $1_n = \{(1, \ldots, m)\}$ and μ is the Möbius functional in the incidence algebra and $E_\pi(\cdots)$ is the partition-depending moment of $x_{j_1}, \ldots, x_{j_m}$ (See [18]).

For instance, $\pi = \{(1, 4), (2, 3)\}$ is in $NC(4)$. We say that the elements (1, 4) and (2, 3) of π are blocks of π, and write $(1, 4) \in \pi$ and $(2, 3) \in \pi$. In this case, the partition-depending moment $E_\pi(x_{j_1}, \ldots, x_{j_4})$ is determined by

$$E_\pi\left(x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}\right) = E\left(x_{j_1} E(x_{j_2} x_{j_3}) x_{j_4}\right).$$

The ordering on $NC(m)$ is defined by

$$\pi \leq \theta \Longleftrightarrow \text{for any block } B \in \pi, \text{ there is } V \in \theta \text{ such that } B \subseteq V$$

for π, $\theta \in NC(m)$, where "$\subseteq$" means the usual set-inclusion.

Suppose M_1 and M_2 are W^*-subalgebras of M containing their common subalgebra N. The W^*-subalgebras M_1 and M_2 are said to be free over N in (M, E), if all mixed cumulants of M_1 and M_2 vanish. The subsets X_1 and X_2 of M are said to be free over N in (M, E), if the W^*-subalgebras $vN(X_1, N)$ and $vN(X_2, N)$ are free over N in (M, E), where $vN(S_1, S_2)$ is the von Neumann algebra generated by arbitrary sets S_1 and S_2. In particular, we say that the N-valued random variables x and y are free over N in (M, E) if and only if $\{x\}$ and $\{y\}$ are free over N in (M, E). Notice that the N-freeness is totally depending on the conditional expectation E. If M_1 and M_2 are free over N in (M, E), then the N-free product von Neumann algebra $M_1 *_N M_2$ is a W^*-subalgebra of M, where

$$M_1 *_N M_2 = N \oplus \left(\oplus_{n=1}^{\infty} \left(\underset{i_1 \neq i_2,\, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n}{\oplus} (M_{i_1}^o \otimes \cdots \otimes M_{i_n}^o) \right) \right)$$

with

$$M_{i_j}^o = M_{i_j} \ominus N \text{ for all } j = 1, \ldots, n.$$

Here, all algebraic operations $\oplus$, $\otimes$ and $\ominus$ are defined under W^*-topology.

Also, if (M_1, E_1) and (M_2, E_2) are N-valued W^*-probability space with their conditional expectation $E_j : M_j \to N$ for $j = 1, 2$. Then we can construct the free product conditional expectation $E = E_1 * E_2 : M_1 *_N M_2 \to N$ making its cumulant $k_n^{(E)}(\cdots)$ vanish for mixed n-tuples of M_1 and M_2 (See [18]).

The main result of this paper is that if $G_1 * G_2$ is a free product of groups G_1 and G_2, then

$$M \times_\alpha (G_1 * G_2) = (M \times_\alpha G_1) *_M (M \times_\alpha G_2), \tag{0.5}$$

where M is a von Neumann algebra and $\alpha : G_1 * G_2 \to \text{Aut}M$ is an action. This shows that the group-freeness implies a certain freeness on von Neumann algebras with amalgamation. Also, this shows that, under the crossed product structure, the amalgamated freeness determines the group freeness.

Acknowledgment. The author specially thanks to Prof. F. Radulescu, who is his Ph. D. thesis advisor in Univ. of Iowa, for the valuable discussion and advice. Also, the author appreciate all supports from St. Ambrose Univ.. In particular, the author thanks to Prof. V. Vega and Prof. T. Anderson, for the useful discussion and for the kind encouragement and advice.

1. Crossed product probability spaces

In this chapter, we will introduce the free probability information of crossed product algebras. Throughout this chapter, let M be a von Neumann algebra and G, a group and let $\alpha : G \to \text{Aut}M$ be an action of G on M, where $\text{Aut}M$ is the automorphism group of M.

Denote the group identity of G by e_G. Consider the trivial subgroup $G_0 = \langle e_G \rangle$ of G and the crossed product algebra $\mathbb{M}_0 = M \times_\alpha G_0$. Then this algebra $\mathbb{M}_0$ is a W^*-subalgebra of $\mathbb{M}$ and it satisfies that

$$\mathbb{M}_0 = M, \tag{1.1}$$

where the equality "=" means "$*$-isomorphic". Indeed, there exists a linear map sending $m \in M$ to $m\, u_{e_G}$ in $\mathbb{M}_0$. This is the $*$-isomorphism from M onto $\mathbb{M}_0$, since

$$m_1 m_2 \longmapsto \begin{cases} (m_1 m_2) u_{e_G} & = m_1 \alpha_{e_G}(m_2) u_{e_G} \\ & = m_1 u_{e_G} m_2 u_{e_G} u_{e_G} \\ & = (m_1 u_{e_G})(m_2 u_{e_G}) \end{cases} \tag{1.2}$$

for all $m_1, m_2 \in M$. The first equality of the above formula holds, because α_{e_G} is the identity automorphism on M satisfying that $\alpha_{e_G}(m) = m$ for all $m \in M$. Also, the third equality holds, because $u_{e_G}\, u_{e_G} = u_{e_G^2} = u_{e_G}$ on G_0 (and also on G).

Proposition 1.1. *Let $G_0 = \langle e_G \rangle$ be the trivial subgroup of G and let $\mathbb{M}_0 = M \times_\alpha G_0$ be the crossed product algebra, where α is the given action of G on M. Then the von Neumann algebra $\mathbb{M}_0$ and M are $*$-isomorphic, i.e., $\mathbb{M}_0 = M$.*

From now, we will identify M and $\mathbb{M}_0$, as $*$-isomorphic von Neumann algebras.

Definition 1. Let $\mathbb{M} = M \times_\alpha G$ be the given crossed product algebra. Define a canonical conditional expectation $E_M : \mathbb{M} \to M$ by

$$E_M\left(\sum_{g \in G} m_g u_g\right) = m_{e_G} \text{ for all } \sum_{g \in G} m_g u_g \in \mathbb{M}. \tag{1.3}$$

By (0.4), we have $u_{e_G} m = \alpha_{e_G}(m)\, u_{e_G} = m\, u_{e_G}$. So, indeed, the $\mathbb{C}$-linear map E is a conditional expectation; By the very definition, E is continuous and

(i) $E_M(m) = E_M(m u_{e_G}) = E_M(u_{e_G} m) = m$ for all $m \in M$,

(ii) $E_M(m_1(m u_g)m_2) = m_1 E_M(m_g u_g) m_2$

$$= \begin{cases} m_1 m_g m_2 = m_1 E_M(m u_g) m_2 & \text{if } g = e_G \\ \\ 0_M \ = m_1 E_M(m u_g) m_2 & \text{otherwise} \end{cases}$$

for all $m_1, m_2 \in M$ and $m u_g \in \mathbb{M}$. Therefore, we can conclude that

$$E_M(m_1 x m_2) = m_1 E_M(x) m_2 \text{ for } m_1, m_2 \in M \text{ and } x \in \mathbb{M}.$$

(iii) For $\sum_{g \in G} m_g u_g \in \mathbb{M}$,

$$E_M\left((\sum_{g \in G} m_g u_g)^* \right) = E_M\left(\sum_{g \in G} u_g^* m_g^* \right)$$
$$= E_M\left(\sum_{g \in G} \alpha_g(m_g^*) u_{g^{-1}} \right)$$
$$= \alpha_{e_G}(m_{e_G}^*) = m_{e_G}^* = \left(E_M\left(\sum_{g \in G} m_g u_g \right) \right)^*.$$

Therefore, by (i), (ii) and (iii), the map E_M is a conditional expectation. Thus the pair $(\mathbb{M}, E_M)$ is a M-valued W^*-probability space.

Definition 2. The M-valued W^*-probability space $(\mathbb{M}, E_M)$ is called the M-valued crossed product probability space.

It is trivial that $\mathbb{C}\cdot 1_M$ is a W^*-subalgebra of M. Consider the crossed product $\mathbb{M}_G = \mathbb{C} \times_\alpha G$, as a W^*-subalgebra of $\mathbb{M}$. Recall the group von Neumann algebra $L(G)$ is defined by

$$L(G) = \overline{\mathbb{C}[G]}^w.$$

Since every element y in $\mathbb{M}_G$ has its Fourier expansion $y = \sum_{g \in G} t_g u_g$ and since every element in $L(G)$ has its Fourier expansion $\sum_{g \in G} r_g u_g$, there exists a $*$-isomorphism, which is the generator-preserving linear map, between $\mathbb{M}_G$ and $L(G)$.

Proposition 1.2. *Let $\mathbb{M}_G \equiv \mathbb{C}\cdot 1_M \times_\alpha G$ be the crossed product algebra. Then* $\mathbb{M}_G = L(G)$.

2. Moments and cumulants on $(\mathbb{M}, E_M)$

In the previous chapter, we defined an amalgamated W^*-probability space for the given crossed product algebra $\mathbb{M} = M \times_\alpha G$. As in Chapter 1, throughout this chapter, we will let M be a von Neumann algebra and G, a group and let $\alpha : G \to \text{Aut}M$ be an action of G on M. We will compute the amalgamated moments and cumulnats of operators in $\mathbb{M}$. These computations will play a key role to get our main results (0.5), in Chapter 3. Let $(\mathbb{M}, E_M)$ be the M-valued crossed product probability space.

Notation. From now, we denote $\alpha_g(m)$ by m^g for convenience.

Consider group von Neumann algebras $L(G)$, which are $*$-isomorphic to $\mathbb{M}_G = \mathbb{C} \times_\alpha G$, with its canonical trace tr on it. On $L(G)$, we can always define its canonical trace tr as follows,

$$\text{tr}\left(\sum_{g\in G} r_g u_g\right) = r_{e_G} \text{ for all } \sum_{g\in G} r_g u_g \in L(G), \tag{2.1}$$

where $r_g \in \mathbb{C}$, for $g \in G$. So, the pair $(L(G), \text{tr})$ is a $\mathbb{C}$-valued W^*-probability space. We can see that the unitary representations $\{u_g\}_{g\in G}$ in $(\mathbb{M}, E)$ and $\{u_g\}_{g\in G}$ in $(L(G), \text{tr})$ are identically distributed.

By using the above new notation, we have

$$\begin{aligned} &(m_{g_1}u_{g_1})(m_{g_2}u_{g_2})\cdots(m_{g_n}u_{g_n}) \\ &= \left(m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}\cdots m_{g_n}^{g_1g_2\cdots g_{n-1}}\right)u_{g_1\cdots g_n} \end{aligned} \tag{2.2}$$

for all $m_{g_j}u_{g_j} \in \mathbb{M}$, $j = 1, \ldots, n$, where $n \in \mathbb{N}$. The following lemma shows us that a certain collection of M-valued random variables in $(\mathbb{M}, E_M)$ and the generators of group von Neumann algebra $(L(G), \text{tr})$ are identically distributed (over $\mathbb{C}$).

Lemma 2.1. *Let $u_{g_1}, \ldots, u_{g_n} \in \mathbb{M}$ (i.e., $u_{g_k} = 1_M \cdot u_{g_k}$ in $\mathbb{M}$ for $k = 1, \ldots, n$). Then*

$$E_M(u_{g_1}\cdots u_{g_n}) = \text{tr}(u_{g_1}\cdots u_{g_n}) \cdot 1_M, \tag{2.3}$$

where tr *is the canonical trace on the group von Neumann algebra $L(G)$.*

Proof. By definition of E_M,

$$\begin{aligned} E_M(u_{g_1}\cdots u_{g_m}) &= E_M\left((1_M \cdot 1_M^{g_1} \cdot 1_M^{g_1g_2}\cdots 1_M^{g_1g_2\cdots g_{n-1}})u_{g_1\cdots g_n}\right) \\ &= E_M(u_{g_1\cdots g_n}) \\ &= \begin{cases} 1_M & \text{if } g_1\cdots g_n = e_G \\ 0_M & \text{otherwise,} \end{cases} \end{aligned}$$

since $1_M^g = u_g 1_M u_{g^{-1}} = u_g u_{g^{-1}} = u_{gg^{-1}} = u_{e_G} = 1_M$ for all $g \in G$ and $n \in \mathbb{N}$. By definition of tr on $L(G)$, we have that

$$\text{tr}(u_{g_1}\cdots u_{g_n}) = \text{tr}(u_{g_1\cdots g_n}) = \begin{cases} 1 & \text{if } g_1\cdots g_n = e_G \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$. ∎

We want to compute the M-valued cumulant $k_n^{E_M}(m_{g_1}u_{g_1}, \ldots, m_{g_n}u_{g_n})$, for all $m_{g_k}u_{g_k} \in \mathbb{M}$ and $n \in \mathbb{N}$. If this M-valued cumulant has a "good" relation with the cumulant $k_n^{\text{tr}}(u_{g_1}, \ldots, u_{g_n})$, then we might find the relation between a group free product in G and M-valued free product in $\mathbb{M}$. The following three lemmas are the preparation for computing the M-valued cumulant $k_n^{E_M}$ $(m_{g_1}u_{g_1}, \ldots, m_{g_n}u_{g_n})$.

Lemma 2.2. *Let* $(\mathbb{M},\ E_M)$ *be the* M*-valued crossed product probability space and let* $m_{g_1}u_{g_1}, \ldots, m_{g_n}u_{g_n}$ *be* M*-valued random variables in* $(\mathbb{M},\ E_M)$ *for* $n \in \mathbb{N}$. *Then*

$$E_M\left(m_{g_1}u_{g_1}\cdots m_{g_n}u_{g_n}\right) \tag{2.4}$$
$$= \begin{cases} m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}\cdots m_{g_n}^{g_1\cdots g_{n-1}} & \text{if } g_1\cdots g_n = e_G \\ 0_M & \text{otherwise,} \end{cases}$$

in M.

Proof. By the straightforward computation, we can get that

$$\begin{aligned} & E_M\left(m_{g_1}u_{g_1}\cdots m_{g_n}u_{g_n}\right) \\ &= E_M\left(m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}\cdots m_{g_n}^{g_1\cdots g_{n-1}}\cdot u_{g_1}u_{g_2}\cdots u_{g_n}\right) \text{ by (0.2)} \\ &= E_M\left((m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}\cdots m_{g_n}^{g_1\cdots g_{n-1}})u_{g_1\cdots g_n}\right) \\ &= \left(m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}\cdots m_{g_n}^{g_1\cdots g_{n-1}}\right)E_M\left(u_{g_1\cdots g_n}\right) \\ & \quad \text{since } E_M : \mathbb{M} \to M = M \times_\alpha \langle e_G\rangle \text{ is a conditional expectation} \\ &= \begin{cases} m_{g_1}m_{g_2}^{g_1}\cdots m_{g_n}^{g_1\cdots g_{n-1}} & \text{if } g_1\cdots g_n = e_G \\ 0_M & \text{otherwise,} \end{cases} \end{aligned}$$

by the previous lemma. □

Based on the previous lemma, we will compute the partition-depending moments of M-valued random variables. But first, we need the following observation.

Lemma 2.3. *Let* $mu_g \in (\mathbb{M},\ E_M)$ *be a* M*-valued random variable. Then* $E_M(u_g m) = m^g E_M(u_g)$.

Proof. Compute

$$E_M(u_g m) = E_M\left(u_g m u_{g^{-1}} u_g\right) = E_M(m^g u_g) = m^g E_M(u_g).$$

□

Since E_M is a conditional expectation, $E_M(u_g m) = E_M(u_g)\, m$, too. So, by the previous lemma, we have that

$$E_M(u_g)\, m = E(u_g m) = m^g E(u_g). \tag{2.6}$$

In the following lemma, we will extend this observation (2.6) to the general case. Notice that since E_M is a M-valued conditional expectation, we have to consider the insertion property (See [18]), i.e., in general,

$$E_{M,\pi}(x_1,\ldots,x_n) \neq \prod_{V\in\pi} E_{M,V}(x_1,\ldots,x_n)$$

for $x_1, \ldots, x_n \in \mathbb{M}$, where $E_{M,V}(\cdots)$ is the block-depending moments. But, if $x_k = u_{g_k} = 1_M \cdot u_{g_k}$ in $\mathbb{M}$, then we can have that

$$E_{M,\,\pi}\left(u_{g_1}, \ldots, u_{g_n}\right) = \prod_{B \in \pi} E_{M,V}\left(u_{g_1}, \ldots, u_{g_n}\right)$$

since

$$E_M\left(u_g\right) = \begin{cases} 1 \in \mathbb{C} \cdot 1_M & \text{if } g = e_G \\ 0 \in \mathbb{C} \cdot 1_M & \text{otherwise,} \end{cases}$$

and hence

$$E_{M,\,\pi}\left(u_{g_1}, \ldots, u_{g_n}\right) = \prod_{B \in \pi} \left(\operatorname{tr}_V\left(u_{g_1}, \ldots, u_{g_n}\right) \cdot 1_M\right) \text{ by (2.3).}$$

Suppose that $\pi \in NC(n)$ is a partition which is not 1_n and by $[V \in \pi]$, denote the relation [V is a block of π]. We say that a block $V = (j_1, \ldots, j_p)$ is inner in a block $B = (i_1, \ldots, i_k)$, where $V,\, B \in \pi$, if there exists $k_0 \in \{2, \ldots, k-1\}$ such that $i_{k_0} < j_t < i_{k_0+1}$ for all $t = 1, \ldots, p$. In this case, we also say that B is outer than V. Also, we say that V is innerest if there is no other block inner in V. For instance, if we have a partition

$$\pi = \{(1,6), (2,5), (3,4)\} \text{ in } NC(6).$$

Then the block (2, 5) is inner in the block (1, 6) and the block (3, 4) is inner in the block (2, 5). Clearly, the block (3, 4) is inner in both (2, 5) and (1, 6), and there is no other block inner in (3, 4). So, the block (3, 4) is an innerest block in π. Remark that it is possible there are several innerest blocks in a certain noncrossing partition. Also, notice that if V is an innerest block, then there exists j such that $V = (j,\, j+1, \ldots, j+|V|-1)$, where $|V|$ means the cardinality of entries of V.

Lemma 2.4. *Let $n \in \mathbb{N}$ and $\pi \in NC(n)$, and let $m_{g_1}u_{g_1}, \ldots, m_{g_n}u_{g_n} \in (\mathbb{M}, E_M)$ be the M-valued random variables. Then*

$$\begin{aligned} &E_{M,\pi}\left(m_{g_1}u_{g_1}, \ldots, m_{g_n}u_{g_n}\right) \\ &= \left(m_{g_1} m_{g_2}^{g_1} \cdots m_{g_n}^{g_1 \cdots g_{n-1}}\right) \operatorname{tr}_\pi\left(u_{g_1}, \ldots, u_{g_n}\right), \end{aligned} \tag{2.7}$$

where tr *is the canonical trace on the group von Neumann algebra $L(G)$.*

Proof. If $\pi = 1_n$, then $E_{M,1_n}(\cdots) = E_M(\cdots)$ and $\operatorname{tr}_{1_n}(\cdots) = \operatorname{tr}(\cdots)$, and hence we are done, by (2.3) and (2.4). Assume that $\pi \neq 1_n$ in $NC(n)$. Assume that $V = (j,\, j+1, \ldots, j+k)$ is an innerest block of π. Then

$$\begin{aligned} T_V &\overset{def}{=} E_{M,\,V}\left(m_{g_1}u_{g_1}, \ldots, m_{g_n}u_{g_n}\right) \\ &= E_M\left(m_{g_j}u_{g_j} m_{g_{j+1}}u_{g_{j+1}} \cdots m_{g_{j+k}}u_{g_{j+k}}\right) \\ &= \left(m_{g_j} m_{g_{j+1}}^{g_j} m_{g_{j+2}}^{g_j g_{j+1}} \cdots m_{g_{j+k}}^{g_j g_{j+1} \cdots g_{j+k-1}}\right) \cdot \operatorname{tr}\left(u_{g_j \cdots g_{j+k}}\right). \end{aligned}$$

Suppose V is inner in a block B of π and B is inner in all other blocks B', where V is inner in B'. Let $B = (i_1, \ldots, i_k)$ and assume that there is $k_0 \in$

$\{2, \ldots, k-1\}$ such that $i_{k_0} < t < i_{k_0+1}$ for all $t = j, j+1, \ldots, j+k$. Then the B-depending moment goes to

$$\begin{aligned}
&E_M\left(m_{g_{i_1}} u_{g_{i_1}} \cdots m_{g_{k_0}} u_{g_{k_0}} (T_V) m_{g_{k_0+1}} u_{g_{k_0+1}} \cdots m_{g_{i_k}} u_{g_{i_k}}\right) \\
&= E_M\Big(\Big(m_{g_{i_1}} m_{g_{i_2}}^{g_{i_1}} \cdots m_{g_{k_0}}^{g_{i_1} \cdots g_{i_{k_0-1}}} \\
&\quad \cdot \left(m_{g_j}^{g_{i_1} \cdots g_{i_{k_0}}} m_{g_{j+1}}^{g_{i_1} \cdots g_{i_{k_0}} g_j} \cdots m_{g_{j+1}}^{g_{i_1} \cdots g_{i_{k_0}} g_j \cdots g_{j+k-1}}\right) \\
&\quad \cdot m_{g_{i_{k_0}}}^{g_{i_1} \cdots g_{i_{k_0}} g_j \cdots g_{j+k}} \cdots m_{g_{i_k}}^{g_{i_1} \cdots g_j \cdots g_{j+1} \cdots g_{i_{k-1}}}\Big) u_{g_{i_1} \cdots g_{i_{k_0}} g_j \cdots g_{j+k} g_{i_{k_0+1}} \cdots g_{i_k}}\Big) \\
&= \Big(m_{g_{i_1}} m_{g_{i_2}}^{g_{i_1}} \cdots m_{g_{k_0}}^{g_{i_1} \cdots g_{i_{k_0-1}}} \left(m_{g_j}^{g_{i_1} \cdots g_{i_{k_0}}} m_{g_{j+1}}^{g_{i_1} \cdots g_{i_{k_0}} g_j} \cdots m_{g_{j+1}}^{g_{i_1} \cdots g_{i_{k_0}} g_j \cdots g_{j+k-1}}\right) \\
&\quad \cdot m_{g_{i_{k_0}}}^{g_{i_1} \cdots g_{i_{k_0}} g_j \cdots g_{j+k}} \cdots m_{g_{i_k}}^{g_{i_1} \cdots g_j \cdots g_{j+1} \cdots g_{i_{k-1}}}\Big) E_M\left(u_{g_{i_1} \cdots g_{i_{k_0}} g_j \cdots g_{j+k} g_{i_{k_0+1}} \cdots g_{i_k}}\right).
\end{aligned}$$

By doing the above process for all block-depending moments in the π-depending moments, we can get that

$$E_{M,\pi}\left(m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n}\right) = \left(m_{g_1} m_{g_2}^{g_1} m_{g_3}^{g_1 g_2} \cdots m_{g_n}^{g_1 \cdots g_{n-1}}\right) E_\pi\left(u_{g_1}, \ldots, u_{g_n}\right).$$

By (2.3), we know $E_\pi\left(u_{g_1}, \ldots, u_{g_n}\right) = \mathrm{tr}_\pi(u_{g_1}, \ldots, u_{g_n}) \cdot 1_M$, where tr is the canonical trace on the group von Neumann algebra $L(G)$. □

By the previous lemmas and proposition, we have the following theorem.

Theorem 2.5. *Let $m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n} \in (\mathbb{M}, E_M)$ be the M-valued random variables for $n \in \mathbb{N}$. Then*

$$\begin{aligned}
&k_n^{E_M}\left(m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n}\right) \\
&= \left(m_{g_1} m_{g_2}^{g_1} m_{g_3}^{g_1 g_2} \cdots m_{g_n}^{g_1 \cdots g_{n-1}}\right) k_n^{\mathrm{tr}}\left(u_{g_1}, \ldots, u_{g_n}\right).
\end{aligned} \tag{2.8}$$

Proof. Observe that

$$\begin{aligned}
&k_n^M\left(m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n}\right) \\
&= \sum_{\pi \in NC(n)} E_{M,\pi}\left(m_{g_1} u_{g_1}, \ldots, m_{g_n} u_{g_n}\right) \mu(\pi, 1_n) \\
&= \sum_{\pi \in NC(n)} \left(\left(m_{g_1} m_{g_2}^{g_1} \cdots m_{g_n}^{g_1 \cdots g_{n-1}}\right) \mathrm{tr}_\pi\left(u_{g_1}, \ldots, u_{g_n}\right)\right) \mu(\pi, 1_n) \text{ by (2.7)} \\
&= \left(m_{g_1} m_{g_2}^{g_1} m_{g_3}^{g_1 g_2} \cdots m_{g_n}^{g_1 \cdots g_{n-1}}\right) \left(\sum_{\pi \in NC(n)} \mathrm{tr}_\pi(u_{g_1}, \ldots, u_{g_n}) \mu(\pi, 1_n)\right) \\
&= \left(m_{g_1} m_{g_2}^{g_1} m_{g_3}^{g_1 g_2} \cdots m_{g_n}^{g_1 \cdots g_{n-1}}\right) k_n^{\mathrm{tr}}\left(u_{g_1}, \ldots, u_{g_n}\right).
\end{aligned}$$

□

The above theorem shows us that there is close relation between the M-valued cumulant on $(\mathbb{M}, E_M)$ and $\mathbb{C}$-valued cumulant on $(L(G), \mathrm{tr})$.

Example 1. *In this example, instead of using (2.7) directly, we will compute the π-depending moment of $m_{g_1}\ u_{g_1}, \ldots, m_{g_n}\ u_{g_n}$ in $\mathbb{M}$, only by using the* simple computations (0.1)∼(0.4). By doing this, we can understand why (2.7) holds concretely. Let $\pi = \{(1,4),(2,3),(5)\}$ in $NC(5)$. Then

$$
\begin{aligned}
& E_{M,\pi}\left(m_{g_1}u_{g_1}, \dots, m_{g_5}u_{g_5}\right) \\
&= E_M\left(m_{g_1}u_{g_1}E_M\left(m_{g_2}u_{g_2}m_{g_3}u_{g_3}\right)m_{g_4}u_{g_4}\right)E_M\left(m_{g_5}u_{g_5}\right) \\
&= m_{g_1}E_M\left(u_{g_1}E_M\left(m_{g_2}m_{g_3}^{g_2}u_{g_2g_3}\right)m_{g_4}u_{g_4}\right)\left(m_{g_5}E_M(u_{g_5})\right) \\
&= m_{g_1}E_M\left(u_{g_1}(m_{g_2}m_{g_3}^{g_2})E_M(u_{g_2g_3})m_{g_4}u_{g_4}\right)\left(m_{g_5}E_M(u_{g_5})\right) \\
&= m_{g_1}E_M\left(m_{g_2}^{g_1}m_{g_3}^{g_1g_2}u_{g_1}E_M(u_{g_2g_3})m_{g_4}u_{g_4}\right)\left(m_{g_5}E_M(u_{g_5})\right) \\
&= m_{g_1}E_M\left(m_{g_2}^{g_1}m_{g_3}^{g_1g_2}u_{g_1}m_{g_4}^{g_2g_3}E_M(u_{g_2g_3})u_{g_4}\right)\left(m_{g_5}E_M(u_{g_5})\right) \\
&= m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}E_M\left(u_{g_1}m_{g_4}^{g_2g_3}E_M(u_{g_2g_3})u_{g_4}\right)\left(m_{g_5}E_M(u_{g_5})\right) \\
&= m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}E_M\left(m_{g_4}^{g_1g_2g_3}u_{g_1}E_M(u_{g_2g_3})u_{g_4}\right)\left(m_{g_5}E_M(u_{g_5})\right) \\
&= m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}m_{g_4}^{g_1g_2g_3}E_M\left(u_{g_1}E_M(u_{g_2g_3})u_{g_4}\right)m_{g_5}\left(E_M(u_{g_5})\right) \\
&= m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}m_{g_4}^{g_1g_2g_3}E_M\left(u_{g_1}E_M(u_{g_2g_3})u_{g_4}m_{g_5}\right)\left(E_M(u_{g_5})\right) \\
&= m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}m_{g_4}^{g_1g_2g_3}E_M\left(u_{g_1}E_M(u_{g_2g_3})m_{g_5}^{g_4}u_{g_4}\right)\left(E_M(u_{g_5})\right) \\
&= m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}m_{g_4}^{g_1g_2g_3}E_M\left(u_{g_1}m_{g_5}^{g_2g_3g_4}E_M(u_{g_2g_3})u_{g_4}\right)\left(E_M(u_{g_5})\right) \\
&= m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}m_{g_4}^{g_1g_2g_3}E_M\left(m_{g_5}^{g_1g_2g_3g_4}u_{g_1}E_M(u_{g_2g_3})u_{g_4}\right)\left(E_M(u_{g_5})\right) \\
&= \left(m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}m_{g_4}^{g_1g_2g_3}m_{g_5}^{g_1g_2g_3g_4}\right)\left(\left(E_M(u_{g_1}E_M(u_{g_2}u_{g_3})u_{g_4})\right)\left(E_M(u_{g_5})\right)\right) \\
&= \left(m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}m_{g_4}^{g_1g_2g_3}m_{g_5}^{g_1g_2g_3g_4}\right)\left(\mathrm{tr}\left(u_{g_1}\mathrm{tr}(u_{g_2}u_{g_3})u_{g_4}\right)\left(\mathrm{tr}(u_{g_5})\right)\right) \\
&= \left(m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}m_{g_4}^{g_1g_2g_3}m_{g_5}^{g_1g_2g_3g_4}\right)\left(\mathrm{tr}_\pi(u_{g_1},u_{g_2},u_{g_3},u_{g_4},u_{g_5})\right).
\end{aligned}
$$

Example 2. We can compute the following M-valued cumulant, by applying (2.8).

$$
\begin{aligned}
& k_3^{E_M}\left(m_{g_1}u_{g_1}, m_{g_2}u_{g_2}, m_{g_3}u_{g_3}\right) \\
&= \left(m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}\right)\cdot k_3^{\mathrm{tr}}\left(u_{g_1},u_{g_2},u_{g_3}\right) \\
&= \left(m_{g_1}m_{g_2}^{g_1}m_{g_3}^{g_1g_2}\right)\left(\mathrm{tr}\left(u_{g_1g_2g_3}\right) - \mathrm{tr}(u_{g_1})\mathrm{tr}(u_{g_2}u_{g_3})\right. \\
&\quad \left. -\mathrm{tr}(u_{g_1}u_{g_2})\mathrm{tr}(u_{g_3}) + 2\mathrm{tr}(u_{g_1})\mathrm{tr}(u_{g_2})\mathrm{tr}(u_{g_3})\right).
\end{aligned}
$$

3. The main result (0.5)

In this chapter, we will prove our main result (0.5). Like before, throughout this chapter, let M be a von Neumann algebra and G, a group and let $\alpha : M \to \mathrm{Aut}M$ be an action of G on M. Assume that a group G is a group free product $G_1 * G_2$ of groups G_1 and G_2. (Also, we can assume that there is a subgroup $G_1 * G_2$ in the group G, and $M \times_\alpha (G_1 * G_2)$ is a W^*-subalgebra of $\mathbb{M}$.) Recall that, by Voiculescu, it is well-known that

$$L\left(G_1 * G_2\right) = L(G_1) * L(G_2),$$

where "$*$" in the left-hand side is the group free product and "$*$" in the right-hand side is the von Neumann algebra free product, where $L(K)$ is a group von Neumann algebra of an arbitrary group K. This says that the $\mathbb{C}$-freeness on $(L(G), \mathrm{tr})$ is depending on the group freeness on $G = G_1 * G_2$, whenever tr is a canonical trace on $L(G)$. In other words, if the groups G_1 and G_2 are free in $G = G_1 * G_2$, then the group von Neumann algebras $L(G_1)$ and $L(G_2)$ are free in $(L(G), \mathrm{tr})$. Also, if two group von Neumann algebras $L(G_1)$ and $L(G_2)$ are given and if we construct the $\mathbb{C}$-free product $L(G_1) * L(G_2)$ of them, with respect to the canonical trace $\mathrm{tr}_G = \mathrm{tr}_{G_1} * \mathrm{tr}_{G_2}$, where tr_{G_k} is the canonical trace on $L(G_k)$, for $k = 1, 2$, then this $\mathbb{C}$-free product is $*$-isomorphic to a group von Neumann algebra $L(G)$, where G is the group free product $G_1 * G_2$ of G_1 and G_2.

Theorem 3.1. *Let $\mathbb{M} = M \times_\alpha G$ be a crossed product algebra, where $G = G_1 * G_2$ is the group free product of G_1 and G_2. Then*

$$\mathbb{M} = (M \times_\alpha G_1) *_M (M \times_\alpha G_2), \tag{3.1}$$

where "$*_M$" is the M-valued free product of von Neumann algebras.

Proof. Let $G = G_1 * G_2$ be the group free product of G_1 and G_2. By Chapter 1, the crossed product algebra $\mathbb{M}$ has its W^*-subalgebra

$$M = M \times_\alpha \langle e_G \rangle,$$

where $\langle e_G \rangle$ is the trivial subgroup of G generated by the group identity $e_G \in G$. Define the canonical conditional expectation $E_M : \mathbb{M} \to M$ by

$$E_M\left(\sum_{g \in G} m_g u_g\right) = m_{e_G} \text{ for all } \sum_{g \in G} m_g u_g \in \mathbb{M}.$$

By (2.8), if $m_{g_1}u_{g_1}, \ldots, m_{g_n}u_{g_n} \in (\mathbb{M}, E_M)$ are M-valued random variables, then

$$k_n^{E_M}(m_{g_1}u_{g_1}, \ldots, m_{g_n}u_{g_n}) = \left(m_{g_1} m_{g_2}^{g_1} m_{g_3}^{g_1 g_2} \cdots m_{g_n}^{g_1 \cdots g_{n-1}}\right) k_n^{\mathrm{tr}}(u_{g_1}, \ldots, u_{g_n})$$

for all $n \in \mathbb{N}$, where tr is the canonical trace on $L(G)$. As we mentioned in the previous paragraph, the $\mathbb{C}$-freeness on $L(G)$ is completely determined by the group freeness of G_1 and G_2 on G and vice versa. By the previous cumulant relation, the M-freeness on $\mathbb{M}$ is totally determined by the $\mathbb{C}$-freeness on $L(G)$. Therefore, the M-freeness on $\mathbb{M}$ is determined by the group freeness on G. Thus, we can conclude that

$$M \times_\alpha (G_1 * G_2) = (M \times_\alpha G_1) *_M (M \times_\alpha G_2).$$

□

Recall that, if F_N is the free group with N-generators, then

$$L(F_N) = *_{k=1}^{N} L(\mathbb{Z})_k,$$

where $L(\mathbb{Z})_k = L(\mathbb{Z})$ for all $k = 1, \ldots, N$ (see [22]). Also, $L(F_N) = L(F_{k_1}) * L(F_{k_2})$ for all $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = N$.

Corollary 3.2. *Let F_N be the free group with N-generators for $N \in \mathbb{N}$. Then*

$$M \times_\alpha F_N = \underbrace{(M \times_\alpha \mathbb{Z}) *_M \cdots *_M (M \times_\alpha \mathbb{Z})}_{N\text{-times}} \tag{3.2}$$

and

$$M \times_\alpha F_N = (M \times_\alpha F_{k_1}) *_M (M \times_\alpha F_{k_2}), \tag{3.3}$$

whenever $k_1 + k_2 = N$ for k_1, $k_2 \in \mathbb{N}$.

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