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CORE STABILITY OF DOMINATING SET GAMES

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Abstract. In this paper, we study the core stability of the dominating set game which has arisen from the cost allocation problem related to domination problem on graphs. Let G be a graph whose neighborhood matrix is balanced. Applying duality theory of linear programming and graph theory, we prove that the dominating set game corresponding to G has the stable core if and only if every vertex belongs to a maximum 2-packing in G. We also show that for dominating set games corresponding to G, the core is stable if it is large, the game is extendable, or the game is exact. In fact, the core being large, the game being extendable and the game being exact are shown to be equivalent.

1. Introduction

It is one of the scopes of cooperative game theory to study how to distribute the total revenue or cost among the participants in a fair way when they work in cooperation. Von Neumann and Morgenstern [14] claimed that stable sets are very useful in the analysis of a lot of bargaining situations. However, it does not seem to be easy to drive basic properties of the stable set because some transferable utilities games do not have even the stable set. Deng and Papadimitriou [3] pointed out that determining the existence of the stable set for a given cooperative game is not known to be computable, and it is still unsolved. For this reason, the stable sets for some specific games have been studied.

While the core and the stable set are different, Shapley [9] has proved that for convex games, the core is the unique stable set. His result motivated us to study when the core and the stable set coincide, that is, when the core is stable. As far as the core stability for concrete cooperative game model is concerned, only a few results have been obtained. Solymosi and Raghavan [11] studied the core stability for assignment games, and Bietenhader and Okamoto [1] studied core stability for minimum coloring games defined on perfect graphs.

The relaxed dominating set games was first studied in Velzen [13] and Kim and Fang [7]. In this paper we focus the core stability on the relaxed dominating

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set games. Especially, we identify those dominating set games whose cores are stable. Let G be a graph whose neighborhood matrix is balanced. We prove that the dominating set game corresponding to G has the stable core if and only if every vertex belongs to a maximum 2-packing of the graph. As a corollary, it follows that checking whether or not the dominating set game corresponding to G has the stable core is polynomially solvable. Furthermore, we discuss about for dominating set games corresponding to G, the core is stable if it is large, the game is extendable, or the game is exact. In fact, the core being large, the game being extendable and the game being exact are shown to be equivalent.

2. Definitions and preliminaries

In this section, we introduce the definition of dominating set game and some known results. We begin with some concepts and notions in cooperative game theory.

2.1. Cooperative game and core stability

A transferable cost cooperative game $\Gamma = (N, c)$ consists of a player set $N = \{1, 2, \ldots, n\}$ and a characteristic function $c : 2^N \to R$. The function c specifies the cost of every coalition $S \subseteq N$. The main problems on cooperative games are how to distribute the total cost $c(N)$ among all the players in a fair way. A vector $x = (x_1, x_2, \ldots, x_n)$ is called an *imputation* if $\sum_{i\in N} x_i = c(N)$ and $\forall\ i \in N$: $x_i \leq c(\{i\})$ (*individual rationality*). We denote $\mathcal{I}(\Gamma)$ the set of imputations of Γ. The *core* of Γ is defined as:

$$\mathcal{C}(\Gamma) = \{x \in R^n : x(N) = c(N) \text{ and } x(S) \leq c(S),\ \forall S \subseteq N\},$$

where $x(S) = \sum_{i\in S} x_i$. The second constraint $x(S) \leq c(S)$ for a vector belonging to $\mathcal{C}(\Gamma)$ is called *coalitional rationality.*

The game Γ is called *balanced* if $\mathcal{C}(\Gamma)$ is not empty and *totally balanced* if every subgame (i.e., the game obtained by restricting the player set to a coalition and the characteristic function to the power set of that coalition) is balanced.

The concept of stability is introduced by von Neumann and Morgenstern [14]. Let $\Gamma = (N, c)$ be a cost game, and $x, y \in \mathcal{I}(\Gamma)$. We say that x *dominates* y if there is a coalition S such that $x(S) \geq c(S)$ and for each $i \in S$, $x_i < y_i$. For convenience, we say that x *dominates* y on S. A set $\mathcal{F} \subseteq \mathcal{I}(\Gamma)$ is a *stable set* if any two imputations in $\mathcal{F}$ do not dominate each other and any imputation not in $\mathcal{F}$ is dominated by an imputation in $\mathcal{F}$. Since the imputations in the core do not dominate each other, the core being stable means that any imputation not in the core is dominated by a core element. It can be formally stated as follows: the core of a balanced cost game Γ is *stable* if for every $y \in \mathcal{I}(\Gamma) \setminus \mathcal{C}(\Gamma)$, there exists $x \in \mathcal{C}(\Gamma)$ and a non-empty coalition $S \subset N$ such that $x(S) = c(S)$ and $x_i < y_i$ for each $i \in S$.

2.2. Dominating set game

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. For any non-empty set $V' \subseteq V$, *the subgraph induced by* V', denoted by $G[V']$, is a subgraph of G whose vertex set is V' and edge set is the set of edges having both endpoints in V'. The *closed neighborhood* of a vertex $v \in V$ is denoted by $N[v] = \{u \in V : (u, v) \in E\} \bigcup \{v\}$.

A *dominating set* of a graph G is a set of vertices $D \subseteq V$ such that $N[v] \bigcap D \neq \emptyset$, $\forall v \in V$. Finding a minimum dominating set of a graph G is an important dominating set problem. The cardinality of a minimum dominating set is called the *domination number*, denoted by $\gamma(G)$.

Given a graph $G = (V, E)$, the *dominating set game* (DS game) $\Gamma_G = (V, c)$ corresponding to G is defined as:

1. The player set is $V = \{1, 2, \ldots, n\}$;
2. For each coalition $S \subseteq V$, $c(S) = \min\{\gamma(G[T]) : S \subseteq T \subseteq V\}$.

Velzen [13] independently introduced the notion of a DS game and called it a relaxed dominating games. In fact, DS games belongs to the class of combinatorial optimization games studied in Deng *et al.* [2]. With the technique of integer program and duality theory of linear programming, Deng *et al.* [2] and Velzen [13] presented a necessary and sufficient condition for the balancedness of DS games.

Let $G = (V, E)$ be a graph with $|V| = n$. The *neighborhood matrix* of G, denoted by $A(G) = [a_{ij}]$, is a $n \times n$-matrix with rows and columns indexed by the vertices in V respectively, where $a_{ij} = 1$ if vertex $i \in N[j]$, and $a_{ij} = 0$ otherwise. Then the domination number $\gamma(G)$ can be formulated as the following 0-1 program:

$$\text{P}: \quad \gamma(G) = \min\{\sum_{i=1}^{n} x_i : A(G)x \geq 1, x \in \{0,1\}^n\}.$$

Consider the linear program relaxation (LP) and its dual (DP) of (P):

$$\text{LP}: \ \min\{\sum_{i=1}^{n} x_i : A(G)x \geq 1, x \geq 0\};$$

$$\text{DP}: \ \max\{\sum_{i=1}^{n} y_i : y^t A(G) \leq 1, y \geq 0\}.$$

Theorem 2.1 ([2, 13]). *Let $\Gamma_G = (V, c)$ be the DS game corresponding to a graph $G = (V, E)$. Then $\mathcal{C}(\Gamma_G) \neq \emptyset$ if and only if the linear program relaxation (LP) has an integer optimal solution. In such case, $x = (x_1, x_2, \ldots, x_n) \in \mathcal{C}(\Gamma_G)$ if and only if it is an optimal solution to (DP).*

As the problem of determining $\gamma(G)$ is NP-hard for general graphs, Theorem 2.1 implies that it is difficult to determine whether or not the DS game Γ_G corresponding to G is balanced. However, for some class of graphs, the domination problem is relatively easy to solve, and we focus on one of such graphs classes: A $\{0,1\}$-matrix M is called *balanced* if M has no square submatrix of odd order with exactly two 1's in each row and in each column. It was shown by Fulkerson et al. [5] that if M is balanced, then both polyhedra $\{x \in R^n : Mx \leq 1, x \geq 0\}$ and $\{x \in R^n : Mx \geq 1, x \geq 0\}$ are integral. It follows that if the neighborhood matrix of G is balanced, then both sets of feasible solutions of (LP) and (DP) are integral. There are several classes of graphs whose neighborhood matrices are balanced such as trees, interval graphs and block graphs [4, 8]. That is to say, the DS games defined on this class of graphs are balanced. Moreover, since every submatrix of a balanced matrix is balanced, the corresponding DS games are also totally balanced.

Denoted by $\mathcal{G}_B$ the class of graphs with balanced neighborhood matrices. Throughout the next two sections, we restrict our attention to a DS game corresponding to G in $\mathcal{G}_B$.

3. Dominating set games with stable core

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is called a *2-packing* of G if for every pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$. In other words, $S \subseteq V$ is a 2-packing if for every $v \in V$, $|N[v] \cap S| \leq 1$. For $v \in V$, *v belongs to a 2-packing of G* means that there is some 2-packing S of G such that $v \in S$.

A 2-packing is called *maximal* if adding any other vertex to the set makes it no longer a 2-packing and called *maximum* if it has the maximum cardinality among all the 2-packings. The cardinality of a maximum 2-packing of G is called the *2-packing number* of G, denoted by $p_2(G)$. In terms of an integer program (P_2), it can be formulated as follows:

$$\mathrm{P}_2: \quad p_2(G) = \max\{\sum_{i=1}^{n} y_i : y^t A(G) \leq 1, y \in \{0,1\}^n\},$$

where $A(G)$ is the neighborhood matrix of G. Obviously, (DP) is the linear program relaxation of (P_2), and by duality theorem,

$$p_2(G) \leq \mathrm{opt(DP)} = \mathrm{opt(LP)} \leq \gamma(G).$$

When $G \in \mathcal{G}_B$, equalities hold both inequalities above.

Before we discuss the core stability of DS games, we prove the following lemmas. Here the notion of indicator vector is utilized. Given a subset $S \subseteq V$, the *indicator vector* of S is denoted by $\mathbf{e}_S \in \{0,1\}^{|V|}$ with a component being 1 if and only if the corresponding vertex belongs to S.

Lemma 3.1. *Let $\Gamma_G = (V, c)$ be the* DS *game corresponding to a graph $G = (V, E) \in \mathcal{G}_B$. Then $\mathcal{C}(\Gamma_G)$ is the convex hull of the indicator vectors of the maximum 2-packings of G.*

Proof. Since the neighborhood matrix $A(G)$ is balanced for $G \in \mathcal{G}_B$, the set of optimal solutions of (DP) is the convex hull of integral optimal solutions, i.e., the optimal solutions of (P_2) as shown in [5]. This together with Theorem 2.1 and the fact that the optimal solutions of (P_2) are the indicator vectors of the maximum 2-packings of G imply that $\mathcal{C}(\Gamma_G)$ is the convex hull of the indicator vectors of the maximum 2-packings. □

Lemma 3.2. *Let $\Gamma_G = (V, c)$ be the* DS *game corresponding to a graph $G = (V, E) \in \mathcal{G}_B$. If Γ_G has the stable core, then for every $i \in V$ there exists a core element x such that $x_i > 0$.*

Proof. Assume that there exists a vertex $k \in V$ such that $x_k = 0$ for all $x \in \mathcal{C}(\Gamma_G)$. Since $c(V) > 0$, there exist $l \in V$ and $\hat{x} \in \mathcal{C}(\Gamma_G)$ such that $\hat{x}_l > 0$. Construct a vector $y \in R^n$ based on $\hat{x}$ as follows:

$$y_i = \begin{cases} \hat{x}_i & \text{if } i \notin \{k, l\} \\ \hat{x}_l & \text{if } i = k \\ 0 & \text{if } i = l. \end{cases}$$

By the assumption on $\mathcal{I}(\Gamma_G)$, $y \in \mathcal{I}(\Gamma_G) \setminus \mathcal{C}(\Gamma_G)$. Since $\mathcal{C}(\Gamma_G)$ is stable, there exist $x^* \in \mathcal{C}(\Gamma_G)$ and a non-empty set $T \subseteq V$ such that x^* dominates y on T. Note that it must be $l \notin T$. In fact, suppose that $l \in T$. Then $x_l^* < y_l$. Since $y_l = 0$ by definition of y and $0 \le x_l^*$ by definition of dominating games, it follows that $0 \le x_l^* < y_l = 0$, which is a contradiction.

Moreover, it must be $k \in T$. To see why, suppose $k \notin T$. Since $l \notin T$. Since $\hat{x} \in \mathcal{C}(\Gamma_G)$ and $y_i = \hat{x}_i$ for all $i \in T$, x^* can not dominate y on T either.

Therefore, we have $c(T) \le x^*(T) = x^*(T \setminus \{k\}) < y(T \setminus \{k\}) = \hat{x}(T \setminus \{k\}) \le c(T \setminus \{k\}) \le c(T)$, which is a contradiction. □

Now we present the main result of this section.

Theorem 3.3. *Let $G = (V, E) \in \mathcal{G}_B$. The* DS *game $\Gamma_G = (V, c)$ corresponding to G has the stable core if and only if every vertex $i \in V$ belongs to a maximum 2-packing of G.*

Proof. We first prove the 'only if' part. Assume that $\Gamma_G = (V, c)$ has the stable core. By Lemma 3.2, for every $i \in V$, there exists a core element x such that $x_i > 0$. In addition, by Lemma 3.1, x is a convex combination of the indicator vectors of some maximum 2-packings. Hence, for each $i \in V$, i belongs to at least one maximum 2-packing of G.

Conversely, suppose that every vertex $i \in V$ belongs to a maximum 2-packing of G. Given $y \in \mathcal{I}(\Gamma_G) \setminus \mathcal{C}(\Gamma_G)$. Since $y \in \mathcal{I}(\Gamma_G)$, according to the definition of $\mathcal{I}(\Gamma_G)$ we have $\sum_{i=1}^{n} y_i = c(V)$, where $c(V)$ is the optimal value of (DP).

If y is a feasible solution to (DP), then, by Theorem 2.1, $y \notin C(\Gamma_G)$, which is a contradiction. Thus, y is not a feasible solution to (DP). Hence, there exists a vertex $i_0 \in V$ such that $y(N[i_0]) > 1$. Let $S = N[i_0] \setminus \{i \in N[i_0] : y_i \le 0\} = \{i_1, i_2, \ldots, i_k\}$. Since a neighbor of i_0 belongs to S that $S \neq \emptyset$, $y(S) > 1$, and $y_{i_j} > 0$ $(j = 1, 2, \ldots, k)$. By our assumption, each vertex i_j belongs to a maximum 2-packing of G, namely P_j $(j = 1, 2, \ldots, k)$. Since S is a subset of $N[i_0]$, it holds that each vertex $i_j \in S$ is contained in the unique 2-packing P_j in the set $\{P_1, P_2, \ldots, P_k\}$.

Let

$$z = \lambda_1 \mathbf{e}_{P_1} + \lambda_2 \mathbf{e}_{P_2} + \cdots + \lambda_k \mathbf{e}_{P_k} \in R^n,$$

where $\lambda_j = \frac{y_{i_j}}{y(S)}$ and $\mathbf{e}_{P_j}$is the indicator vector of P_j $(j = 1, 2, \ldots, k)$. Obviously, $\lambda_j > 0$ and $\sum_{j=1}^{k} \lambda_j = 1$. For convenience, we denote by $\mathbf{e}_{P_j}(S) = \sum_{i \in S} e_{j_i}$, where $\mathbf{e}_{P_j} = (e_{j_1}, e_{j_2}, \ldots, e_{j_n})$. By Lemma 3.1, we conclude that $z \in \mathcal{C}(\Gamma_G)$, and

$$z(S) = \sum_{j=1}^{k} \frac{y_{i_j}}{y(S)} \mathbf{e}_{P_j}(S) = \sum_{j=1}^{k} \frac{y_{i_j}}{y(S)} = \frac{y(S)}{y(S)} = 1,$$
$$z_{i_j} = \lambda_j = \frac{y_{i_j}}{y(S)} < y_{i_j}, \quad \forall j = 1, 2, \ldots, k.$$

It follows that z dominates y on S. Therefore, $\mathcal{C}(\Gamma_G)$ is stable. □

In the rest of this section, we consider the algorithm for checking whether or not a DS game has the stable core. The problem is stated as:

Problem A: Checking core stability of DS game corresponding to a graph G in $\mathcal{G}_B$
Instance: A DS game Γ_G corresponding to a graph $G \in \mathcal{G}_B$
Question: Does Γ_G possess the stable core?

Followed from Theorem 3.3, the above problem is equivalent to checking whether or not every vertex belongs to a maximum 2-packing of G. For each $i \in V$, we define a weight function $\beta^i : V \to Z^+$ such that $\beta^i(i) = M$ and $\beta^i(j) = 1$ for $j \in V \setminus \{i\}$, where M is a sufficiently large integer. Consider the following integer program:

$$\mathrm{P}_2(\beta^i): \quad p_2(G, \beta^i) = \max\{\sum_{j=1}^{n} \beta^i_j y_j : y^t A(G) \leq 1, y \in \{0,1\}^n\}.$$

If there exists a vertex $k \in V$ such that $p_2(G, \beta^k) - p_2(G) < M - 1$, then it is easy to conclude that k is not contained in a maximum 2-packing of G, meaning that $\mathcal{C}(\Gamma_G)$ is not stable; otherwise, $\mathcal{C}(\Gamma_G)$ is stable.

Since $G \in \mathcal{G}_B$, $p_2(G)$ and $p_2(G, \beta^i)$ can be obtained by solving the linear program relaxations of (P_2) and $\mathrm{P}_2(\beta^i)$ $(\forall i \in V)$, respectively [5]. Hence, we have

Theorem 3.4. *The problem of checking core stability of* DS *game corresponding to a graph $G \in \mathcal{G}_B$ can be solved in polynomial time.*

4. Exactness, extendability and core largeness

Let $\Gamma = (N, c)$ be a cost game with $|N| = n$. The game Γ is said to have a *large core* if for every $y \in R^n$ satisfying that $y(S) \leq c(S)$ $(\forall S \subseteq N)$, there exists $x \in \mathcal{C}(\Gamma)$ such that $x \geq y$. The game Γ is called *extendable* if for every nonempty set $S \subset N$ and every core element y of the subgame (S, c_S), there exists $x \in \mathcal{C}(\Gamma)$ such that $x_i = y_i$ for all $i \in S$. The game Γ is called *exact* if for every $S \subset N$ there exists $x \in \mathcal{C}(\Gamma)$ such that $x(S) = c(S)$.

The three concepts are closely related to the core stability. Kikuta and Shapley [6] showed that if a balanced game has the large core, then it is extendable. Moreover, if a balanced game is extendable, then it has the stable core. Sharkey [10] showed that if a totally balanced game has the large core, then it is exact. We summarize these results in the following theorem.

Theorem 4.1 ([6, 10]). *Let $\Gamma = (N, c)$ be a totally balanced game. Then*

$$\text{core largeness} \Rightarrow \text{extendability} \Rightarrow \begin{cases} \text{exactness} \\ \text{core Stability.} \end{cases}$$

In the next theorem, we prove that the core being large, the game being extendable, and the game being exact are shown to be equivalent for DS games corresponding to a graph $G \in \mathcal{G}_B$.

Theorem 4.2. *Let $G = (V, E) \in \mathcal{G}_B$ and $\Gamma_G = (V, c)$ be the* DS *game corresponding to G. Then the following conditions are equivalent:*

(1) *the core $\mathcal{C}(\Gamma_G)$ is large;*
(2) *the game Γ_G is extendable;*
(3) *the game Γ_G is exact;*
(4) *Every 2-packing is contained in a maximum 2-packing in graph G.*

In order to show Theorem 4.2, we need the following lemmas.

For a cost game $\Gamma = (N, c)$ with $|N| = n$, the set of lower vectors is defined as:

$$L(\Gamma) = \{y \in R^n : y(S) \leq c(S),\ \forall S \subseteq N\}.$$

Van Gellekom, et al. [12] showed the following:

Lemma 4.3 ([12]). *Let $\Gamma = (N, c)$ be a balanced cost game. Then $\Gamma = (N, c)$ has a large core if and only if $y(N) \geq c(N)$ for all extreme points y of $L(\Gamma)$.*

Now, we characterize the extreme points of $L(\Gamma_G)$. In order to do that, we need to give an alternative description of $L(\Gamma_G)$.

Let $\mathcal{S} = \bigcup_{i \in V} \{T : T \subseteq N[i]\}$, define

$$L'(\Gamma_G) = \{y \in R^n : y(S) \leq 1,\ \forall\, S \in \mathcal{S}\}.$$

Lemma 4.4. *Let $G = (V, E) \in \mathcal{G}_B$ and $\Gamma_G = (V, c)$ be the* DS *game corresponding to G. Then $L(\Gamma_G) = L'(\Gamma_G)$.*

Proof. Since $c(S) \le 1, \forall s \in S$, $L(\Gamma_G) \subseteq L'(\Gamma_G)$. To show $L'(\Gamma_G) \subseteq L(\Gamma_G)$, take $y \in L'(\Gamma_G)$. We have to check that $y(S) \le c(S)$ for every $S \subseteq V$. Assume that $c(S) = k$. Then we can divide S into k disjoint sets, namely, $S_1, S_2, \ldots, S_k$. Then $S_i \in \mathcal{S}$ $(i = 1, 2, \ldots, k)$. Since $y \in L'(\Gamma_G)$, $y(S_i) \le 1$ for each $i = 1, 2, \ldots, k$. Therefore, $y(S) = \sum_{i=1}^{k} y(S_i) \le k = c(S)$. Thus, $y \in L(\Gamma_G)$ and so $L'(\Gamma_G) \subseteq L(\Gamma_G)$. □

Lemma 4.5. *Let $G = (V, E) \in \mathcal{G}_B$ and $\Gamma_G = (V, c)$ be the* DS *game corresponding to G. Then*

(1) *the extreme points of $L'(\Gamma_G)$ are non-negative;*

(2) *each extreme point of $L'(\Gamma_G)$ is the indicator vector of a maximal 2-packing of G.*

Proof. (1) We prove by contradiction. Suppose that y is an extreme point of $L'(\Gamma_G)$ with at least one negative component i. Now, we define two vectors y^1 and y^2 as follows:

$$y_i^1 = \begin{cases} y_i & \text{if } y_i \ge 0 \\ 0 & \text{if } y_i < 0 \end{cases} \quad \text{and} \quad y_i^2 = \begin{cases} y_i & \text{if } y_i \ge 0 \\ 2y_i & \text{if } y_i < 0. \end{cases}$$

It is easy to see that $y^1, y^2 \in L'(\Gamma_G)$. Since $y \ne y^1, y \ne y^2$ and $y = \frac{y^1+y^2}{2}$, y is not an extreme point of $L'(\Gamma_G)$.

(2) Consider the following polyhedron:

$$L''(\Gamma_G) = \{y \in R^n : y(S) \le 1, \forall S \in \mathcal{S};\ y \ge 0\}.$$

Then $L''(\Gamma_G) \subseteq L'(\Gamma_G)$. Thus, by (1), each extreme point of $L'(\Gamma_G)$ is also an extreme point of $L''(\Gamma_G)$. Thus, it is sufficient to show each extreme point of $L''(\Gamma_G)$ is the indicator vector of a maximal 2-packing of G.

By the definition of $\mathcal{S}$ and $A(G)$, it is easy to verify that

$$\begin{aligned} L''(\Gamma_G) &= \{y \in R^n : y(N[i]) \le 1, \forall i \in V;\ y \ge 0\} \\ &= \{y \in R^n : y^t A(G) \le 1, y \ge 0\}. \end{aligned}$$

That is, $L''(\Gamma_G)$ is exactly the set of feasible solutions to (DP). Let y^* be an extreme point of $L''(\Gamma_G)$. In the theory of linear programming, it is well known that there exists a non-negative function $\omega : V \to Z^+$, such that y^* is the unique optimal solution of the following linear program:

$$\text{LP}^*: \quad \max\{\sum_{j=1}^{n} \omega_j y_j : y^t A(G) \le 1,\ y \ge 0\}.$$

Since $A(G)$ is balanced for $G \in \mathcal{G}_B$, (LP*) has an integer optimal solution (see [5]). Thus, y^* is a $\{0,1\}$-vector which is an indicator vector of some 2-packing P^* of G.

Assume that P^* is not a maximal 2-packing. We will reach a contradition. Then there exists a 2-packing P' such that $P^* \subset P'$. It follows that the indicator vector of P' is also an optimal solution of (LP*), which contradicts to the fact that y^* is the unique optimal solution. Therefore, P^* is a maximal 2-packing. □

Proof of Theorem 4.2. By Theorem 4.1, "(1) $\Rightarrow$ (2) $\Rightarrow$ (3)" is true. It remains to prove "(3) $\Rightarrow$ (4)" and "(4) $\Rightarrow$ (1)".

We show "(3) $\Rightarrow$ (4)". Suppose that Γ_G is exact. Let S be a 2-packing of G. By the definition of the exactness, there exists $x \in \mathcal{C}(\Gamma_G)$ such that $x(S) = c(S) = |S|$. We denote $\mathcal{P}$ the set of maximum 2-packings of G. By from Lemma 3.1, x can be expressed as

$$x = \sum_{P\in\mathcal{P}} \lambda_P \mathbf{e}_P,$$

where $\mathbf{e}_P$ is the indicator vector of the 2-packing P, $\lambda_P \geq 0$ for each $P \in \mathcal{P}$, and $\sum\limits_{P\in\mathcal{P}} \lambda_P = 1$. Then we have

$$x(S) = \sum_{P\in\mathcal{P}} \lambda_P \mathbf{e}_P(S) = \sum_{P\in\mathcal{P}} \lambda_P |P \cap S| \leq \sum_{P\in\mathcal{P}} \lambda_P |S| = |S|.$$

Since $x(S) = |S|$, the equality holds in the above formula, meaning that $P \cap S = S$ for any $P \in \mathcal{P}$ with $\lambda_P > 0$. That is, S is contained in at least one maximum 2-packing of G.

Finally, we shoe that "(4) $\Rightarrow$ (3)". Suppose every 2-packing is contained in a maximum 2-packing in graph G. Let y be an extreme point of $L(\Gamma_G)$. By Lemma 4.4 and 4.5, y is the indicator vector of some maximal 2-packing P. By the hypothesis, P is a maximum 2-packing of G. Therefore, $y(V) = |P| = c(V)$. By Lemma 4.2, Γ_G has the large core. □

There are problems regading algorithms related to the core largeness, exactness and extendability for DS games corresponding to a graph G. The problems concerned are stated as:

Problem B: Checking extendability, exactness and core largeness of DS game corresponding to G on $\mathcal{G}_B$

Instance: A DS game Γ_G defined on a graph $G \in \mathcal{G}_B$

Question: Is the game Γ_G extendable, exact and with large core?

Theorem 4.2 tell us that, these problems are all equivalent to determining whether or not every 2-packing is contained in a maximum 2-packing of G. These can be restated as follows:

Problem C: Size equality of maximum 2-packing and minimum maximal 2-packing in graph $G \in \mathcal{G}_B$

Instance: A graph $G \in \mathcal{G}_B$

Question: Do a maximum 2-packing and a minimum maximal 2-packing have the same size?

Suppose that G is a "YES" instance of Problem C. Then for each maximal 2-packing, its cardinality is not less than that of the maximum 2-packing. It means that a maximal 2-packing is exactly a maximum 2-packing. On the other hand, assume that G is a "NO" instance of Problem C. There exists a maximal 2-packing such that its cardinality is less than that of a maximum 2-packing, meaning that none of the maximum 2-packings contain this maximal 2-packing.

Although we are not aware of a polynomial-time algorithm for Problem C for general graphs in $\mathcal{G}_B$, there are some graphs for which Problem C can be solved in polynomial time. For example, sun-free chordal graphs, included in $\mathcal{G}_B$, satisfy this property [4, 8] (trees, line graph of trees, interval graphs, and block graphs are examples of sun-free chordal graphs). For these classes of graphs, we can conclude the following.

Corollary 4.6. *Let $G = (V, E)$ be a graph such that the corresponding Problem C can be solved in polynomial time. Then the problems of checking extendability, exactness and core largeness of the DS game corresponding to G on $\mathcal{G}_B$ can be solved in polynomial time.*

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