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PRIMITIVE EVEN 2-REGULAR POSITIVE QUATERNARY QUADRATIC FORMS

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Abstract. In this paper, we provide a complete list of 177 equivalence classes of primitive even 2-regular quaternary positive definite quadratic forms and their discriminants. All of them have class number 1.

1. Introduction

A positive definite integral quadratic form f is called regular if f represents all integers that are represented by the genus of f. Regular quadratic forms were first studied systematically by Dickson in [3], where the term "regular" was coined. Jones and Pall in [6] classified all ternary regular positive definite diagonal quadratic forms. In the last chapter of his doctoral thesis [13], Watson showed by arithmetic arguments that there are only finitely many equivalence classes of primitive positive definite regular ternary quadratic forms. The problem of enumerating the equivalence classes of the primitive positive definite regular ternary quadratic forms was recently resurrected by Kaplansky and his collaborators [5]. Their algorithm relies on the complete list of those regular ternary quadratic forms with square free discriminant [16] and a method of descent set forth by Watson in [13]. This method of descent involves a collection of transformations which change a regular ternary form to another one with smaller discriminant and simpler local structure, and it is this method which enables Watson to obtain the explicit discriminant bounds for regular ternary quadratic forms.

The study of higher dimensional analogs of regular quadratic forms is first initiated by Earnest in [4]. A positive definite quadratic form f of m variables is called n-regular if f represents all quadratic forms of n variables that are represented by the genus of f. In particular, a 1-regular quadratic form is simply a regular from defined in the previous paragraph. Earnest showed that there exist only finitely many equivalence classes of primitive positive definite 2-regular quaternary quadratic forms. His method is an extension of Watson's analytic argument (see [14]) which seems to be inadequate to classify such

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lattices. Recently, Chan and the author [2] proved that there exist only finitely many primitive n-regular quadratic forms of rank $n+2$ and $n+3$ for $n \geq 2$. In that paper, we turned the stage back to an arithmetic setting and bring back Watson's transformations into the arsenal. In this paper, we determine all primitive even positive definite 2-regular quaternary quadratic forms by using Watson's arithmetic method. More precisely, we show that there are exactly 177 equivalence classes of such quadratic forms and all of them have class number 1.

The subsequent discussion will be conducted in the better adapted geometric language of quadratic spaces and lattices, and any unexplained notations and terminologies can be found in [8] or [11]. The term lattice will always refer to an integral $\mathbb{Z}$-lattice on an n-dimensional positive definite quadratic space over $\mathbb{Q}$. The scale and the norm ideal of a lattice L are denoted by $\mathfrak{s}(L)$ and $\mathfrak{n}(L)$ respectively. The discriminant and the class number of L are denoted by dL and $h(L)$ respectively. For any positive rational number a, L^a is the lattice whose quadratic map is scaled by a. The successive minima of L are denoted by $\mu_1(L) \leq \cdots \leq \mu_n(L)$ and we define $\overline{\mu}(L) := \mu_n(L)$. Let

$$L = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_n$$

be a $\mathbb{Z}$-lattice of rank n. We write

$$L \simeq (B(x_i, x_j)).$$

The right hand side matrix is called a *matrix presentation* of L. For $\mathbb{Z}$-sublattices L_1, L_2 of L, we write $L = L_1 \perp L_2$ when $L = L_1 \oplus L_2$ and $B(v_1, v_2) = 0$ for all $v_1 \in L_1, v_2 \in L_2$. If L admits an orthogonal basis $\{x_1, x_2, \ldots, x_n\}$, we call L *diagonal* and simply write

$$L \simeq \langle Q(x_1), Q(x_2), \ldots, Q(x_n)\rangle.$$

We also write

$$L \simeq [a_1, b, a_2],$$
$$L \simeq [a_1, a_2, a_3, b, c, d]$$

or

$$L \simeq [a_1, a_2, a_3, a_4, b, c, d, e, f, g]$$

if

$$L \simeq \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix}, \quad L \simeq \begin{pmatrix} a_1 & d & c \\ d & a_2 & b \\ c & b & a_3 \end{pmatrix} \quad \text{or} \quad L \simeq \begin{pmatrix} a_1 & b & c & e \\ b & a_2 & d & f \\ c & d & a_3 & g \\ e & f & g & a_4 \end{pmatrix}$$

for $n = 2, 3, 4$ respectively.

For the tables of ternary and quaternary $\mathbb{Z}$-lattices, see [1] and [9] or Sloane's electronic home page http://www.research.att.com/~njas/.

Throughout this paper, we always assume that every $\mathbb{Z}$-lattice L is *positive definite* and is *primitive*, i.e., $\mathfrak{s}(L) = \mathbb{Z}$. L is called *even* if $\mathfrak{n}(L) = 2\mathbb{Z}$, and *odd* otherwise. For a prime p, the group of units in $\mathbb{Z}_p$ is denoted by $\mathbb{Z}_p^\times$. Unless confusion arises, we will simply use Δ_p to denote a nonsquare element in $\mathbb{Z}_p^\times$. When we discuss lattices over $\mathbb{Z}_p$, $\mathbb{H}$ stands for hyperbolic plane $A(0,0)$ and $\mathbb{A}$ stands for the binary anisotropic $\mathbb{Z}_p$-lattice $\langle 1, -\Delta_p\rangle$ when $p > 2$, and $A(2,2)$ if $p = 2$.

Since most of the proofs require laborious computation, we simply outline the ideas behind. A version of this paper but with complete proofs of all stated results is available upon request to the author.

2. Some definitions and lemmas

Let L be a $\mathbb{Z}$-lattice. For any positive integer m, define

$$\Lambda_m(L) = \{x \in L : Q(x+z) \equiv Q(z) \pmod{m} \text{ for all } z \in L\}.$$

Let $\lambda_m(L)$ be the primitive lattice obtained from $\Lambda_m(L)$ by scaling $L \otimes \mathbb{Q}$ by a suitable rational number. Note that the scaling factor depends on the lattice structure of L_p for $p \mid m$. For the properties of this transformation, see [15] or [2]. For $M \in \text{gen}(L)$, it is known that $\lambda_m(M) \in \text{gen}(\lambda_m(L))$. Furthermore, as a map from $\text{gen}(L)$ to $\text{gen}(\lambda_m(L))$, λ_m is a surjective map (see [15]).

Lemma 2.1. *Let L be a $\mathbb{Z}$-lattice and $\mathfrak{l}_p$ be the scale of the last component of a Jordan decomposition of L_p. Let m be any positive integer such that*

$$\text{ord}_p(m) \geq \text{ord}_p(\mathfrak{l}_p) + 2\delta_{2,p}$$

for any prime p. Then, the map $\lambda_m : \text{gen}(L) \to \text{gen}(\lambda_m(L))$ is a bijective map.

Proof. Since λ_m is a surjective map, it suffices to show that $\lambda_m(\lambda_m(M)) = M$ for all $M \in \text{gen}(L)$. Since $\lambda_p \circ \lambda_q(L) = \lambda_q \circ \lambda_p(L)$ for distinct primes p and q, we may assume that $m = p^r$. Let $M_p = \mathfrak{M}_0 \perp \cdots \perp \mathfrak{M}_t$ ($\mathfrak{M}_t \neq 0$) be a Jordan decomposition such that $\mathfrak{s}(\mathfrak{M}_i) = p^i\mathbb{Z}_p$ or $\mathfrak{M}_i = 0$. Then one may easily check that

$$\Lambda_{p^r}(M_p) = p^{r-\delta_{2,p}}\mathfrak{M}_0 \perp \cdots \perp p^{r-\delta_{2,p}-t}\mathfrak{M}_t.$$

From this and the Weak Approximation Theorem follows the lemma. □

Remark 2.2. The above bound for $\text{ord}_p(m)$ is best possible, for example,

$$h(\langle 1,1,36\rangle) = 2 \quad \text{but} \quad h(\lambda_8(\langle 1,1,36\rangle)) = 1.$$

Lemma 2.3. *Let L be a quaternary $\mathbb{Z}$-lattice and $\ell := \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3$ be any ternary sublattice of L satisfying $Q(x_i) = \mu_i(L)$. For any $\mathbb{Z}$-lattice M such that $M \to L$ and rank(M) ≤ 2, we have:*

(1) *ℓ is always a primitive sublattice of L and $2|B(x_i,x_j)| \leq Q(x_i)$ for $1 \leq i \leq j \leq 3$.*
(2) *$dL \leq d\ell \cdot \mu_4(L)$. Furthermore, if the equality holds, then*

$$L \simeq \ell \perp \langle \mu_4(L)\rangle.$$

(3) *If $\overline{\mu}(M) < \overline{\mu}(\ell)$, then $M \to \ell$.*

(4) *If* $M \twoheadrightarrow \ell$, *then* $\overline{\mu}(L) \le \overline{\mu}(M)$.

Proof. Let $\tilde{\ell}$ be any ternary $\mathbb{Z}$-lattice such that $\ell \subset \tilde{\ell} \subset L$. Then $\mu_i(L) \le \mu_i(\tilde{\ell}) \le \mu_i(\ell)$. Since $\mu_i(L) = \mu_i(\ell)$, $\{x_1, x_2, x_3\}$ is a basis of $\tilde{\ell}$ by [12]. Therefore, ℓ is a primitive sublattice of L. Since the proofs of the other statements are quite trivial, they are left to the readers. □

Let L be a 2-regular quaternary $\mathbb{Z}$-lattice. L is called *p-stable* (or *stable at p*) if

$$\mathfrak{n}(L_p) = 2\mathbb{Z}_p \quad \text{and} \quad \mathbb{H} \rightarrow L_p \text{ or } L_p \simeq \mathbb{A} \perp \mathbb{A}^p$$

and is called *stable* if L is p-stable for any prime p. By applying λ_p-transformations (or λ_4, if $p = 2$) successively to L, we can transform it to a p-stable 2-regular $\mathbb{Z}$-lattice. The first p-stable 2-regular $\mathbb{Z}$-lattice obtained by applying such transformations successively to L is denoted by $\lambda_{(p)}(L)$ and the stable 2-regular $\mathbb{Z}$-lattice obtained by the above process for any prime p is denoted by $\lambda(L)$. For useful properties of the $\lambda_{(p)}$ and λ-transformations, we refer the readers to [2].

Lemma 2.4. *Let L be a 2-regular quaternary $\mathbb{Z}$-lattice. Then*

$$\{2,3\} \cup \{p : p \,|\, dL\} = \{2,3\} \cup \{p : p \,|\, d(\lambda(L))\}.$$

Proof. Let p be a prime dividing dL. For $p \ge 5$, let $L_p = \mathfrak{L}_0 \perp \mathfrak{L}_1 \perp \cdots \perp \mathfrak{L}_t$ ($\mathfrak{L}_t \ne 0$) be a Jordan decomposition such that $\mathfrak{s}(\mathfrak{L}_i) = p^i\mathbb{Z}_p$ or $\mathfrak{L}_i = 0$. Note that there exists an $\epsilon_q \in \mathbb{Z}_p^\times$ such that $(\lambda_q(L))_p \simeq L_p^{\epsilon_q}$ for all $q \ne p$. Hence we may assume that L is q-stable for all $q \ne p$. To prove the assertion, it suffices to show that $\text{rank}(\mathfrak{L}_0) \ge 3$ or $\text{rank}(\mathfrak{L}_1) \ne 0$.

Suppose that $\text{rank}(\mathfrak{L}_0) = 1$ and $\text{rank}(\mathfrak{L}_1) = 0$. Define

$$T_p := \{2t : 1 \le t \le p-1, \langle 2t \rangle_p \simeq \mathfrak{L}_0\}$$

and

$$X_p := \{x \in L : Q(x) \in T_p\}.$$

Since L is also 1-regular, L represents all elements in T_p. Let M be the $\mathbb{Z}$-sublattice of L generated by the vectors in X_p. Note that $\mu_1(M) \le p-1$. If $\text{rank}(M) = k \ge 3$, then

$$p^{2k-2} \le dM \le \prod_{i=1}^{k} \mu_i(M) \le (p-1)(2p-2)^{k-1},$$

which is a contradiction. Now, suppose that M is a binary lattice. Put $M = [a, b, c]$ $(0 \le 2b \le a \le c,\ 2 \le a \le p-1)$. First, assume that $p \ge 31$. For all $2t \in T_p$, the equation

$$2ta = (ax + by)^2 + (ac - b^2)y^2$$

should have an integer solution. Since $d\ell = ac - b^2 \ge p^2$, $y = 0, \pm 1$. Therefore, the number of t that has an integer solution is less than or equal to $(\sqrt{2} +$

$1)\sqrt{p-1}+1$, which is less than $\frac{p-1}{2}$. This is a contradiction. If $p<31$, one may show a contradiction by a direct calculation. Since the case when $\mathrm{rank}(M)=1$ is easier than the above case, the proof is left to the readers.

Now, suppose that $\mathrm{rank}(\mathfrak{L}_0)=2$ and $\mathrm{rank}(\mathfrak{L}_1)=0$. In this case, L represents all elements in $\{2t : 1\le t\le p-1\}$. Let N be a sublattice of L generated by the vectors x with $Q(x)\le 2(p-1)$. Since other cases can be done in a similar manner to the above, we only provide the proof of the case when $\mathrm{rank}(N)=2$. Note that N is isometric to one of $[2,1,2],[2,0,2],[2,1,4],[2,0,4]$. These binaries do not represent $4,6,6,10$ respectively. Hence if $p\ne 5$, this is a contradiction. If $p=5$, then we may assume that $N\simeq[2,0,4]$. Since $\langle 14\rangle \not\to [2,0,4]$, $\mu_3(L)\le 14$. Furthermore, since the discriminant of a ternary sublattice ℓ generated by $\{x_1,x_2,x_3\}$ such that $Q(x_i)=\mu_i(L)$ is divisible by 50, $\ell\simeq[2,4,14,2,1,0]$. Since the binary $\mathbb{Z}$-lattice $[2,1,2]$ that is represented by L is not represented by ℓ, this is a contradiction by Lemma 2.3 (3). $\square$

3. Basic strategy for computations

In this section, we explain the method classifying all even 2-regular quaternary $\mathbb{Z}$-lattices.

First we classify all stable 2-regular quaternary $\mathbb{Z}$-lattices. Let L be such a $\mathbb{Z}$-lattice. Since L represents all even integers, $dL\le 4292$ by [7]. Furthermore, if we use the fact that L is 2-regular, we may have more effective upper bound and we may also prove the following lemma by using Lemma 2.3 and 2.4:

Lemma 3.1. *For any 2-regular quaternary $\mathbb{Z}$-lattice L,*

$$\{p : p\mid dL\}\subset\{2,3,5,7,11,13,17,23\}.$$

Theorem 3.2. *There are exactly 48 equivalence classes of stable 2-regular quaternary $\mathbb{Z}$-lattices. They are the $\mathcal{L}_i$'s in Section 4, for $1\le i\le 48$.*

Let L be an even primitive 2-regular quaternary $\mathbb{Z}$-lattice and p be a prime.

Definition 3.3. Assume that $\lambda_{(p)}(L)$ is a stable 2-regular lattice. Let

$$L_p=\mathfrak{L}_0\perp\mathfrak{L}_1\perp\cdots\perp\mathfrak{L}_t\quad(\mathfrak{L}_t\ne 0)$$

be a Jordan decomposition such that $\mathfrak{s}(\mathfrak{L}_i)=p^i\mathbb{Z}_p$ or $\mathfrak{L}_i=0$. We define $r_i=r_i(p,L)$ as follows: For $p\ne 2$, we define $r_i:=\mathrm{rank}(\mathfrak{L}_i)$. Assume that $p=2$. If $\mathfrak{s}(\mathfrak{L}_i)=\mathfrak{n}(\mathfrak{L}_i)$, then $r_i:=\mathrm{rank}(\mathfrak{L}_i)$ and if $L_i\simeq\mathbb{A}^{2^i}$ $(\mathbb{H}^{2^i})$ then $r_i=a$ (h, respectively). $S_p(r_0,\dots,r_t)$ is defined by the set of all such $\mathbb{Z}$-lattices L satisfying $r_i=r_i(p,L)$ for all i. We also define

$$\mathcal{S}_p:=\{S_p(r_0,\dots,r_t): S_p(r_0,\dots,r_t)\ne\emptyset\}.$$

If $L\notin S_p(2,2)\cup S_p(1,3)$ when p is odd or $L\notin S_2(a,h)$ when $p=2$, then L is called *absolutely unstable at p*.

Assume that L is unstable at p but is not absolutely unstable at p. If p is odd, then $\lambda_p(L)$ is a stable $\mathbb{Z}$-lattice. Therefore, by Lemma 2.1,

$$L \in S_p(1,3) = \{\lambda_p(L) : L \in S_p(3,1)\} \quad \text{or} \quad L \simeq \lambda_p(L'),$$

where L' is a stable lattice such that

$$L'_p \simeq \langle 1, -1, p, -p\Delta_p \rangle.$$

If p is even, $\lambda_4(L)$ is a stable lattice and $L \in S_2(a,h) = \{\lambda_4(L) : L \in S_2(h,a)\}$. Therefore, $L \in S_p(1,3)$ or L is isometric to

$$\lambda_3(\mathcal{L}_{\mathbf{25}}) \simeq \mathcal{L}'_{\mathbf{25}} = [2,1,2] \perp [2,1,8] \quad \text{or} \quad \lambda_4(\mathcal{L}_{\mathbf{9}}) = [2,2,2,6,1,1,0,1,0,0].$$

We classify every 2-regular quaternary $\mathbb{Z}$-lattice that has only one absolutely unstable prime. In the list of Section 4, every quaternary $\mathbb{Z}$-lattice except the $\mathcal{L}_{\mathbf{i}}$'s and the $\mathcal{M}_{\mathbf{i}}$'s has only one unstable prime.

Now let L be an arbitrary even primitive 2-regular quaternary $\mathbb{Z}$-lattice and p be a prime. By applying suitable numbers of λ_q (or λ_4)-transformations, L can be transformed to a 2-regular q-stable $\mathbb{Z}$-lattice for every prime $q \neq p$. We denote by $\overline{\lambda}_p(L)$ such a $\mathbb{Z}$-lattice. Then $\overline{\lambda}_p(L)$ is isometric to one of $\mathbb{Z}$-lattices having at most one absolutely unstable prime p. Since $\lambda_{(p)}(\overline{\lambda}_p(L))$ is stable everywhere, it is isometric to one of the stable lattices $\mathcal{L}_{\mathbf{i}}$'s. We define

$$t(L) := \{p : \overline{\lambda}_p(L) \text{ is absolutely unstable at } p\}.$$

Theorem 3.4. *If $|t(L)| \geq 2$ and L is stable at p for every $p \notin t(L)$, then $t(L) = \{2,3\}$ and L is isometric to one of $\mathbb{Z}$-lattices in Table 3.1. All of them have class number* 1.

Table 3.1

Name	$\mathcal{M}_i$	$d(\mathcal{M}_i)$	Name	$\mathcal{M}_i$	$d(\mathcal{M}_i)$
$\mathcal{M}_{\mathbf{1}}$	$[2,8.10,14,0,1,4,1,0,5]$	$2^4 \cdot 3^4$	$\mathcal{M}_{\mathbf{2}}$	$[10,16,18,58,4,3,-6,2,8,-3]$	$2^4 \cdot 3^8$
$\mathcal{M}_{\mathbf{3}}$	$[6,6.10,10,3,3,0,3,3,5]$	$2^4 \cdot 3^4$	$\mathcal{M}_{\mathbf{4}}$	$[2,8,10,14,0,1,2,0,4,1]$	$2^6 \cdot 3^3$
$\mathcal{M}_{\mathbf{5}}$	$[10,10,16,16,1,2,2,4,4,8]$	$2^6 \cdot 3^5$	$\mathcal{M}_{\mathbf{6}}$	$[2,2,10,12,0,1,1,0,0,0]$	$2^4 \cdot 3^3$
$\mathcal{M}_{\mathbf{7}}$	$[2,12,14,14,0,1,0,1,0,5]$	$2^4 \cdot 3^5$	$\mathcal{M}_{\mathbf{8}}$	$[4,10,10,12,2,2,1,0,0,0]$	$2^4 \cdot 3^5$

Theorem 3.5. *There are exactly* 177 *equivalence classes of even primitive* 2-*regular quaternary $\mathbb{Z}$-lattices given by next section. The largest discriminant appearing in the table is* $2^4 3^3 11^3 = 574992$, *and all of them have class number* 1.

4. List of primitive even 2-regular quaternary $\mathbb{Z}$-lattices

In this section, the notation $L = (\mathbf{r_0}, \mathbf{r_1}, \ldots, \mathbf{r_t}; \mathbf{k})$ in the row beginning with $\mathcal{L}_{\mathbf{i}}$ implies that $\lambda_{(p)}(L) = \mathcal{L}_{\mathbf{i}}$ and $L \in S_p(r_0, r_1, \ldots, r_t)$ for a suitable prime p. If such a $\mathbb{Z}$-lattice uniquely exists, then $\mathbf{k}$ will be omitted. In all cases, the prime p can easily be deduced from the discriminant of $(\mathbf{r_0}, \mathbf{r_1}, \ldots, \mathbf{r_t}; \mathbf{k})$.

Name	Stable 2-regular lattices	$d(\mathcal{L}_i)$	L	dL	Notation
$\mathcal{L}_1$	[2,2,2,2,0,0,0,1,1,1]	$2^2 = 4$	[2,2,2,4,1,1,0,0,0,0]	$2^4 = 16$	$(\mathbf{a},\mathbf{0},\mathbf{2})$
			[2,2,6,6,1,1,0,1,1,3]	$2^6 = 64$	$(\mathbf{a},\mathbf{0},\mathbf{0},\mathbf{a})$
			[2,2,4,6,1,0,0,1,0,0]	$2^6 = 64$	$(\mathbf{a},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{1})$
			[4,10,10,10,2,2,1,2,1,1]	$2^2 \cdot 3^6 = 2916$	$(\mathbf{1},\mathbf{0},\mathbf{3};\mathbf{1})$
			[2,14,14,14,1,1,5,1,5,−4]	$2^2 \cdot 3^6 = 2916$	$(\mathbf{1},\mathbf{0},\mathbf{3};\mathbf{2})$
			[2,2,10,10,0,1,1,1,1,1]	$2^2 \cdot 3^4 = 324$	$(\mathbf{2},\mathbf{0},\mathbf{2})$
$\mathcal{L}_2$	[2,2,2,2,1,0,0,1,0,1]	5	[2,2,6,6,1,1,0,0,1,1]	$2^4 \cdot 5 = 80$	$(\mathbf{a},\mathbf{0},\mathbf{h})$
			[4,10,10,16,2,2,1,1,5,5]	$5 \cdot 3^6 = 3645$	$(\mathbf{1},\mathbf{0},\mathbf{3})$
			[4,4,4,4,1,1,−1,−1,1,1]	$5^3 = 125$	$\lambda_5(\mathcal{L}_2)$
			[2,2,4,14,1,0,1,1,1,2]	$5^3 = 125$	$(\mathbf{2},\mathbf{1},\mathbf{1})$
			[4,6,14,14,2,1,3,1,−2,−1]	$5^5 = 3125$	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})$
			[2,8,18,18,1,1,3,1,3,−7]	$5^5 = 3125$	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{2})$
			[14,14,26,26,4,7,2,2,7,1]	$3^6 \cdot 5^3 = 91125$	$\lambda_5((\mathbf{1},\mathbf{0},\mathbf{3}))$
			[6,6,6,14,1,1,1,−2,3,3]	$2^4 \cdot 5^3 = 2000$	$\lambda_5((\mathbf{a},\mathbf{0},\mathbf{h}))$
$\mathcal{L}_3$	[2,2,2,2,0,0,0,1,1,0]	$2^3 = 8$	[2,2,8,8,1,1,0,0,0,0]	$2^5 = 32$	$(\mathbf{a},\mathbf{0},\mathbf{1},\mathbf{1})$
			[2,2,6,8,1,1,0,0,0,0]	$2^7 = 128$	$(\mathbf{a},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1})$
			[2,2,6,10,1,1,0,1,1,3]	$2^7 = 128$	$(\mathbf{a},\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{1})$
$\mathcal{L}_4$	[2,2,2,2,1,0,0,0,0,1]	$3^2 = 9$	[2,2,8,8,1,0,0,0,0,4]	$2^4 \cdot 3^2 = 144$	$(\mathbf{a},\mathbf{0},\mathbf{a})$
			[2,2,6,6,1,0,0,0,0,3]	$3^4 = 81$	$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1})$
			[4,4,4,4,1,1,1,2,2,−1]	$3^4 = 81$	$(\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{2})$
			[4,4,4,16,1,1,1,2,2,−1]	$3^6 = 729$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1})$
			[4,4,10,10,2,1,−1,1,2,4]	$3^6 = 729$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{2})$
			[2,8,8,8,1,1,−1,1,2,2]	$3^6 = 729$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{3})$
			[2,2,4,8,0,1,1,1,0,2]	$3^4 = 81$	$(\mathbf{2},\mathbf{1},\mathbf{0},\mathbf{1})$
			[4,10,16,16,2,1,5,1,−4,−2]	$3^8 = 6561$	$(\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2})$
			[2,8,10,14,0,1,4,1,0,5]	$2^4 \cdot 3^4 = 1296$	$\mathcal{M}_1$
			[10,16,18,58,4,3,−6,2,8,−3]	$2^4 \cdot 3^8 = 104976$	$\mathcal{M}_2$
			[6,6,10,10,3,3,0,3,3,5]	$2^4 \cdot 3^4 = 1296$	$\mathcal{M}_3$
$\mathcal{L}_5$	[2,2,2,4,1,1,0,1,0,0]	$2^2 \cdot 3 = 12$	[2,2,2,12,1,1,0,0,0,0]	$2^4 \cdot 3 = 48$	$(\mathbf{a},\mathbf{0},\mathbf{2})$
			[2,2,6,14,1,1,0,1,1,3]	$2^6 \cdot 3 = 192$	$(\mathbf{a},\mathbf{0},\mathbf{0},\mathbf{2})$
			[2,2,4,10,0,1,1,1,1,1]	$2^2 \cdot 3^3 = 108$	$(\mathbf{2},\mathbf{1},\mathbf{1})$
			[2,8,8,12,1,1,−1,0,3,3]	$2^2 \cdot 3^5 = 972$	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{1})$
			[4,4,10,10,2,1,−1,1,−1,1]	$2^2 \cdot 3^5 = 972$	$(\mathbf{1},\mathbf{1},\mathbf{2};\mathbf{2})$
			[4,4,4,4,2,1,−1,−1,1,−1]	$2^2 \cdot 3^3 = 108$	$\lambda_3(\mathcal{L}_5)$
			[2,8,10,14,0,1,2,0,4,1]	$2^6 \cdot 3^3 = 1728$	$\mathcal{M}_4$
			[10,10,16,16,1,2,2,4,4,8]	$2^6 \cdot 3^5 = 15552$	$\mathcal{M}_5$
			[2,2,10,12,0,1,1,0,0,0]	$2^4 \cdot 3^3 = 432$	$\mathcal{M}_6$
			[2,12,14,14,0,1,0,1,0,5]	$2^4 \cdot 3^5 = 3888$	$\mathcal{M}_7$
			[4,10,10,12,2,2,1,0,0,0]	$2^4 \cdot 3^5 = 3888$	$\mathcal{M}_8$
			[4,6,6,6,0,0,0,0,3,3]	$2^4 \cdot 3^3 = 432$	$\lambda_3((\mathbf{a},\mathbf{0},\mathbf{2}))$
			[6,6,10,10,3,3,0,3,0,2]	$2^6 \cdot 3^3 = 1728$	$\lambda_3((\mathbf{a},\mathbf{0},\mathbf{0},\mathbf{2}))$
$\mathcal{L}_6$	[2,2,2,2,1,0,0,0,0,0]	$2^2 \cdot 3 = 12$	[2,2,4,4,1,0,0,0,0,0]	$2^4 \cdot 3 = 48$	$(\mathbf{a},\mathbf{0},\mathbf{2})$
			[2,2,6,6,1,0,0,0,0,0]	$2^2 \cdot 3^3 = 108$	$\lambda_3(\mathcal{L}_6)$
			[2,2,12,12,1,0,0,0,0,0]	$2^4 \cdot 3^3 = 432$	$\lambda_3((\mathbf{a},\mathbf{0},\mathbf{2}))$
$\mathcal{L}_7$	[2,2,2,4,1,1,0,0,1,0]	13	[4,10,10,10,1,−1,3,1,−3,3]	$13^3 = 2197$	$\lambda_{13}(\mathcal{L}_7)$
$\mathcal{L}_8$	[2,2,2,4,1,0,0,1,0,1]	17	[6,10,12,12,3,2,1,2,1,−5]	$17^3 = 4913$	$\lambda_{17}(\mathcal{L}_8)$
$\mathcal{L}_9$	[2,2,2,4,0,0,0,1,1,1]	$2^2 \cdot 5 = 20$	[2,2,2,6,1,1,0,1,0,0]	$2^2 \cdot 5 = 20$	$(\mathbf{a},\mathbf{h})$
			[2,8,8,8,1,1,−2,1,3,3]	$2^2 \cdot 5^3 = 500$	$\lambda_5(\mathcal{L}_9)$
			[4,6,6,6,2,2,1,2,1,1]	$2^2 \cdot 5^3 = 500$	$\lambda_5((\mathbf{a},\mathbf{h}))$
$\mathcal{L}_{10}$	[2,2,2,4,1,0,0,1,0,0]	$2^2 \cdot 5 = 20$	[4,4,4,10,1,1,−1,0,0,0]	$2^2 \cdot 5^3 = 500$	$\lambda_5(\mathcal{L}_{10})$
$\mathcal{L}_{11}$	[2,2,2,4,1,0,0,0,0,1]	$3 \cdot 7 = 21$	[2,2,6,12,1,0,0,0,0,3]	$3^3 \cdot 7 = 189$	$\lambda_3(\mathcal{L}_{11})$
			[2,4,14,14,1,0,0,0,0,7]	$3 \cdot 7^3 = 1029$	$\lambda_7(\mathcal{L}_{11})$
			[6,12,14,14,3,0,0,0,0,7]	$3^3 \cdot 7^3 = 9261$	$\lambda_3(\lambda_7(\mathcal{L}_{11}))$

Name	Stable 2-regular lattices	$d(\mathcal{L}_i)$	L	dL	Notation
$\mathcal{L}_{12}$	$[2,2,2,6,1,1,0,0,1,0]$	$3\cdot 7=21$	$[4,4,4,4,1,1,1,1,1,1]$	$3^3\cdot 7=189$	$\lambda_3(\mathcal{L}_{12})$
			$[6,6,6,6,1,1,-1,-1,1,1]$	$3\cdot 7^3=1029$	$\lambda_7(\mathcal{L}_{12})$
			$[4,16,16,16,1,1,-5,1,-5,-5]$	$3^3\cdot 7^3=9261$	$\lambda_3(\lambda_7(\mathcal{L}_{12}))$
			$[2,2,8,8,0,1,0,0,1,3]$	$3^3\cdot 7=189$	$(\mathbf{2,1,1})$
			$[2,8,12,12,1,0,3,0,3,3]$	$3^5\cdot 7=1701$	$(\mathbf{1,1,2;1})$
			$[4,10,10,10,1,1,1,2,5,-4]$	$3^5\cdot 7=1701$	$(\mathbf{1,1,2;2})$
			$[10,10,10,20,3,3,-4,2,2,-5]$	$3^3\cdot 7^3=9261$	$\lambda_7((\mathbf{2,1,1;2}))$
			$[12,14,20,38,0,3,7,6,7,5]$	$3^5\cdot 7^3=83349$	$\lambda_7((\mathbf{1,1,2;1}))$
			$[10,10,34,34,4,5,2,2,5,1]$	$3^5\cdot 7^3=83349$	$\lambda_7((\mathbf{1,1,2;2}))$
$\mathcal{L}_{13}$	$[2,2,2,4,0,0,0,1,1,0]$	$2^3\cdot 3=24$	$[2,2,6,6,0,1,1,1,1,0]$	$2^5\cdot 3=96$	$(\mathbf{a,0,1,1})$
			$[2,6,6,8,1,1,0,0,2,2]$	$2^7\cdot 3=384$	$(\mathbf{a,0,1,0,0,1})$
			$[2,6,6,8,1,1,-1,0,-2,2]$	$2^7\cdot 3=384$	$(\mathbf{a,0,0,1,1})$
			$[4,4,4,6,2,1,-1,0,0,0]$	$2^3\cdot 3^3=216$	$\lambda_3(\mathcal{L}_{13})$
			$[4,6,6,10,0,0,0,2,3,3]$	$2^5\cdot 3^3=864$	$\lambda_3((\mathbf{a,0,1,1}))$
			$[4,6,10,22,0,2,3,2,3,4]$	$2^7\cdot 3^3=3456$	$\lambda_3((\mathbf{a,0,1,0,0,1}))$
			$[6,10,10,10,3,0,1,3,5,-1]$	$2^7\cdot 3^3=3456$	$\lambda_3((\mathbf{a,0,0,1,1}))$
$\mathcal{L}_{14}$	$[2,2,4,4,1,1,0,1,1,2]$	$5^2=25$	$[2,8,8,8,1,1,-2,-1,-3,2]$	$5^4=625$	$(\mathbf{1,2,1;1})$
			$[4,4,4,14,1,1,-1,1,-1,-1]$	$5^4=625$	$(\mathbf{1,2,1;2})$
$\mathcal{L}_{15}$	$[2,2,4,4,1,1,0,0,1,1]$	$2^2\cdot 7=28$	$[2,2,4,10,1,0,0,1,0,0]$	$2^4\cdot 7=112$	$(\mathbf{a,0,2})$
			$[6,6,6,10,1,1,-1,2,-2,-2]$	$2^2\cdot 7^3=1372$	$\lambda_7(\mathcal{L}_{15})$
			$[6,6,6,28,1,1,-1,0,0,0]$	$2^4\cdot 7^3=5488$	$\lambda_7((\mathbf{a,0,2}))$
$\mathcal{L}_{16}$	$[2,2,2,6,1,0,0,1,0,0]$	$2^5=32$			
$\mathcal{L}_{17}$	$[2,2,4,4,0,0,0,1,1,2]$	$2^5=32$			
$\mathcal{L}_{18}$	$[2,2,4,4,1,1,0,1,0,0]$	$2^5=32$			
$\mathcal{L}_{19}$	$[2,2,4,4,0,1,1,1,0,2]$	$3\cdot 11=33$	$[2,6,6,10,1,1,-1,1,1,0]$	$2^4\cdot 3\cdot 11=528$	$(\mathbf{a,0,a})$
			$[4,4,4,10,2,1,-1,1,2,1]$	$3^3\cdot 11=297$	$\lambda_3(\mathcal{L}_{19})$
			$[4,12,12,14,2,2,1,1,6,6]$	$3\cdot 11^3=3993$	$\lambda_{11}(\mathcal{L}_{19})$
			$[12,14,20,20,6,-3,4,3,7,2]$	$3^3\cdot 11^3=35937$	$\lambda_3(\lambda_{11}(\mathcal{L}_{19}))$
			$[6,10,10,10,3,0,1,0,-1,2]$	$2^4\cdot 3^3\cdot 11=4752$	$\lambda_3((\mathbf{a,0,a}))$
			$[14,14,16,26,3,2,2,1,1,8]$	$2^4\cdot 3\cdot 11^3=63888$	$\lambda_{11}((\mathbf{a,0,a}))$
			$[14,26,26,78,1,1,-7,-6,9,9]$	$2^4\cdot 3^3\cdot 11^3=574992$	$\lambda_3(\lambda_{11}((\mathbf{a,0,a})))$
$\mathcal{L}_{20}$	$[2,2,2,10,1,1,0,1,0,0]$	$2^2\cdot 3^2=36$			
$\mathcal{L}_{21}$	$[2,2,2,6,0,0,0,1,1,1]$	$2^2\cdot 3^2=36$			
$\mathcal{L}_{22}$	$[2,2,2,6,1,0,0,0,0,0]$	$2^2\cdot 3^2=36$	$[4,4,4,6,1,1,1,0,0,0]$	$2^2\cdot 3^4=324$	$(\mathbf{1,2,1})$
			$[4,10,10,12,1,1,-5,0,3,-3]$	$2^2\cdot 3^6=2916$	$(\mathbf{1,1,1,1})$
$\mathcal{L}_{23}$	$[2,2,4,4,1,0,0,0,0,2]$	$2^2\cdot 3^2=36$	$[2,2,12,12,1,0,0,0,0,6]$	$2^2\cdot 3^4=324$	$(\mathbf{1,2,1;1})$
			$[4,4,6,6,2,0,0,0,0,3]$	$2^2\cdot 3^4=324$	$(\mathbf{1,2,1;2})$
			$[4,6,6,28,0,0,3,2,0,0]$	$2^2\cdot 3^6=2916$	$(\mathbf{1,1,1,1;1})$
			$[2,12,12,14,0,0,6,1,0,0]$	$2^2\cdot 3^6=2916$	$(\mathbf{1,1,1,1;2})$
$\mathcal{L}_{24}$	$[2,2,4,4,0,1,1,1,1,1]$	$2^2\cdot 3^2=36$	$[4,4,4,10,2,1,-1,-1,1,-1]$	$2^2\cdot 3^4=324$	$(\mathbf{1,2,1;1})$
			$[2,6,6,8,0,0,0,1,3,3]$	$2^2\cdot 3^4=324$	$(\mathbf{1,2,1;2})$
$\mathcal{L}_{25}$	$[2,2,4,4,1,0,0,0,0,1]$	$3^2\cdot 5=45$	$[2,2,2,8,1,0,0,0,0,1]$	$3^2\cdot 5=45$	$\mathcal{L}'_{25}$
			$[4,4,4,10,1,1,1,-2,-2,1]$	$3^4\cdot 5=405$	$(\mathbf{1,2,1})$
			$[2,8,12,26,1,0,3,1,2,6]$	$3^6\cdot 5=3645$	$(\mathbf{1,1,1,1})$
			$[2,2,6,24,1,0,0,0,0,3]$	$3^4\cdot 5=405$	$\mathcal{L}'_{25}(\mathbf{1,2,1;1})$
			$[4,4,6,6,1,0,0,0,0,3]$	$3^4\cdot 5=405$	$\mathcal{L}'_{25}(\mathbf{1,2,1;2})$
			$[4,10,10,16,1,1,-5,1,-2,-2]$	$3^6\cdot 5=3645$	$\mathcal{L}'_{25}(\mathbf{1,1,1,1})$
			$[2,8,10,10,1,0,0,0,0,5]$	$3^2\cdot 5^3=1125$	$\lambda_5(\mathcal{L}_{25})$
			$[4,4,10,10,1,0,0,0,0,5]$	$3^2\cdot 5^3=1125$	$\lambda_5(\mathcal{L}'_{25})$
			$[6,14,14,14,3,3,4,3,4,-1]$	$3^4\cdot 5^3=10125$	$\lambda_5((\mathbf{1,2,1}))$
			$[10,18,22,40,0,-5,9,5,0,5]$	$3^6\cdot 5^3=91125$	$\lambda_5((\mathbf{1,1,1,1}))$
			$[10,10,12,12,5,0,0,0,0,3]$	$3^4\cdot 5^3=10125$	$\lambda_5(\mathcal{L}'_{25}(\mathbf{1,2,1;1}))$
			$[2,8,30,30,1,0,0,0,0,15]$	$3^4\cdot 5^3=10125$	$\lambda_5(\mathcal{L}'_{25}(\mathbf{1,2,1;2}))$
			$[6,14,20,74,3,0,5,3,4,10]$	$3^6\cdot 5^3=91125$	$\lambda_5(\mathcal{L}'_{25}(\mathbf{1,1,1,1}))$

Name	Stable 2-regular lattices	$d(\mathcal{L}_1)$	L	dL	Notation
$\mathcal{L}_{26}$	$[2,2,4,4,0,1,0,0,1,1]$	$3^2 \cdot 5 = 45$	$[4,6,6,14,2,2,1,1,3,3]$	$3^2 \cdot 5^3 = 1125$	$\lambda_5(\mathcal{L}_{26})$
$\mathcal{L}_{27}$	$[2,2,4,6,1,1,0,0,1,1]$	$2^4 \cdot 3 = 48$	$[2,4,8,10,1,0,2,1,1,2]$	$2^4 \cdot 3^3 = 432$	$(\mathbf{2,1,1})$
			$[4,10,10,12,1,1,1,0,3,3]$	$2^4 \cdot 3^5 = 3888$	$(\mathbf{1,1,2})$
			$[4,4,4,10,1,1,1,2,2,2]$	$2^4 \cdot 3^3 = 432$	$\lambda_3(\mathcal{L}_{27})$
$\mathcal{L}_{28}$	$[2,2,4,4,0,1,1,0,0,0]$	$2^4 \cdot 3 = 48$	$[4,4,4,12,2,1,-1,0,0,0]$	$2^4 \cdot 3^3 = 432$	$\lambda_3(\mathcal{L}_{28})$
$\mathcal{L}_{29}$	$[2,4,4,4,0,0,0,1,2,2]$	$2^4 \cdot 3 = 48$	$[2,6,8,8,0,1,3,1,3,2]$	$2^4 \cdot 3^3 = 432$	$\lambda_3(\mathcal{L}_{29})$
$\mathcal{L}_{30}$	$[2,2,4,4,0,1,0,0,1,0]$	$7^2 = 49$	$[2,6,6,16,1,1,-1,0,2,2]$	$2^4 \cdot 7^2 = 784$	$(\mathbf{a,0,a})$
			$[6,6,10,10,1,2,-2,2,-2,3]$	$7^4 = 2401$	$(\mathbf{1,2,1})$
$\mathcal{L}_{31}$	$[2,4,4,4,1,0,2,0,1,2]$	$2^2 \cdot 3 \cdot 5 = 60$	$[2,4,6,6,0,1,0,1,0,0]$	$2^4 \cdot 3 \cdot 5 = 240$	$(\mathbf{a,0,2})$
			$[4,4,6,10,2,0,0,2,1,3]$	$2^4 \cdot 3^3 \cdot 5 = 2160$	$\lambda_3(\mathcal{L}_{31})$
			$[2,8,8,18,1,1,-2,1,3,3]$	$2^2 \cdot 3 \cdot 5^3 = 1500$	$\lambda_5(\mathcal{L}_{31})$
			$[6,14,14,20,3,3,4,0,5,-5]$	$2^2 \cdot 3^3 \cdot 5^3 = 13500$	$\lambda_3(\lambda_5(\mathcal{L}_{31}))$
			$[4,6,10,12,0,2,3,0,0,0]$	$2^4 \cdot 3^3 \cdot 5 = 2160$	$\lambda_3((\mathbf{a,0,2}))$
			$[2,10,18,20,0,1,5,0,0,0]$	$2^4 \cdot 3 \cdot 5^3 = 6000$	$\lambda_5((\mathbf{a,0,2}))$
			$[6,14,14,60,3,3,4,0,0,0]$	$2^4 \cdot 3^3 \cdot 5^3 = 54000$	$\lambda_3(\lambda_5((\mathbf{a,0,2})))$
$\mathcal{L}_{32}$	$[2,2,4,6,0,1,0,1,1,0]$	$3 \cdot 23 = 69$	$[4,4,6,10,1,0,0,2,2,3]$	$3^3 \cdot 23 = 621$	$\lambda_3(\mathcal{L}_{32})$
			$[10,14,20,20,5,4,2,4,2,-3]$	$3 \cdot 23^3 = 36501$	$\lambda_{23}(\mathcal{L}_{32})$
			$[14,20,38,42,2,7,1,-6,9,-3]$	$3^3 \cdot 23^3 = 328509$	$\lambda_3(\lambda_{23}(\mathcal{L}_{32}))$
$\mathcal{L}_{33}$	$[2,4,4,4,1,1,0,1,0,0]$	$2^4 \cdot 5 = 80$	$[4,14,14,14,1,1,-6,1,-6,-6]$	$2^4 \cdot 5^3 = 2000$	$\lambda_5(\mathcal{L}_{33})$
$\mathcal{L}_{34}$	$[2,2,4,8,0,1,0,0,0,2]$	$2^5 \cdot 3 = 96$	$[4,4,6,10,1,0,0,1,1,0]$	$2^5 \cdot 3^3 = 864$	$\lambda_3(\mathcal{L}_{34})$
$\mathcal{L}_{35}$	$[2,4,4,4,0,0,0,1,1,1]$	$2^5 \cdot 3 = 96$	$[4,4,10,10,1,2,-1,1,1,5]$	$2^5 \cdot 3^3 = 864$	$\lambda_3(\mathcal{L}_{35})$
$\mathcal{L}_{36}$	$[2,2,6,6,0,1,1,1,1,1]$	$2^2 \cdot 5^2 = 100$	$[4,6,6,26,2,2,1,2,1,1]$	$2^2 \cdot 5^4 = 2500$	$(\mathbf{1,2,1})$
$\mathcal{L}_{37}$	$[2,4,4,6,1,0,2,0,1,2]$	$2^2 \cdot 5^2 = 100$			
$\mathcal{L}_{38}$	$[2,4,4,6,0,0,2,-1,1,-1]$	$2^2 \cdot 3^3 = 108$	$[2,4,6,12,0,1,2,0,0,0]$	$2^4 \cdot 3^3 = 432$	$(\mathbf{a,0,2})$
$\mathcal{L}_{39}$	$[2,4,4,6,1,0,2,0,1,0]$	$2^4 \cdot 7 = 112$	$[6,10,12,12,2,3,1,3,1,-2]$	$2^4 \cdot 7^3 = 5488$	$\lambda_7(\mathcal{L}_{39})$
$\mathcal{L}_{40}$	$[2,2,6,8,0,1,1,1,0,3]$	$5^3 = 125$			
$\mathcal{L}_{41}$	$[2,4,4,8,1,0,2,0,2,0]$	$2^7 = 128$			
$\mathcal{L}_{42}$	$[2,4,4,6,1,1,0,0,1,1]$	$2^7 = 128$			
$\mathcal{L}_{43}$	$[2,4,4,8,1,1,0,1,2,2]$	$2^4 \cdot 3^2 = 144$	$[4,4,10,10,1,1,1,1,1,-2]$	$2^4 \cdot 3^4 = 1296$	$(\mathbf{1,2,1})$
$\mathcal{L}_{44}$	$[2,4,4,8,1,0,1,1,1,2]$	$13^2 = 169$			
$\mathcal{L}_{45}$	$[2,4,6,6,0,1,1,1,2,1]$	$3^3 \cdot 7 = 189$	$[6,6,10,34,1,2,-2,1,-1,5]$	$3^3 \cdot 7^3 = 9261$	$\lambda_7(\mathcal{L}_{45})$
$\mathcal{L}_{46}$	$[2,4,6,6,1,0,1,0,-1,2]$	$2^6 \cdot 3 = 192$	$[4,4,10,16,1,1,1,2,2,-4]$	$2^6 \cdot 3^3 = 1728$	$\lambda_3(\mathcal{L}_{46})$
$\mathcal{L}_{47}$	$[2,4,6,10,0,1,2,0,2,1]$	$2^2 \cdot 3^4 = 324$			
$\mathcal{L}_{48}$	$[2,4,6,12,0,1,0,0,2,0]$	$2^2 \cdot 11^2 = 484$			

Primitive even 2-regular quaternary $\mathbb{Z}$-lattices

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