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SOLVING SINGULAR NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS IN THE REPRODUCING KERNEL SPACE

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Abstract. In this paper, we present a new method for solving a nonlinear two-point boundary value problem with finitely many singularities. Its exact solution is represented in the form of series in the reproducing kernel space. In the mean time, the n-term approximation $u_n(x)$ to the exact solution $u(x)$ is obtained and is proved to converge to the exact solution. Some numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

1. Introduction

In this paper, we consider the following nonlinear second order ordinary differential equation with finitely many singularities in the reproducing kernel space

$$\begin{cases} H(u(x))u''(x) + \frac{1}{p(x)}u'(x) + \frac{1}{q(x)}N(u(x)) = f(x), \ \ 0 < x < 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases} \tag{1.1}$$

where $u(x) \in W_2^3[0,1]$, $f(x) \in W_2^1[0,1]$, $p(x)$, $q(x)$ are continuous and may be equal to zero at $\{x_i\}_{i=1}^m \in [0,1]$, H and N are continuous functions of u. It is easy to see that the problem may have singularities not only at points $\{x_i\}_{i=1}^m$, but also at $u = 0$. The singular boundary value problem arises in a variety of differential applied mathematics and physics such as gas dynamics, nuclear physics, chemical reaction, studies of atomic structures and atomic calculations. Therefore, the problem has attracted much attention and has been studied by many authors. The existence and uniqueness of the equation have been widely investigated (see [1,3,5,6,10–12]). In general, classical numerical methods fail to produce good approximations for the equations. Hence one has to go for

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non-classical method. Mohan and Vivek treated the homogeneous equation via Chebyshev polynomial and B-spline (see [2]). Kanth and Reddy studied a particular singular boundary value problem $u''(x) + \frac{k}{x}u'(x) + q(x)u(x) = r(x)$ by applying higher order finite difference method and cubic spline method (see [8,9]). Mohchty and his co-workers considered such an singular boundary value problem $u''(x) + \frac{a}{x}u'(x) + \frac{a}{x^2}u(x) = r(x)$ using a four order accurate cubic spine method (see [7]). However, in most of the present references, the problems discussed only have singularities at boundary points, not at interior points, and there are few valid methods of solving the equation (1.1).

In this paper, we will give the representation of exact solution to the equation (1.1) and approximate solution in the reproducing kernel space under the assumption that the solution to the equation (1.1) is unique.

After multiplying the equation (1.1) by $p(x)q(x)$, we find that

$$(1.2)\qquad \begin{cases} p(x)q(x)H(u(x))u''(x) + q(x)u'(x) + p(x)N(u(x)) \\ = p(x)q(x)f(x), \quad 0 < x < 1, \\ u(0) = 0, \\ u(1) = 0. \end{cases}$$

Clearly, the solution of the equation (1.2) is the solution of the equation (1.1). So we only need to obtain the solution of the equation (1.2).

Put $Lu \equiv q(x)u'(x)$ and write

$$F(x, u(x), u''(x)) = p(x)q(x)f(x) - p(x)q(x)H(u(x))u''(x) - p(x)N(u(x))$$

simply. Then the equation (1.2) can further be converted into the following form

$$(1.3)\qquad \begin{cases} Lu = F(x, u(x), u''(x)), x \in [0, 1] \\ u(0) = 0, \\ u(1) = 0, \end{cases}$$

where $u \in W_2^3[0,1], F(x, u(x), u''(x)) \in W_2^1[0,1]$. $W_2^1[0,1]$ and $W_2^3[0,1]$ are defined in the following section.

2. Several reproducing kernel spaces

1. The reproducing kernel space $W_2^3[0,1]$.

The inner product space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x) \mid u, u', u''$ are absolutely continuous real valued functions, $u, u', u'', u^{(3)} \in L^2[0,1], u(0) = 0, u(1) = 0\}$. The inner product in $W_2^3[0,1]$ is given by

$$(2.1)\qquad (u(y), v(y))_{W_2^3} = \int_0^1 (36uv + 49u'v' + 14u''v'' + u^{(3)}v^{(3)})dy,$$

and the norm $\| u \|_{W_2^3}$ is denoted by $\| u \|_{W_2^3} = \sqrt{(u,u)_{W_2^3}}$, where $u, v \in W_2^3[0,1]$.

Theorem 2.1. *The space $W_2^3[0,1]$ is a reproducing kernel space. That is, there exists $R_x(y) \in W_2^3[0,1]$ for any $u(y) \in W_2^3[0,1]$ and each fixed $x \in [0,1]$, $y \in [0,1]$, such that $(u(y), R_x(y))_{W_2^3} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$(2.2)\quad R_x(y) = \begin{cases} c_1e^y + c_2e^{-y} + c_3e^{2y} + c_4e^{-2y} + c_5e^{3y} + c_6e^{-3y}, & y \le x, \\ d_1e^y + d_2e^{-y} + d_3e^{2y} + d_4e^{-2y} + d_5e^{3y} + d_6e^{-3y}, & y > x. \end{cases}$$

The coefficients of the reproducing kernel $R_x(y)$ and the proof of Theorem 2.1 are given in section 5.

2. The reproducing kernel space $W_2^1[0,1]$.

The inner product space $W_2^1[0,1]$ is defined by $W_2^1[0,1] = \{u(x) \mid u$ is absolutely continuous real valued function, $u, u' \in L^2[0,1]\}$. The inner product and norm in $W_2^1[0,1]$ are given respectively by

$$(u(x), v(x))_{W_2^1} = \int_0^1 (uv + u'v')dx, \ \| u \|_{W_2^1} = \sqrt{(u,u)_{W_2^1}},$$

where $u(x), v(x) \in W_2^1[0,1]$. In [4], the authors proved that $W_2^1[0,1]$ is a complete reproducing kernel space and its reproducing kernel is

$$\overline{R}_x(y) = \frac{1}{2\sinh(1)}[\cosh(x+y-1) + \cosh(|x-y|-1)].$$

3. The solution of the equation (1.3)

In this section, we will give the representation of exact solution of the equation (1.3) and implementation method in the reproducing kernel space $W_2^3[0,1]$.

In the equation (1.3), it is clear that $L : W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{R}_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the adjoint operator of L . The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$(3.1)\qquad \overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), (\beta_{ii} > 0, i = 1,2,\ldots).$$

Theorem 3.1. *For the equation (1.3), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$ and $\psi_i(x) = L_yR_x(y)|_{y=x_i}$.*

Proof. We have

$$\begin{aligned} \psi_i(x) &= (L^*\varphi_i)(x) = ((L^*\varphi_i)(y), R_x(y)) \\ &= (\varphi_i(y), L_yR_x(y)) = L_yR_x(y)|_{y=x_i}. \end{aligned}$$

The subscript y by the operator L indicates that the operator L applies to the function of y.

Clearly, $\psi_i(x) \in W_2^3[0,1]$.

For each fixed $u(x) \in W_2^3[0,1]$, let $(u(x), \psi_i(x)) = 0, (i = 1, 2, \ldots)$, which means that,

$$(u(x), (L^*\varphi_i)(x)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(x_i) = 0. \tag{3.2}$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of the Theorem 3.1 is complete. □

Theorem 3.2. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$ *and the solution of the equation* (1.3) *is unique, then the solution of the equation* (1.3) *satisfies the form*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), u''(x_k))\overline{\psi}_i(x). \tag{3.3}$$

Proof. Applying Theorem 3.1, it is easy to see that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^3[0,1]$.

Note that $(v(x), \varphi_i(x)) = v(x_i)$ for each $v(x) \in W_2^1[0,1]$, hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}(u(x), \overline{\psi}_i(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(u(x), L^*\varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(Lu(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(F(x, u(x), u''(x)), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), u''(x_k))\overline{\psi}_i(x) \end{aligned} \tag{3.4}$$

and the proof of the theorem is complete. □

The implementation method

(3.3) can be denoted by

$$u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x), \tag{3.5}$$

where $A_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), u''(x_k))$. Let $x_1 = 0$, it follows that

$$F(x_1, u(x_1), u''(x_1))$$

is known. Considering the numerical computation, we put $u_0(x_1) = u(x_1)$, $u_0''(x_1) = u''(x_1)$ and define the n-term approximation to $u(x)$ by

$$u_n(x) = \sum_{i=1}^{n} B_i \overline{\psi}_i(x), \tag{3.6}$$

where

$$
\begin{array}{lll}
B_1 & = & \beta_{11}F(x_1, u_0(x_1), u_0''(x_1)), \\
u_1(x) & = & B_1\overline{\psi}_1(x), \\
B_2 & = & \sum\limits_{k=1}^{2} \beta_{2k}F(x_k, u_{k-1}(x_k), u_{k-1}''(x_k)), \\
u_2(x) & = & \sum\limits_{i=1}^{2} B_i\overline{\psi}_i(x), \\
\vdots & & \\
u_{n-1}(x) & = & \sum\limits_{i=1}^{n-1} B_i\overline{\psi}_i(x), \\
B_n & = & \sum\limits_{k=1}^{n} \beta_{nk}F(x_k, u_{k-1}(x_k), u_{k-1}''(x_k)).
\end{array} \tag{3.7}
$$

Next, the convergence of $u_n(x)$ will be proved.
Now, two lemmas are given first.

Lemma 3.1. *If* $u(x) \in W_2^3[0,1]$, *then* $|u(x)| \leq \sqrt{3} \parallel u(x) \parallel_{W_2^3}$, $|u'(x)| \leq \sqrt{3} \parallel u(x) \parallel_{W_2^3}$ *and* $|u''(x)| \leq \sqrt{3} \parallel u(x) \parallel_{W_2^3}$.

Proof. Noting that

$$u(x) - u(y) = \int_y^x u'(t)dt,$$

we get

$$|u(x)|^2 \leq |u(y)|^2 + (\int_0^1 |u'(t)|dt)^2 + 2|u(y)| \int_0^1 |u'(t)|dt. \tag{3.8}$$

Integrating (3.8) with respect to y from 0 to 1 and by Hölder's inequality, then

$$
\begin{array}{lll}
|u(x)|^2 & \leq & \int_0^1 |u(y)|^2dy + \int_0^1 |u'(t)|^2dt + 2\int_0^1 |u'(t)|dt \int_0^1 |u(y)|dy \\
& \leq & \parallel u \parallel_{W_2^3[0,1]}^2 + 2 \parallel u \parallel_{W_2^3} \parallel u \parallel_{W_2^3} \\
& = & 3 \parallel u \parallel_{W_2^3}^2 \\
& \leq & 3 \parallel u \parallel_{W_2^3}^2 .
\end{array}
$$

That is, $|u(x)| \leq \sqrt{3} \parallel u(x) \parallel_{W_2^3}$. In the same way, we obtain that

$$|u'(x)| \leq \sqrt{3} \parallel u(x) \parallel_{W_2^3}, |u''(x)| \leq \sqrt{3} \parallel u(x) \parallel_{W_2^3} .$$

□

By Lemma 3.1, it is easy to obtain the following Lemma 3.2.

Lemma 3.2. *If* $u_n \xrightarrow{\|\cdot\|} \overline{u}(n \to \infty)$, $\parallel u_n \parallel$ *is bounded, that is,* $\sum\limits_{i=1}^{n} B_i^2 < \infty$, $x_n \to y(n \to \infty)$ *and* $F(x, u(x), u''(x))$ *is continuous, then*

$$F(x_n, u_{n-1}(x_n), u_{n-1}''(x_n)) \to f(y, \overline{u}(y), \overline{u}''(y))(n \to \infty).$$

Theorem 3.3. *Suppose that* $\| u_n \|$ *is bounded in* (3.6) *and the equation* (1.3) *has a unique solution. If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$, *then the n-term approximate solution* $u_n(x)$ *derived from the above method converges to the exact solution* $u(x)$ *of the equation* (1.3) *and*

$$u(x) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i(x), \tag{3.9}$$

where B_i *is given by* (3.7).

Proof. First of all, we will prove the convergence of $u_n(x)$. From (3.6), we infer that

$$u_{n+1}(x) = u_n(x) + B_{n+1}\overline{\psi}_{n+1}(x). \tag{3.10}$$

The orthonormality of $\{\overline{\psi}_i\}_{i=1}^{\infty}$ yields that

$$\| u_{n+1} \|^2 = \| u_n \|^2 + (B_{n+1})^2 = \cdots = \sum_{i=1}^{n+1} (B_i)^2. \tag{3.11}$$

In terms of (3.11), it holds that $\| u_{n+1} \| \geq \| u_n \|$. Due to the condition that $\| u_n \|$ is bounded, $\| u_n \|$ is convergent and there exists a constant c such that

$$\sum_{i=1}^{\infty} (B_i)^2 = c.$$

This implies that

$$\{B_i\}_{i=1}^{\infty} \in l^2.$$

If $m > n$, then

$$\| u_m - u_n \|^2 = \| u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n \|^2 .$$

In view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\| u_m - u_n \|^2 = \| u_m - u_{m-1} \|^2 + \cdots + \| u_{n+1} - u_n \|^2 .$$

Furthermore

$$\| u_m - u_{m-1} \|^2 = (B_m)^2.$$

Consequently,

$$\| u_m - u_n \|^2 = \sum_{l=n+1}^{m} (B_l)^2 \to 0 \quad \text{as } n \to \infty.$$

The completeness of $W_2^3[0,1]$ shows that $u_n \to \overline{u}$ as $n \to \infty$ in the sense of $\| \cdot \|_{W_2^3}$.

Secondly, we will prove that $\overline{u}$ is the solution of the equation (1.3). Taking limits in (3.6), we get

$$\overline{u}(x) = \sum_{i=1}^{\infty} B_i \overline{\psi}_i(x). \tag{3.12}$$

Note here that

$$L\overline{u}(x) = \sum_{i=1}^{\infty} B_i L\overline{\psi}_i(x)$$

and

$$\begin{aligned}(L\overline{u})(x_n) &= \sum_{i=1}^{\infty} B_i(L\overline{\psi}_i, \varphi_n) \\ &= \sum_{i=1}^{\infty} B_i(\overline{\psi}_i, L^*\varphi_n) \\ &= \sum_{i=1}^{\infty} B_i(\overline{\psi}_i, \psi_n).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{n} \beta_{nj}(L\overline{u})(x_j) &= \sum_{i=1}^{\infty} B_i(\overline{\psi}_i, \sum_{j=1}^{n} \beta_{nj}\psi_j) \\ &= \sum_{i=1}^{\infty} B_i(\overline{\psi}_i, \overline{\psi}_n) \\ &= B_n.\end{aligned} \tag{3.13}$$

If $n = 1$, then

$$(L\overline{u})(x_1) = F(x_1, u_0(x_1), u_0''(x_1)).$$

If $n = 2$, then

$$\begin{aligned}&\beta_{21}(L\overline{u})(x_1) + \beta_{22}(L\overline{u})(x_2) \\ &= \beta_{21}F(x_1, u_0(x_1), u_0''(x_1)) + \beta_{22}F(x_2, u_1(x_2), u_1''(x_2)).\end{aligned}$$

It is clear that

$$(L\overline{u})(x_2) = F(x_2, u_1(x_2), u_1''(x_2)).$$

Moreover, it is easy to see by induction that

$$(L\overline{u})(x_j) = F(x_j, u_{j-1}(x_j), u_{j-1}''(x_j)), j = 1, 2, \ldots. \tag{3.14}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$ for $\forall\, Y \in [0, 1]$, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that

$$x_{n_j} \to Y \quad \text{as } j \to \infty.$$

From (3.14), it is easy to see that $(L\overline{u})(x_{n_j}) = F(x_{n_j}, u_{n_j-1}(x_{n_j}), u_{n_j-1}''(x_{n_j}))$. Let $j \to \infty$, by Lemma 3.2 and the continuity of $F(x, u(x), u''(x))$, we have

$$(L\overline{u})(Y) = F(Y, \overline{u}(Y), \overline{u}''(Y)). \tag{3.15}$$

From (3.15), it follows that $\overline{u}(x)$ satisfies the equation (1.3).

Since $\overline{\psi}_i(x) \in W_2^3[0, 1]$. Clearly, $\overline{u}(Y)$ satisfies the boundary conditions of the equation (1.3).

That is, $\overline{u}(x)$ is the solution of the equation (1.3). The application of the uniqueness of solution to the equation (1.3) then yields that

$$u(x) = \sum_{i=1}^{\infty} B_i\overline{\psi}_i(x). \tag{3.16}$$

The proof is complete. □

4. Numerical example

In this section, some numerical examples are studied to demonstrate the accuracy of the present method. The examples are computed using Mathematica 4.2. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Example 1. Consider the singular equation

$$\begin{cases} u^2(x)u''(x) + \dfrac{20u'(x)}{x^3(1-x)^{2.5}(x-0.4)^2} + \dfrac{\sin(u(x))}{x\sqrt{1-x}} = f(x), \quad 0 < x < 1, \\ u(0) = 0, \ u(1) = 0, \end{cases}$$

where $f(x) = \frac{20-40\,x-2\,(1-x)^{2.5}\,(-1+x)^2\,(-0.4+x)^2\,x^5+(-1+x)^4\,(-0.4+x)^2\,x^4\,\sin(x-x^2)}{(1-x)^{\frac{5}{2}}\,(-0.4+x)^2\,x^3}$.

The true solution is $x - x^2$. Using our method, we choose 26 points on $[0,1]$. The numerical results are given in the following table 1.

Example 2. Consider the singular equation

$$\begin{cases} \sqrt{u(x)}u''(x) + \frac{30u'(x)}{\sin(x)^3(1-x)^{2.2}(x-0.4)^2(x-0.6)} + \frac{\sqrt{u(x)}}{1-x} = f(x), \quad 0 < x < 1, \\ u(0) = 0, \ u(1) = 0, \end{cases}$$

where

$$\begin{aligned} f(x) = \{&\csc(x)^3 \Big(30\,\pi\,\cos(\pi\,x) + (1-x)^{1.2}\,(-0.6+x)\,(-0.4+x)^2\sin(x)^3 \\ &\times \sin(\pi\,x) - \pi^2\,(1-x)^{2.2}\,(-0.6+x)\,(-0.4+x)^2\sin(x)^3\sin(\pi\,x)^{1.5}\Big)\} \\ &/\{(1-x)^{2.2}\,(-0.6+x)\,(-0.4+x)^2\}. \end{aligned}$$

The true solution is $\sin(\pi x)$. Using our method, we choose 26 points on $[0,1]$. The numerical results are given in the following table 2.

Example 3. Consider the singular equation

$$\begin{cases} u(x)u''(x) + \dfrac{u'(x)}{x^2(1-x)^3} + \dfrac{u^2(x)}{1-x} = f(x), 0 < x < 1, \\ u(0) = 0, \ u(1) = 0, \end{cases}$$

where

$$\begin{aligned} f(x) = &\frac{-1.1752 + \cosh(x) + (1-x)^3\,x^2\,\sinh(x)\,(-1.1752x + \sinh(x))}{(1-x)^3\,x^2} \\ &+ \frac{(1-x)^2\,x^2\,(-1.1752\,x + \sinh(x))^3}{(1-x)^3\,x^2}. \end{aligned}$$

The true solution is $\sinh(x) - \sinh(1)x$. Using our method, we choose 26 points on $[0,1]$. The numerical results are given in the following table 3.

TABLE 1. Numerical results for example 1($n = 26$).

x	True solution $u(x)$	Approximate solution u_{26}	Absolute error
0.001	9.990E-4	9.9899E-4	5.0E-11
0.08	0.073600	0.0736000	3.1E-08
0.16	0.134400	0.1344000	4.8E-08
0.32	0.217600	0.2176000	6.7E-08
0.48	0.249600	0.2496000	8.1E-08
0.64	0.230400	0.2304000	5.9E-08
0.80	0.160000	0.1600000	2.6E-08
0.96	0.038400	0.0384000	1.3E-08
1.00	0	0	0

TABLE 2. Numerical results for example 2($n = 26$).

x	True solution $u(x)$	Approximate solution u_{26}	Absolute error
0.001	0.0031415	0.0031414	1.7E-07
0.08	0.2486900	0.2486450	4.4E-05
0.16	0.4817540	0.4817060	4.7E-05
0.32	0.8443280	0.8442490	4.8E-05
0.48	0.9980270	0.9979780	4.8E-05
0.64	0.9048270	0.9047790	4.8E-05
0.80	0.5877850	0.5877370	4.8E-05
0.96	0.1253330	0.1252720	6.0E-05
1.00	0	0	0

TABLE 3. Numerical results for example 3($n = 26$).

x	True solution $u(x)$	Approximate solution u_{26}	Absolute error
0.001	-1.7520E-4	-1.7519E-4	1.0E-08
0.08	-0.0139307	-0.0139391	9.4E-06
0.16	-0.0273486	-0.0273614	1.2E-05
0.32	-0.0505750	-0.0505831	8.0E-06
0.48	-0.0654511	-0.0654407	1.0E-05
0.64	-0.0675345	-0.0675128	2.1E-05
0.80	-0.0520550	-0.0520382	1.6E-05
0.96	-0.0137914	-0.0137858	5.5E-06
1.00	0	0	0

5. Appendix

Appendix A. The coefficients of the reproducing kernel $\mathbf{R_x(y)}$

$$\begin{aligned}\Delta_1 = \; & 48(-1+e)e^{3x}(57121 + 171363e + 287970e^2 + 409502e^3 + 283644e^4 \\ & +283644e^5 + 409502e^6 + 287970e^7 + 171363e^8 + 57121e^9)\end{aligned}$$

$$\Delta_2 = 60\,(-1+e)\,e^{3x}\,(57121 + 114242\,e + 173728\,e^2 + 235774\,e^3 + 47870\,e^4 + 235774\,e^5 + 173728\,e^6 + 114242\,e^7 + 57121\,e^8)$$

$$\Delta_3 = 5\Delta_1$$

$$c_1 = \frac{1}{\Delta_1}(-6318\,e^4 - 19548\,e^5 - 19548\,e^6 - 19548\,e^7 - 6318\,e^8 - 55926\,e^{4x} - 7648\,e^{5x} + 6453\,e^{6x} + 7488\,e^{3+x} + 30816\,e^{4+x} + 30816\,e^{5+x} + 57121\,e^{2(5+x)} + 30816\,e^{6+x} + 30816\,e^{7+x} + 7488\,e^{8+x} + 54756\,e^{2+2x} + 108232\,e^{3+2x} + 165353\,e^{4+2x} + 222474\,e^{5+2x} + 39495\,e^{6+2x} + 229764\,e^{7+2x} + 171363\,e^{8+2x} + 114242\,e^{9+2x} - 111852\,e^{1+4x} - 167778\,e^{2+4x} - 223704\,e^{3+4x} - 37080\,e^{4+4x} - 223704\,e^{5+4x} - 167778\,e^{6+4x} - 111852\,e^{7+4x} - 55926\,e^{8+4x} - 15296\,e^{1+5x} - 22944\,e^{2+5x} - 46272\,e^{3+5x} - 46272\,e^{4+5x} - 22944\,e^{5+5x} - 15296\,e^{6+5x} - 7648\,e^{7+5x} + 12906\,e^{1+6x} + 19359\,e^{2+6x} + 32724\,e^{3+6x} + 19359\,e^{4+6x} + 12906\,e^{5+6x} + 6453\,e^{6+6x})$$

$$c_2 = \frac{1}{\Delta_1}(6453\,e^4 + 12906\,e^5 + 19359\,e^6 + 32724\,e^7 + 19359\,e^8 + 12906\,e^9 + 6453\,e^{10} + 57121\,e^{4x} - 7648\,e^{3+x} - 15296\,e^{4+x} - 22944\,e^{5+x} - 55926\,e^{2(5+x)} - 46272\,e^{6+x} - 46272\,e^{7+x} - 22944\,e^{8+x} - 15296\,e^{9+x} - 7648\,e^{10+x} - 55926\,e^{2+2x} - 111852\,e^{3+2x} - 167778\,e^{4+2x} - 223704\,e^{5+2x} - 37080\,e^{6+2x} - 223704\,e^{7+2x} - 167778\,e^{8+2x} - 111852\,e^{9+2x} + 114242\,e^{1+4x} + 171363\,e^{2+4x} + 229764\,e^{3+4x} + 39495\,e^{4+4x} + 222474\,e^{5+4x} + 165353\,e^{6+4x} + 108232\,e^{7+4x} + 54756\,e^{8+4x} + 7488\,e^{2+5x} + 30816\,e^{3+5x} + 30816\,e^{4+5x} + 30816\,e^{5+5x} + 30816\,e^{6+5x} + 7488\,e^{7+5x} - 6318e^{2+6x} - 19548\,e^{3+6x} - 19548\,e^{4+6x} - 19548\,e^{5+6x} - 6318\,e^{6+6x})$$

$$c_3 = -\frac{2}{\Delta_1}(1080e^4 - 105840e^5 + 1080e^6 + 9560e^{4x} - 118305e^{5x} + 51624e^{6x} - 1280\,e^{3+x} + 243745\,e^{4+x} + 55841\,e^{5+x} + 60766\,e^{6+x} + 59486\,e^{7+x} + 57121\,e^{8+x} + 57121\,e^{9+x} - 9360\,e^{2+2x} - 29160\,e^{3+2x} - 9360\,e^{4+2x} - 29160\,e^{5+2x} - 9360\,e^{6+2x} + 9560\,e^{1+4x} + 19120e^{2+4x} + 38720e^{3+4x} + 19120\,e^{4+4x} + 9560\,e^{5+4x} + 9560\,e^{6+4x} - 118305\,e^{1+5x} - 121950\,e^{2+5x} - 121950\,e^{3+5x} - 118305\,e^{4+5x} - 118305\,e^{5+5x} + 51624\,e^{1+6x} + 52704\,e^{2+6x} + 51624\,e^{3+6x} + 51624\,e^{4+6x})$$

$$c_4 = -\frac{1}{\Delta_2}(51624e^5 + 51624e^6 + 52704e^7 + 51624e^8 + 51624e^9 + 57121e^{5x} - 118305e^{4+x} - 118305e^{5+x} - 121950e^{6+x} - 121950e^{7+x} - 118305e^{8+x} - 118305e^{9+x} + 9560e^{3+2x} + 9560e^{4+2x} + 19120e^{5+2x} + 38720e^{6+2x} + 19120e^{7+2x} + 9560e^{8+2x} + 9560e^{9+2x} - 9360e^{3+4x} - 29160e^{4+4x} - 9360e^{5+4x} - 29160e^{6+4x} - 9360e^{7+4x} + 57121e^{1+5x} + 59486e^{2+5x} + 60766e^{3+5x} + 55841e^{4+5x} + 243745e^{5+5x} - 1280e^{6+5x} + 1080e^{3+6x} - 105840e^{4+6x} + 1080e^{5+6x})$$

$$c_5 = \frac{1}{\Delta_3}e^{-3x}(3645e^4 - 179334e^5 - 122213e^6 + 121532e^7 + 116607e^8 + 114242e^9 + 57121e^{10} + 32265e^{4x} - 206496e^{5x} + 117110e^{6x} - 4320e^{3+x} + 419040e^{5+x} - 4320e^{6+x} - 31590e^{2+2x} - 97740e^{3+2x} - 97740e^{4+2x} - 97740e^{5+2x} - 31590e^{6+2x} + 64530e^{1+4x} + 96795e^{2+4x} + 163620e^{3+4x} + 96795e^{4+4x} + 64530e^{5+4x} + 32265e^{6+4x} - 412992e^{1+5x} - 417312e^{2+5x} - 417312e^{3+5x} - 412992e^{4+5x}$$

$-206496e^{5+5x}+234220e^{1+6x}+235500e^{2+6x}+234220e^{3+6x}+117110e^{4+6x})$

$$
\begin{aligned}
c_6 = {} & \tfrac{1}{\Delta_3}e^{-3x}(117110e^6+234220e^7+235550e^8+234220e^9+117110e^{10}\\
&+57121e^{6x}-206496e^{5+x}+32265e^{10+2x}-412992e^{6+x}-417312e^{8+x}\\
&-412992e^{9+x}-206496e^{10+x}+32265e^{4+2x}+64530e^{5+2x}+96795e^{6+2x}\\
&+163620e^{7+2x}+96795e^{8+2x}+64530e^{9+2x}-31590e^{4+4x}-97740e^{7+4x}\\
&-31590e^{8+4x}-4320e^{4+5x}+419040e^{5+5x}+419040e^{6+5x}-4320e^{7+5x}\\
&+114242e^{1+6x}+116607e^{2+6x}+121532e^{3+6x}-122213e^{4+6x}\\
&-179334e^{5+6x}+3645e^{6+6x})\\
d_1 = {} & \tfrac{1}{\Delta_1}(-6318e^4-19548e^5-19548e^6-19548e^7-6318e^8+57121e^{2x}\\
&-55926e^{4x}-7648e^{5x}+6453e^{6x}+7488e^{3+x}+30816e^{4+x}\\
&+30816e^{5+x}+30816e^{6+x}+30816e^{7+x}+7488e^{8+x}+114242e^{1+2x}\\
&+171363e^{2+2x}+229764e^{3+2x}+39495e^{4+2x}+222474e^{5+2x}\\
&+165353e^{6+2x}+108232e^{7+2x}+54756e^{8+2x}-111852e^{1+4x}\\
&-167778e^{2+4x}-223704e^{3+4x}-37080e^{4+4x}-223704e^{5+4x}\\
&-167778e^{6+4x}-111852e^{7+4x}-55926e^{8+4x}-15296e^{1+5x}\\
&-22944e^{2+5x}-46272e^{3+5x}-46272e^{4+5x}-22944e^{5+5x}\\
&-15296e^{6+5x}-7648e^{7+5x}+12906e^{1+6x}+19359e^{2+6x}\\
&+32724e^{3+6x}+19359e^{4+6x}+12906e^{5+6x}+6453e^{6+6x})\\
d_2 = {} & \tfrac{1}{\Delta_1}e^{2-3x}(6453e^2+12906e^3+19359e^4+32724e^5+19359e^6\\
&+12906e^7+6453e^8-55926e^{2x}+54756e^{4x}+7488e^{5x}-6318e^{6x}\\
&-7648e^{1+x}-15296e^{2+x}-22944e^{3+x}-46272e^{4+x}-46272e^{5+x}\\
&-22944e^{6+x}-15296e^{7+x}-7648e^{8+x}-111852e^{1+2x}-167778e^{2+2x}\\
&-223704e^{3+2x}-37080e^{4+2x}-223704e^{5+2x}-167778e^{6+2x}\\
&-111852e^{7+2x}-55926e^{8+2x}+108232e^{1+4x}+165353e^{2+4x}\\
&+222474e^{3+4x}+39495e^{4+4x}+229764e^{5+4x}+171363e^{6+4x}\\
&+114242e^{7+4x}+57121e^{8+4x}+30816e^{1+5x}+30816e^{2+5x}\\
&+30816e^{3+5x}+30816e^{4+5x}+7488e^{5+5x}-19548e^{1+6x}\\
&-19548e^{2+6x}-19548e^{3+6x}-6318e^{4+6x})\\
d_3 = {} & -\tfrac{1}{\Delta_2}(1080e^4-105840e^5+1080e^6+57121e^x+9560e^{4x}-118305e^{5x}\\
&+51624e^{6x}+57121e^{1+x}+59486e^{2+x}+60766e^{3+x}+55841e^{4+x}\\
&+243745e^{5+x}-1280e^{6+x}-9360e^{2+2x}-29160e^{3+2x}-9360e^{4+2x}\\
&-29160e^{5+2x}-9360e^{6+2x}+9560e^{1+4x}+19120e^{2+4x}\\
&+38720e^{3+4x}+19120e^{4+4x}+9560e^{5+4x}+9560e^{6+4x}\\
&-118305e^{1+5x}-121950e^{2+5x}-121950e^{3+5x}-118305e^{4+5x}\\
&-118305e^{5+5x}+51624e^{1+6x}+52704e^{2+6x}+51624e^{3+6x}\\
&+51624e^{4+6x})\\
d_4 = {} & -\tfrac{1}{\Delta_2}e^{3-3x}(51624e^2+51624e^3+52704e^4+51624e^5+51624e^6\\
&+9560e^{2x}-9360e^{4x}-1280e^{5x}+1080e^{6x}-118305e^{1+x}\\
&-118305e^{2+x}-121950e^{3+x}-121950e^{4+x}-118305e^{5+x}\\
&-118305e^{6+x}+9560e^{1+2x}+19120e^{2+2x}+38720e^{3+2x}\\
&+19120e^{4+2x}+9560e^{5+2x}+9560e^{6+2x}-29160e^{1+4x}\\
&-9360e^{2+4x}-29160e^{3+4x}-9360e^{4+4x}+243745e^{1+5x}\\
&+55841e^{2+5x}+60766e^{3+5x}+59486e^{4+5x}+57121e^{5+5x}
\end{aligned}
$$

$$+57121\,e^{6+5\,x} - 105840\,e^{1+6\,x} + 1080\,e^{2+6\,x})$$

$$\begin{aligned} d_5 = \; & \tfrac{1}{\Delta_3}e^{-3x}(57121 + 114242e + 116607e^2 + 121532e^3 - 122213e^4 \\ & -179334e^5 + 3645e^6 + 32265e^{4x} - 206496e^{5x} + 117110e^{6x} \\ & -4320e^{3+x} + 419040e^{4+x} + 419040e^{5+x} - 4320e^{6+x} - 31590e^{2+2x} \\ & -97740e^{3+2x} - 97740e^{4+2x} - 97740e^{5+4x} - 31590e^{6+2x} \\ & +64530e^{1+4x} + 96795e^{2+4x} + 163620e^{3+4x} + 96795e^{4+4x} \\ & +64530e^{5+4x} + 32265e^{6+4x} - 412992e^{1+5x} - 417312e^{2+5x} \\ & -417312e^{3+5x} - 412992e^{4+5x} - 206496e^{5+5x} + 234220e^{1+6x} \\ & +235550e^{2+6x} + 234220e^{3+6x} + 117110e^{4+6x}) \end{aligned}$$

$$\begin{aligned} d_6 = \; & \tfrac{1}{\Delta_3}e^{4-3x}(117110e^2 + 234220e^3 + 235500e^4 + 234220e^5 + 117110e^6 \\ & +32265e^{2x} - 31590e^{4x} - 4320e^{5x} + 3645e^{6x} - 206496e^{1+x} \\ & -412992e^{2+x} - 417312e^{3+x} - 417312e^{4+x} - 412992e^{5+x} \\ & -206496e^{6+x} + 64530e^{1+2x} + 96795e^{2+2x} + 163620e^{3+2x} \\ & +96795e^{4+2x} + 64530e^{5+2x} + 32265e^{6+2x} - 97740e^{1+4x} \\ & -97740e^{2+4x} - 97740e^{3+4x} - 31590e^{4+4x} + 419040e^{1+5x} \\ & +419040e^{2+5x} - 4320e^{3+5x} - 179334e^{1+6x} - 122213e^{2+6x} \\ & +121532e^{3+6x} + 116607e^{4+6x} + 114242e^{5+6x} + 57121e^{6+6x}) \end{aligned}$$

Appendix B. The proof of Theorem 2.1

Through several integrations by parts for (2.1), then

(B.1)
$$\begin{aligned} & (u(y), R_x(y))_{W_2^3} \\ = \; & \int_0^1 u(y)(36R_x(y) - 49R_x^{(2)}(y) + 14R_x^{(4)}(y) - R_x^{(6)}(y))dy + u(y)(49R'_x(y) \\ & -14R_x^{(3)}(y) + R_x^{(5)}(y))|_0^1 + u'(y)(14R_x^{(2)}(y) - R_x^{(4)}(y))|_0^1 + u''(y)R_x^{(3)}(y)|_0^1. \end{aligned}$$

Since $R_x(y) \in W_2^3[0,1]$, it follows that

(B.2) $$R_x(0) = 0, R_x(1) = 0.$$

Since $u \in W_2^3[0,1]$, $u(0) = u(1) = 0$. If

(B.3)
$$14R_x^{(2)}(0) - R_x^{(4)}(0) = 0, 14R_x^{(2)}(1) - R_x^{(4)}(1) = 0, R_x^{(3)}(0) = 0, R_x^{(3)}(1) = 0,$$

then $(B.1)$ implies that

$$(u(y), R_x(y))_{W_2^3} = \int_0^1 u(y)(36R_x(y) - 49R_x^{(2)}(y) + 14R_x^{(4)}(y) - R_x^{(6)}(y))dy.$$

For $\forall x \in [0,1]$, if $R_x(y)$ also satisfies

(B.4) $$36R_x(y) - 49R_x^{(2)}(y) + 14R_x^{(4)}(y) - R_x^{(6)}(y) = \delta(y-x),$$

then

$$(u(y), R_x(y))_{W_2^3} = u(x).$$

Characteristic equation of (B.4) is given by

$$\lambda^6 - 14\lambda^4 + 49\lambda^2 - 36 = 0,$$

then we can obtain characteristic values $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2, \lambda_4 = -2, \lambda_5 = 3$, and $\lambda_6 = -3$. So, let

$$R_x(y) = \begin{cases} c_1 e^y + c_2 e^{-y} + c_3 e^{2y} + c_4 e^{-2y} + c_5 e^{3y} + c_6 e^{-3y}, & y \le x, \\ d_1 e^y + d_2 e^{-y} + d_3 e^{2y} + d_4 e^{-2y} + d_5 e^{3y} + d_6 e^{-3y}, & y > x. \end{cases}$$

On the other hand, for (B.4), let $R_x(y)$ satisfy

$$\text{(B.5)} \qquad R_x^{(k)}(x+0) = R_x^{(k)}(x-0), \quad k = 0, 1, 2, 3, 4.$$

Integrating (B.4) from $x - \varepsilon$ to $x + \varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $R_x^{(5)}(y)$ at $y = x$

$$\text{(B.6)} \qquad R_x^{(5)}(x-0) - R_x^{(5)}(x+0) = 1.$$

From (B.2), (B.3), (B.5), (B.6), the unknown coefficients of (2.2) can be obtained.

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