Commun. Korean Math. Soc. **23** (2008), No. 2, pp. 285–294

EFFICIENT MONTE CARLO ALGORITHM FOR PRICING BARRIER OPTIONS

Kyoung-Sook Moon

Abstract. A new Monte Carlo method is presented to compute the prices of barrier options on stocks. The key idea of the new method is to use an exit probability and uniformly distributed random numbers in order to efficiently estimate the first hitting time of barriers. It is numerically shown that the first hitting time error of the new Monte Carlo method decreases much faster than that of standard Monte Carlo methods.

1. Introduction

A new Monte Carlo method is proposed in order to efficiently compute the prices of barrier stock options based on an exit probability.

European vanilla option is a contract giving the option holder the right to buy or sell one unit of underlying assets at a prescribed price, known as exercise price K, at a prescribed time, known as expiration date T. Barrier options are similar to standard vanilla options except that the option is knocked out or in if the underlying asset price hits the barrier price, B, before expiration date. Since 1967, barrier options have been traded in the over-the-counter (OTC) market and nowadays are the most popular class of exotic options. Therefore it is quite important to develope accurate and efficient methods to evaluate barrier option prices in financial derivative markets.

The evolution of the financial asset price can be written as a stochastic process $\{S_t\}_{t\in[0,T]}$ defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the setting of the classical Black-Scholes option pricing model: the price S_t of the underlying asset is described by a geometric Brownian motion with a constant expected rate of return $\mu > 0$ and a constant volatility $\sigma > 0$ of the asset price, i.e.,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1}$$

where W_t is a standard Brownian motion process, see [6], [14].

Received August 28, 2007; Revised March 14, 2008.
2000 *Mathematics Subject Classification.* Primary 91B28, 60H35.
Key words and phrases. barrier option pricing, Monte Carlo method, exit probability.
This research was supported by the Kyungwon University Research Fund in 2007.

To evaluate barrier option prices, there have been two major directions. The first approach is to solve the Black-Scholes partial differential equation (PDE), see [2], [11]. Merton [11] provided the first analytical formula for a down-and-out call option which was extended for all 8 types of barriers by Reiner and Rubinstein [13], see also Haug [5] for a generalization. However, it is very difficult to price barrier options based on more than two underlying assets. Therefore one should rely on numerical approximations. Instead of solving Black-Scholes PDE, the second approach is to compute an expected value of the discounted terminal payoff under a risk-neutral measure $\mathbb{Q}$, i.e., $\mu = r$ in (1) where $r > 0$ is a constant riskless interest rate. The barrier option price $V(s,t)$ at a present time t can be computed by

$$V(s,t) = E^{\mathbb{Q}}[\Lambda(S_\tau,\tau)|S_t = s], \tag{2}$$

where $\Lambda(S_\tau,\tau)$ is a discounted payoff function and τ is the first hitting time of the given barrier price by the underlying asset process S_t. For example, for up-and-out call barrier options the random variable τ is defined by

$$\tau = \inf\{u \geq t : S_u \geq B\},$$

and have the payoff

$$\Lambda(S_\tau,\tau) = \begin{cases} e^{-r(T-t)}\max(S_T - K, 0) & \text{if } S_t < B, \forall t \leq T, \text{ i.e., } \tau = T \\ e^{-r(\tau-t)}R & \text{at hit } \tau < T, \end{cases} \tag{3}$$

where K is a given exercise price at expiration date T, B is a barrier price, and R is a prescribed cash rebate. To approximate the option price in (2), one may apply either lattice methods or Monte Carlo methods. Since we are more interested in multi-asset cases, we choose Monte Carlo methods.

The Monte Carlo method is very popular and robust numerical method, since it is not only easily extended to multiple underlying assets but also stochastic and simple to coding. On the other hand, one of main drawback of the Monte Carlo method is a slow convergence. The statistical error of the Monte Carlo method is of order $\mathcal{O}(1/\sqrt{M})$ with M simulations. In particular, for continuously monitored barrier options, the hitting time error is of order $\mathcal{O}(1/\sqrt{N})$ with N time steps, see [4], while the European vanilla options have no time discretization error.

The reason why the continuously monitored barrier option gives such a slow convergence for the time error is that the exact path of the underlying asset may hit the barrier between the discrete computational nodes. In this case, the exact value at time t is $e^{-r(\tau-t)}R$, where $\tau < T$ is the hitting time or the first exit time, on the other hand the approximate option price is $E^{\mathbb{Q}}[\Lambda(S_T,T)|S_t = s]$, which may be very different to the exact price. There have been different ideas to reduce this first hitting time error. Dzougoutov et. al. in [3] used adaptive mesh near the barrier to reduce this error and Atiya and Metwally in [12] used a Brownian bridge idea for jump diffusion process. In order to efficiently reduce this hitting time error near the barrier price, inspired by Mannella [10], at each finite time step, we suggest to use an uniformly distributed random variable and a conditional exit probability to correctly check whether the continuous underlying asset price hits the barrier or not. Numerical results show that the new Monte Carlo method converges much faster than the standard Monte Carlo method. For instance, in Section 3, for two-asset barrier option problem, in order to get the same level of accuracy, the standard Monte Carlo method needs 32 times more computational work than the new method, see Fig. 3. This idea of using exit probability for stopped diffusion is well known in physics community, see [8], [10].

The outline of the paper is as follows. In Section 2 we introduce the new Monte Carlo method based on the idea of using uniformly distributed random variable and the conditional exit probability. In Section 3 we present numerical results for barrier options with one or two underlying assets and compare the accuracy and efficiency between the standard and the new Monte Carlo methods. We finally summarize the conclusions in Section 4.

2. Efficient Monte Carlo algorithm

Let us assume that the evolution of the underlying asset price follows the geometric Brownian motion (1). From Ito's formula, the analytic solution satisfies

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}, \tag{4}$$

where r is a constant interest rate and σ is a constant volatility and W_t is a standard Brownian motion. Using the Monte Carlo method, the expected value of the discounted terminal payoff is approximated by a sample average of M simulations

$$V(s,t) = E^{\mathbb{Q}}[\Lambda(S_\tau,\tau)|S_t = s] \approx \widetilde{V}(s,t) := \frac{1}{M}\sum_{j=1}^{M}\Lambda(S_{\tilde{\tau}},\tilde{\tau};\omega_j), \tag{5}$$

where $\tilde{\tau}$ is an approximation of the hitting time τ.

Then the global error can be split into the first hitting time error and statistical error,

$$\begin{aligned} \mathcal{E} := |V(s,t) - \widetilde{V}(s,t)| &= \left(E^{\mathbb{Q}}[\Lambda(S_\tau,\tau) - \Lambda(S_{\tilde{\tau}},\tilde{\tau})|S_t = s]\right) \\ &\quad + \left(E^{\mathbb{Q}}[\Lambda(S_{\tilde{\tau}},\tilde{\tau})] - \frac{1}{M}\sum_{j=1}^{M}\Lambda(S_{\tilde{\tau}},\tilde{\tau};\omega_j)\right) \\ &=: \mathcal{E}_T + \mathcal{E}_S. \end{aligned} \tag{6, 7}$$

From the central limit theorem, the statistical error, $\mathcal{E}_S$ in (7), has the following upper bound

$$|\mathcal{E}_S| \le c_0 \frac{b_M}{\sqrt{M}}, \tag{8}$$

where b_M is a sample standard deviation of the function values $\Lambda(S_{\tilde{\tau}}, \tilde{\tau})$ and c_0 is a positive constant related to confidence interval and M is the number of samples. For instance, $c_0 = 1.96$ for 95 % of confidence interval. On the other hand, the first hitting time error, $\mathcal{E}_T$ in (7), is approximated using an exit probability given the asset prices at each time step.

Let us first discretize the time interval $[0, T]$ into N uniform subinterval $0 = t_0 < t_1 < \cdots < t_N = T$. Then let us compute $S_{n+1} := S_{t_{n+1}}$ at each time step for $n = 0, \ldots, N-1$ by

$$S_{n+1} = S_n e^{(r - \frac{1}{2}\sigma^2)\Delta t_n + \sigma \Delta W_n}. \tag{9}$$

Here Δt_n and ΔW_n denote the time increments $\Delta t_n \equiv t_{n+1} - t_n$ and the Wiener increments $\Delta W_n \equiv W_{t_{n+1}} - W_{t_n}$ for $n = 0, \ldots, N-1$. Then for the up-and-out barrier case, the approximation of the first hitting time τ can be defined by

$$\tilde{\tau} \equiv \inf\{t_n, n = 1, \ldots, N : S_n \geq B\},$$

with the given barrier price B.

The idea is to use an exit probability at each time step. Let P_k denote the probability that a process X exits a domain D at $t \in [t_k, t_{k+1}]$ given the values X_k and X_{k+1} are in D. In one dimensional half interval case, $D = (-\infty, B)$, for a constant B, the probability P_k has a simple expression using the law of Brownian bridge, see [9]

$$\begin{aligned} P_k &= \mathbb{P}\left[\max_{t \in [t_k, t_{k+1}]} X_t \geq B | X_k = x_1, X_{k+1} = x_2\right] \\ &= \exp\left(-2\frac{(B - x_1)(B - x_2)}{\beta(x_1)^2 \Delta t_k}\right), \end{aligned} \tag{10}$$

where $\beta(x_1)$ is the diffusion part of X_k with $x_1 < B$ and $x_2 < B$. For more general domain in higher dimension, the probability can be approximated by an asymptotic expansion in Δt_k, see [1].

Let us consider the up-and-out barrier option case. At each time interval $t \in [t_n, t_{n+1}]$, we compute S_n and S_{n+1} by (9). Though S_n and S_{n+1} do not hit the barrier, i.e., $S_n < B$ and $S_{n+1} < B$, the continuous path $S_t, t \in [t_n, t_{n+1}]$ may hit the barrier at some time $\tau \in (t_n, t_{n+1})$. To approximate this hitting event, we generate an uniformly distributed random variable u_n and compare with the exit probability P_n in (10). If $P_n < u_n$, then we accept that the continuous path S_t does not hit the barrier during this time interval $t \in [t_n, t_{n+1}]$, since the exit probability is very small, i.e., the hitting event is rare to occur. On the other hand, if $P_n > u_n$ then the probability that the continuous path S_t hits the barrier is high therefore we regard that $S_\tau \geq B$ at $\tau \in (t_n, t_{n+1})$. Therefore we have the rebate R and start the next sample path, i.e., the value of the barrier option of this path is $V(S_0, 0) = Re^{-r\tau}$. In this case, as an approximation of the first hitting time τ, we may choose the midpoint $\widetilde{\tau} = (t_n + t_{n+1})/2$.

Let us summarize the standard and the new Monte Carlo methods to compute the price of the up-and-out barrier call option. Without loss of generality let us assume that the cash rebate is zero, i.e., $R = 0$. The following algorithms are readily extended to other types of barrier options with nonzero cash rebate. Here $\mathcal{N}(0,1)$ and $\mathcal{U}(0,1)$ denote that the random variables have the standard normal distribution with mean 0 and variance 1 and a uniform distribution over $(0,1)$, respectively.

`Standard Monte Carlo Method`

for $j = 1, \dots, M$
  for $n = 0, 1, \dots, N-1$
    generate a $\mathcal{N}(0,1)$ sample z_n
    set $S_{n+1} = S_n e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} z_n}$
  end
  if $\max\limits_{0\le n\le N} S_n < B$ then $V_j = e^{-rT}\max(S_T - K, 0)$
  otherwise $V_j = 0$
end
set $\widetilde{V} = \dfrac{1}{M}\sum\limits_{j=1}^{M} V_j$

`New Monte Carlo Method`

for $j = 1, \dots, M$
  for $n = 0, 1, \dots, N-1$
    generate a $\mathcal{N}(0,1)$ sample z_n
    set $S_{n+1} = S_n e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} z_n}$
    set $P_{n+1} = e^{-2\frac{(B-S_n)(B-S_{n+1})}{\sigma^2 S_n^2 \Delta t}}$
  end
  generate $\mathcal{U}(0,1)$ samples $u_n, n = 1, \dots, N$
  if $(S_n < B$ and $P_n < u_n,\ 1 \le \forall n \le N)$ then $V_j = e^{-rT}\max(S_T - K, 0)$
  otherwise $V_j = 0$
end
set $\widetilde{V} = \dfrac{1}{M}\sum\limits_{j=1}^{M} V_j$

3. Numerical experiments

It is numerically shown that the approximate error of the new method converges much faster than that of the standard Monte Carlo method. Let us compare the approximate errors $\mathcal{E}$ in (6) for barrier options with one or two underlying assets.

3.1. One-asset double barrier options

Let us consider an up-and-out-down-and-out barrier call option, namely the option is knocked out if the underlying asset price touches either the lower barrier price L or the upper barrier price U before the expiration date. The payoff at expiry is

$$\Lambda = \max(S_T - K, 0) \text{ if } L < S_t < U \text{ before } T \text{ else } 0.$$

We use the parameters that the present asset price is $S_0 = 100$, the exercise price $K = 100$, the barrier prices $L = 70, U = 130$, the riskless interest rate $r = 0.1$, the expiration date is 6 months, $T = 0.5$ and the volatility $\sigma = 0.25$. Double barrier options can be computed using the Ikeda and Kunitomo formula in [7]. The fair option price with the above parameters is $V(100, 0) = 4.0004$.

Let us apply the algorithms given in Section 2. For the `New Monte Carlo Method`, we compute two exit probabilities P_n^U, P_n^L for up and down barriers respectively and check the criteria

generate $\mathcal{U}(0,1)$ samples $u_n^L, u_n^U, n = 1, \ldots, N$
if $((L < \min_{0\le n\le N} S_n$ and $\max_{0\le n\le N} S_n < U)$ and
$(P_n^L < u_n^L$ and $P_n^U < u_n^U,\ 1 \le \forall n \le N))$
then $V_j = e^{-rT}\max(S_T - K, 0)$
otherwise $V_j = 0$.

Fig. 1 compares the computational errors, $\mathcal{E}$ in (6), between the new and the standard Monte Carlo methods for different numbers of time steps, $N = 2^k, k = 3, \ldots, 7$. As we can see from Fig. 1, the new Monte Carlo method gives much smaller approximate errors than the standard Monte Carlo using the same number of time steps. For instance, using $N = 2^3$ time steps, the new method gives the error $\mathcal{E} = 0.0604$, but the standard method gives $\mathcal{E} = 0.1066$ using $N = 2^{10}$ time steps. In order to compare the hitting time errors correctly, we use $M = 10^7$ samples and make the statistical error $\mathcal{E}_S = 0.0021$ negligible compared to the hitting time error.

Fig. 2 shows the comparison of the CPU times between the standard and the new Monte Carlo methods using $N = 2^k, k = 3, \ldots, 6$ and $M = 10^7$. The CPU times for the new Monte Carlo method take about three times more than the standard method, because of the calculation of the probabilities and the generation of the random numbers, u_n.

For different volatilities $\sigma = 0.15, 0.25, 0.35$ of the market, the new method gives consistently more accurate approximations compared to the standard case, see the Table 1.

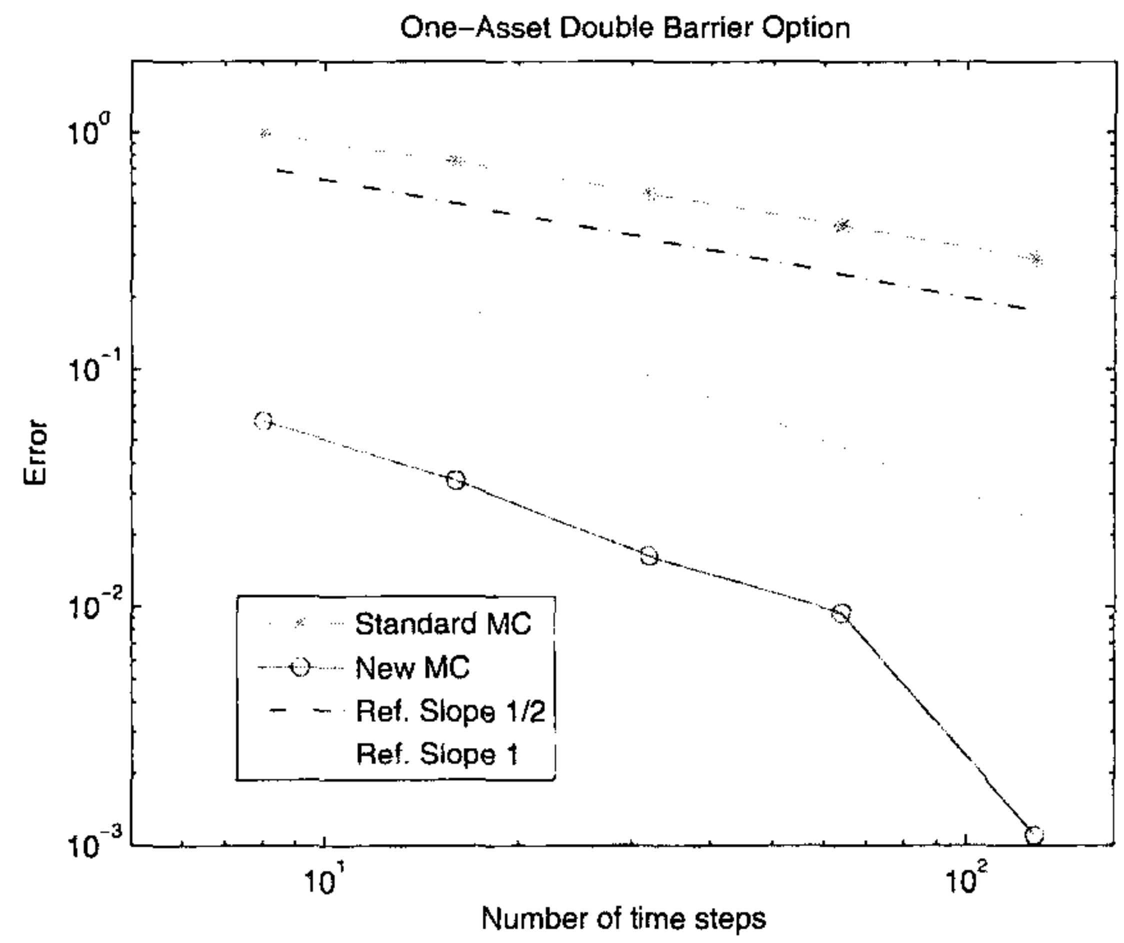

FIGURE 1. One-asset double barrier option: Comparison of the approximation errors between the standard and the new Monte Carlo methods.

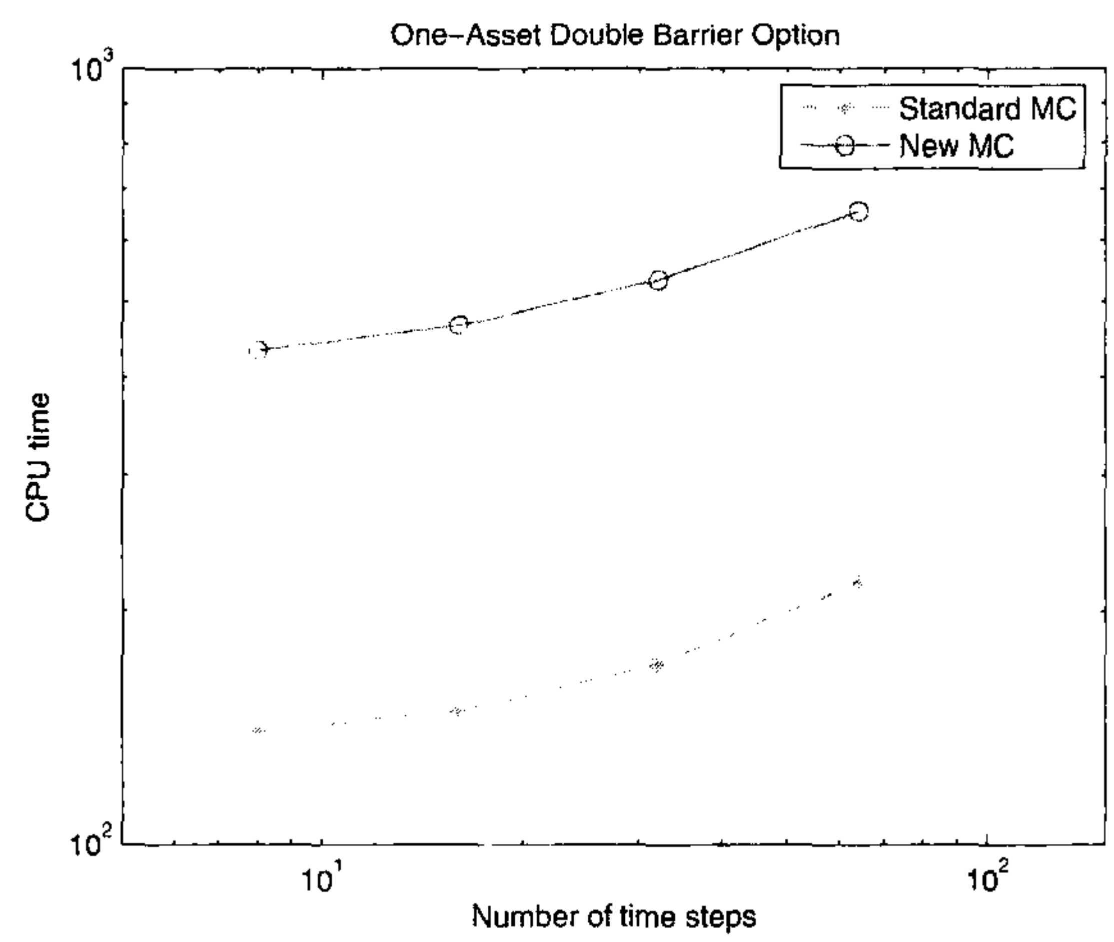

FIGURE 2. One-asset double barrier option: Comparison of the CPU times between the standard and the new Monte Carlo methods.

3.2. Two-asset barrier options

Consider a two-asset up-and-out barrier call option problem, namely the option is knocked out or becomes useless if the asset prices hit the barrier

σ	Standard MC	New MC
0.15	0.1745	0.0059
0.25	0.5528	0.0163
0.35	0.6199	0.0202

TABLE 1. One-asset double barrier option: Comparisons of the computational errors for different volatilities $\sigma = 0.15, 0.25, 0.35$ with $S = K = 100, L = 70, U = 130, r = 0.1, T = 0.5$ using $N = 2^5$ time steps and $M = 10^7$ samples.

before expiration. One of the underlying assets, S^1 determines how much the option is in or out-of-the-money, and the other asset, S^2 is linked to barrier hits. The payoff at expiry is

$$\Lambda = \max(S^1(T) - K, 0) \text{ if } S^2(t) < B \text{ before } T \text{ else } 0 \text{ at hit }.$$

In this case, there exists an exact solution in [5] and we compare the approximate error $\mathcal{E}$ in (6) between the standard and the new Monte Carlo methods. We use the parameters that the present stock prices are $S_0^1 = S_0^2 = 100$, the exercise price $K = 90$, the barrier price $B = 105$, the riskless interest rate $r = 0.08$, the expiration date is 6 months, $T = 0.5$ the volatility $\sigma_1 = \sigma_2 = 0.2$, and correlation between two assets $\rho = -0.5$. Here at each sample path, we generate two independent $\mathcal{N}(0,1)$ random numbers z_1, z_2 and compute the correlated Brownian motions by

$$W_1 = \sqrt{\Delta t} z_1, \; W_2 = \rho \sqrt{\Delta t} z_1 + \sqrt{1 - \rho^2} \sqrt{\Delta t} z_2.$$

The fair option value is $V(100, 100, 0) = 4.66791168$. In order to compare the hitting time error correctly, we use $M = 10^7$ samples and make the statistical error $\mathcal{E}_S = 0.0035$ negligible compared to the time error.

The comparison of the convergence between the standard and the new Monte Carlo methods is shown in Fig. 3. The number of time steps $N = 2^k, k = 3, \ldots, 7$ are used. As we can see from Fig. 3, the hitting time error of the new method decreases much faster than that of the standard Monte Carlo method. For instance, to get the hitting time error around $\mathcal{E}_T = 0.45$, the standard Monte Carlo method needs $N = 2^8$ time steps, while the new method needs $N = 2^3$ time steps, which is around 32 times less work.

4. Conclusions

A new Monte Carlo method has been proposed in order to correctly compute the first hitting time of the barrier price by the underlying asset. The approximate error of the new method converges much faster than that of the standard Monte Carlo method. The future work will be devoted to extend this

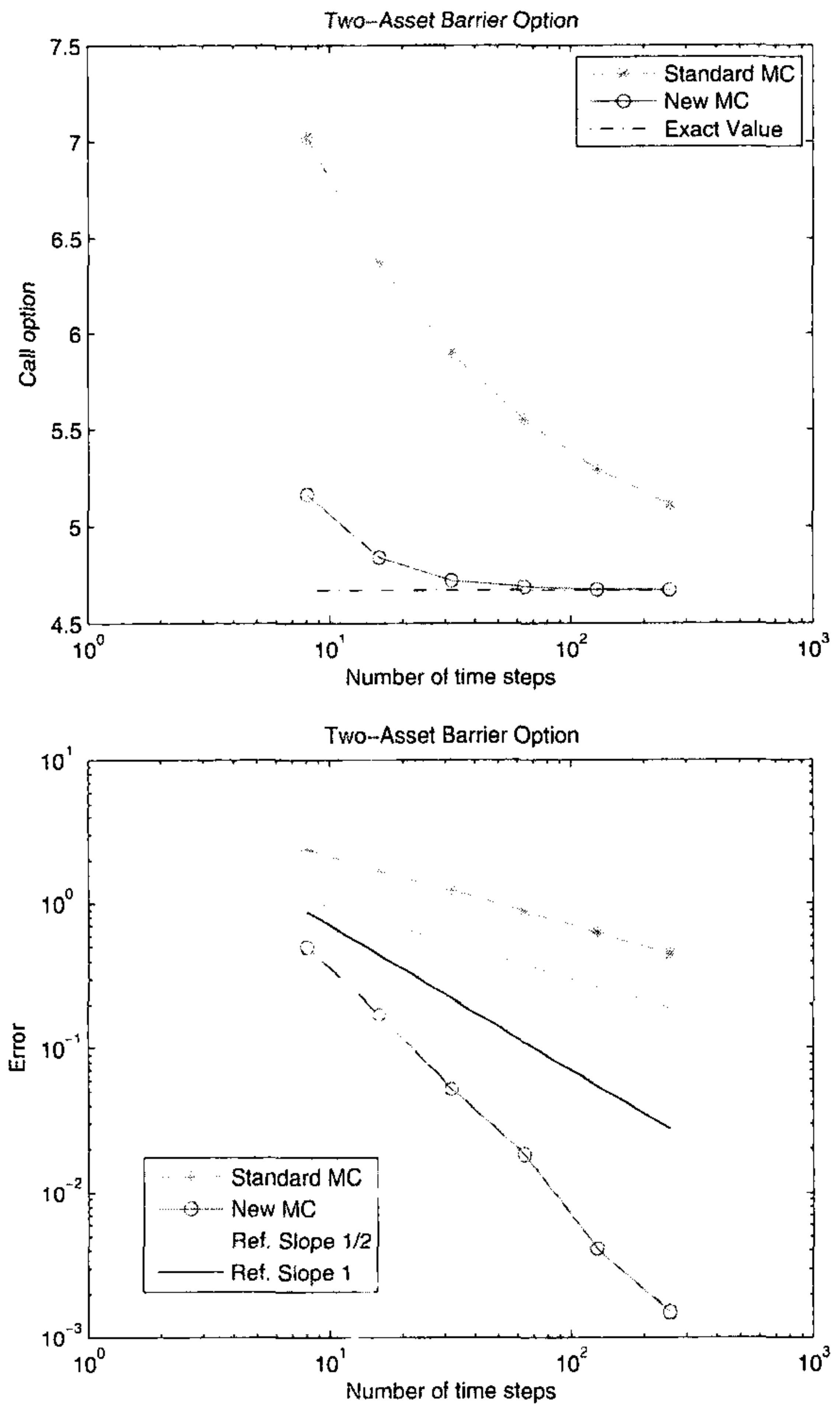

FIGURE 3. Two-asset barrier option: (Up) Comparison of the values of options (Down) Comparison of the approximation errors between the standard and the new Monte Carlo methods.

idea to more general diffusion problems and also theoretically study the rate of convergence of the approximate errors.

References

[1] P. Baldi, *Exact asymptotics for the probability of exit from a domain and applications to simulation*, Ann. Probab. **23** (1995), no. 4, 1644–1670.

[2] F. Black and M. Sholes, *The pricing of options and corporate liabilities*, J. Political Economy **81** (1973), no. 3, 637–659.

[3] A. Dzougoutov, K.-S. Moon, E. von Schwerin, A. Szepessy, and R. Tempone, *Adaptive Monte Carlo algorithms for stopped diffusion*, in Multiscale methods in science and engineering, Lecture notes in computational science and engineering 44, 59–88, Springer, Berlin, 2005.
[4] E. Gobet, *Weak approximation of killed diffusion using Euler schemes*, Stochastic Process. Appl. **87** (2000), no. 2, 167–197.
[5] E. G. Haug *The Complete Guide to Option Pricing Formulas*, MaGraw-Hill, 1997.
[6] J. C. Hull, *Options, Futures and Others*, Prentice Hall, 2003.
[7] M. Ikeda and N. Kunitomo, *Pricing options with curved boundaries*, Mathematical Finance (1992), no. 2, 275–298.
[8] K. M. Jansons and G. D. Lythe, *Efficient numerical solution of stochastic differential equations using exponential timestepping*, J. Stat. Phys. **100** (2000), 1097–1109.
[9] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, 1991.
[10] R. Mannella, *Absorbing boundaries and optimal stopping in a stochastic differential equation*, Physics Letters A **254** (1999), 257–262.
[11] R. C. Merton, *Theory of rational option pricing*, Bell J. Econ. Manag. Sci. **4** (1973), no. 1, 141–183.
[12] S. A. K. Metwally and A. F. Atiya, *Using Brownian bridge for fast simulation of jump-diffusion processes and barrier options*, J. Derivatives Fall (2002), 43–54.
[13] E. Reiner and M. Rubinstein, *Breaking down the barriers*, Risk **4** (1991), no. 8, 28–35.
[14] P. Wilmott, S. Howison, and J. Dewynne, *The Mathematics of Financial Derivatives*, Cambridge University Press, 1995.

Department of Mathematics & Information
Kyungwon University
Gyeonggi-do 461-701, Korea
E-mail address: ksmoon@kyungwon.ac.kr