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RECENT RESULTS AND CONJECTURES IN ANALYTICAL FIXED POINT THEORY

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Abstract. We survey recent results and some conjectures in analytical fixed point theory. We list the known fixed point theorems for Kakutani maps, Fan-Browder maps, locally selectionable maps, approximable maps, admissible maps, and the better admissible class $\mathfrak{B}$ of maps. We also give 16 conjectures related to that theory.

We review some fixed point theorems on multimaps (maps) and related results mainly due to the present author. Some related conjectures (0)-(XV) are also stated. This is a revised and corrected version of parts of our previous works [18-22]. The terminology and notations are standard. For overall historical background, see [15, 18-22].

1. The Schauder conjecture and other problems

A t.v.s. is a Hausdorff topological vector space E with a basis $\mathcal{V}$ of neighborhoods of the origin 0 of E. Multimaps are called simply maps.

We begin with some conjectures related to fixed point theory as follows:

(0) [The Schauder conjecture] *Let E be a t.v.s., C a convex subset of E, and $f : C \to C$ a continuous function. If $f(C)$ is contained in a compact subset of C, then f has a fixed point.*

An infinite dimensional compact convex subset K of a t.v.s. E is said to have the *simplicial approximation property* [7] if for every $V \in \mathcal{V}$ there exists a finite dimensional compact convex set K_0 in K such that if S is any finite dimensional simplex in K then there exists a continuous function $\psi : S \to K_0$ with $\psi(x) - x \in V$ for all $x \in S$.

(I) *A compact convex subset of a metrizable t.v.s. has the simplicial approximation property.*

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Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) [5] provided that, for every nonempty compact subset K of X and every $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

(II) *A compact convex subset of a metrizable t.v.s. is admissible.*

(III) [The compact AR problem] *A compact convex subset of a metrizable t.v.s. is an AR.*

(IV) [The Banach problem] *An infinite dimensional compact convex subset of a metrizable t.v.s. is homeomorphic to the Hilbert cube Q.*

Recall that *Roberts spaces* are compact convex subsets with no extreme points constructed by Roberts' method of needle point spaces; see [27].

(V) *Every Roberts spaces are AR and homeomorphic to the Hilbert cube Q.*

2. Kakutani maps

A *polytope* P in a subset X of a t.v.s. E is a homeomorphic image of a standard simplex.

A nonempty subset K of E is said to be *Klee approximable* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope of E; see [23, 25]. In particular, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

Let X be a nonempty closed convex subset of a t.v.s. E. We say that X is *weakly admissible* (in the sense of Nhu [8] and Arandelović [1]) if for every $V \in \mathcal{V}$ there exist closed convex subsets $X_1, X_2, \dots, X_n$ of X with $X = \mathrm{co}\,(\bigcup_{i=1}^n X_i)$ and continuous functions $f_i : X_i \to X \cap L, i = 1, 2, \dots, n$, where L is a finite dimensional subspace of E, such that $\sum_{i=1}^n (f_i(x_i) - x_i) \in V$ for every $x_i \in X_i$ and $i = 1, 2, \dots, n$.

A subset B of a t.v.s. E is said to be *convexly totally bounded* (simply, c.t.b.) [6] if for every $V \in \mathcal{V}$, there exist a finite subset $\{x_i\}_{i=1}^n \subset B$ and a finite family of convex subsets $\{C_i\}_{i=1}^n$ of V such that $B \subset \bigcup_{i=1}^n (x_i + C_i)$.

A *Kakutani map* is an upper semicontinuous (u.s.c.) map with nonempty compact convex values.

The following in [6, 9, 23] shows some of the most general known partial resolutions of the Schauder conjecture:

Theorem 1. *Let X be a nonempty subset of a t.v.s. E. Then a compact Kakutani map $T : X \multimap X$ has a fixed point if one of the following conditions holds:*

(a) (Idzik) X *is convex and* $\overline{T(X)}$ *is c.t.b.*

(b) (Okon) X *is compact, convex and weakly admissible.*

(c) (Park) $T(X)$ *is Klee approximable into* X.

Note that (c) holds whenever $\overline{T(X)}$ is locally convex or X is convex and admissible (in the sense of Klee). Therefore, Theorem 1 generalizes well-known results due to Kakutani (1941), Bohnenblust and Karlin (1950), Fan (1952), Glicksberg (1952), Granas and Liu (1986), Park (1988), and others. See [10-15,20].

We say that a topological space X has the (*compact*) *fixed point property* (simply, f.p.p.) if any (compact) continuous selfmap $f : X \to X$ has a fixed point $x_0 \in X$.

We say that a subset X of a t.v.s. has the (*compact*) *Kakutani fixed point property* (simply, $\mathbb{K}$-f.p.p.) if any (compact) Kakutani map $T : X \multimap X$ has a fixed point.

The following conjectures are still open:

(VI) *A compact convex subset* X *of a t.v.s. has the* $\mathbb{K}$*-f.p.p.*

(VII) *A convex subset* X *of a metrizable t.v.s. has the compact* $\mathbb{K}$*-f.p.p.*

In view of Theorem 1, (VI) is valid whenever X is c.t.b. or weakly admissible, and that (VII) is valid whenever X is admissible or any compact subset of X is c.t.b. for any t.v.s. (not necessarily metrizable).

In [22], we claimed that the following conjecture follows from (VI):

(VIII) *Let* X *be a compact convex subset of a t.v.s. and* U *an open subset of* X *such that* $0 \in U$. *Let* $T : \overline{U} \multimap X$ *be a Kakutani map. Then one of the following holds*:
(1) T *has a fixed point.*
(2) *There exists an* $x \in \operatorname{Bd} U$ *and a* $\lambda \in (0,1)$ *such that* x *is a fixed point of* λT.

Note that (VIII) is true whenever X is c.t.b. or weakly admissible. Recently, statements like (VIII) are called nonlinear alternatives.

3. The Fan-Browder maps

A map with nonempty convex values and open fibers is called a *Browder map*. The well-known *Fan-Browder fixed point theorem* [4] states that a Browder map from a compact convex subset of a t.v.s. into itself has a fixed point.

For a subset X of a t.v.s. E, a map $T : X \multimap X$ is called a Φ*-map* (or *a Fan-Browder map*) if there exists a map $S : X \multimap X$ such that (1) for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$ and (2) $X = \bigcup\{\operatorname{Int} S^-(x) \mid x \in X\}$.

A subset X of a t.v.s. E is called a Φ*-space* if for each $U \in \mathcal{V}$, there is a Φ-map $T : X \multimap X$ such that $T(x) \subset x + U$ for each $x \in X$.

We say that a convex subset X of a t.v.s. has the (*compact*) Φ-*fixed point property* (simply, Φ-f.p.p.) if any (compact) Φ-map $T : X \multimap X$ has a fixed point.

Since any Φ-map from a paracompact space into a convex subset of a t.v.s. has a continuous selection, we have the following:

Lemma 2. (Komiya) *Let X be a paracompact convex subset of a t.v.s. If X has the f.p.p., then it has the Φ-f.p.p.*

In 1992, Ben-El-Mechaiekh conjectured that the Fan-Browder theorem might be true if we assume that the Browder map is compact instead of the compactness of its domain. More generally the following is still open:

(IX) *A convex subset X of a t.v.s. E has the compact Φ-f.p.p.*

This follows from the above-mentioned selection theorem and Statement **(0)**.

It is well-known that, if K is a compact subset of a t.v.s., then $\operatorname{co} K$ is paracompact. In view of this fact, from Lemma 2 and Theorem 1, we easily have the following:

Theorem 3. *An admissible convex subset X of a t.v.s. E has the following:*
(a) *the compact f.p.p.,*
(b) *the compact Φ-f.p.p.,*
(c) *the compact $\mathbb{K}$-f.p.p.*

Recall that, when E is locally convex, (a) is due to Hukuhara, (b) to Ben-El-Mechaiekh, and (c) to Himmelberg; see [24].

4. Locally selectionable maps

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *selectionable* if it has a continuous selection $f : X \to Y$ (that is, $f(x) \in T(x)$ for all $x \in X$), and *locally selectionable* [16] if for each $x_0 \in X$, there exist an open neighborhood V_0 of x_0 and a continuous function $f_0 : V_0 \to Y$ such that $f_0(x) \in T(x)$ for all $x \in V_0$.

The following conjecture is a consequence of **(0)**:

(X) *Let X be a convex subset of a t.v.s. E and $T : X \multimap X$ a locally selectionable map having convex values. If T is compact, then T has a fixed point.*

It is known that, if X is paracompact and Y is a convex subset of a t.v.s., then any Φ-map $T : X \multimap Y$ is locally selectionable and a locally selectionable map $T : X \multimap Y$ having convex values is selectionable. From this, we have the following:

Theorem 4. *Let E be a t.v.s. whose nonempty paracompact convex subsets have the compact f.p.p., X a nonempty convex subset of E, and $T : X \multimap X$ a* *locally selectionable map having convex values. If T is compact, then it has a fixed point.*

Note that Theorem 4 gives partial solutions to (IX) and (X).

5. Approximable maps

Let X and Y be subsets of t.v.s. E and F, respectively, and $T : X \multimap Y$ a map. Given two open neighborhoods U and V of the origin 0 of E and F, respectively, a (U, V)-*approximative continuous selection* of T is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (T[(x+U) \cap X] + V) \cap Y \quad \text{for every } x \in X.$$

The map T is said to be *approachable* if it admits a (U, V)-approximative continuous selection for every U and V as above; and T *approximable* if its restriction $T|_K$ to any compact subset K of X is approachable. Note that an approachable map is always approximable; see [2,3].

We say that a subset X of a t.v.s. has the (*compact*) *approachable fixed point property* (simply, $\mathbb{A}$-f.p.p.) if any (compact) u.s.c. approachable map $T : X \multimap X$ has a fixed point; and the *approximable fixed point property* (simply, $\mathbb{A}^\kappa$-f.p.p.) if any u.s.c. approximable map $T : X \multimap X$ has a fixed point.

The following are consequences of Conjecture **(0)**:

(XI) *A convex subset X of a t.v.s. E has the compact $\mathbb{A}$-f.p.p.*

(XII) *Let X be a closed convex subset of a t.v.s. E such that $0 \in \operatorname{Int} X$ and $T : X \multimap E$ a compact closed approachable map. Then either*
(1) *T has a fixed point*; *or*
(2) *$\lambda x \in T(x)$ for some $\lambda > 1$ and $x \in \operatorname{Bd} X$.*

(XIII) *A compact convex subset X of a t.v.s. has the $\mathbb{A}^\kappa$-f.p.p.*

The validities of (XI)-(XIII) require certain restrictions as follows:

Theorem 5. *For a subset X of a t.v.s. E, the following are equivalent:*
(a) *X has the compact f.p.p.*
(b) *X has the compact $\mathbb{A}$-f.p.p.*
Moreover, (a) *and* (b) *follow from*
(c) *X has the compact $\mathbb{K}$-f.p.p.*

For the proof, see [23].

From Theorems 1 and 5, we immediately have the following partial solution of (XI):

Corollary 6. *A convex subset X of a t.v.s. E has the compact $\mathbb{A}$-f.p.p. whenever one of the following holds:*

(a) *X is admissible* (*in the sense of Klee*).
(b) *every compact subset of X is c.t.b.*
(c) *X is compact and weakly admissible.*

Note that (XII) holds under one of (a)-(c) of Corollary 6.

Recall that a subset X of a t.v.s. E is said to be *almost convex* if for each $U \in \mathcal{V}$ and for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, there exists a finite subset $\{z_1, z_2, \ldots, z_n\}$ of X such that $z_i - x_i \in U$ for all i and $\operatorname{co}\{z_i\}_{i=1}^n \subset X$.

Corollary 7. *An almost convex subset X of a locally convex t.v.s. E has* (a) *the compact f.p.p.,* (b) *the compact $\mathbb{A}$-f.p.p., and* (c) *the compact $\mathbb{K}$-f.p.p.*

Recall that (c) is due to Park and Tan [26].

The following is a partial solution to (XIII).

Corollary 8. *A compact convex subset X of a t.v.s. E has the $\mathbb{A}^\kappa$-f.p.p. whenever one of the following holds:*
(a) *X has the f.p.p.*
(b) *X is c.t.b.*
(c) *X is weakly admissible.*

It is known by Ben-El-Mechaiekh et al. that, for a compact uniform space X and a convex subset Y of a locally convex t.v.s. E, a Kakutani map $T : X \multimap Y$ is approximable. For non-locally convex t.v.s., we have the following:

Corollary 9. *Let X be a compact subset of a t.v.s. E. Then the following are equivalent:*
(a) *X has the f.p.p.*
(b) *X has the $\mathbb{A}^\kappa$-f.p.p.*
Moreover, (a) *and* (b) *follow from*
(c) *X has the $\mathbb{K}$-f.p.p.*

Examples. ([24]) For a compact convex subset X of a t.v.s. E, we give known cases when (a)-(c) of Corollary 9 hold as follows:

(1) For a Euclidean space E, (a) is due to Brouwer and (a) $\Rightarrow$ (c) to Kakutani by a different method.

(2) For a normed vector space E, (a) is due to Schauder and (c) to Bohnenblust and Karlin.

(3) For a locally convex t.v.s. E, (a) is due to Tychonoff, (b) to Ben-El-Mechaiekh, and (c) to Fan and Glicksberg, independently.

(4) For a t.v.s. E having sufficiently many linear functionals, (a) is due to Fan and (c) to Granas and Liu.

(5) For an admissible set X, (a) is due to Hahn and Pötter and (b) and (c) to Park.

(6) If X is locally convex, (a) is due to Rzepecki and (c) to Idzik.

(7) For a c.t.b. set X, (a) and (c) are particular cases of a result of Idzik.

(8) Further if X is a Φ-space, (a) is due to Horvath and (c) to Park.

(9) For a weakly admissible set X, (a) is due to Nhu and Arandelović and (c) to Okon.

(10) For a Roberts space X, (a) is due to Nhu et al. and (c) to Okon.

6. Better admissible classes of maps

In 1996, the author introduced the 'better' admissible class $\mathfrak{B}$ of maps defined on a subset X of a t.v.s. E into a topological space Y as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that, for each polytope P in X and for any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

Subclasses of $\mathfrak{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$, the Aronszajn maps $\mathbb{M}$ (u.s.c. with R_δ values), the acyclic maps $\mathbb{V}$, the Powers maps $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the Fan-Browder maps (Φ-maps), locally selectionable maps $\mathbb{L}$ having convex values, u.s.c. approachable maps $\mathbb{A}$ (whose domains are uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, the class $\mathbb{K}_c^\sigma$ of Lassonde, the class $\mathbb{V}_c^\sigma$ of Park et al., u.s.c. approximable maps of Ben-El-Mechaiekh and Idzik, and others. Those subclasses are examples of the admissible class $\mathfrak{A}_c^\kappa$ due to the author [12-15]. There are maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$, for example, the connectivity map due to Nash and Girolo. Moreover, compact closed maps in the KKM class due to Chang and Yen and in the s-KKM class due to Chang et al. also belong to the class $\mathfrak{B}$. For details on $\mathfrak{B}$, see [14, 15, 23-25].

In 1997, we obtained the following [14, 15]:

Theorem 10. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

Moreover, in [14], we listed more than sixty papers in chronological order, from which we could deduce particular forms of Theorem 10.

In view of Conjectures (II) or (III), we have the following conjectures:

(XIV) *Let K be a compact convex subset of a metrizable t.v.s. E. Then any $T \in \mathfrak{A}_c^\kappa(K,K)$ has a fixed point.*

(XV) *Let K be a compact convex subset of a metrizable t.v.s. E. Then any closed $T \in \mathfrak{B}(K,K)$ has a fixed point.*

In view of Theorem 10, Statements (XIV) and (XV) hold for any t.v.s. if K is admissible. Note that conjectures (III)-(XV) are consequences of (0), (I) or (VII) with the aid of other results.

In 2004, we obtained a generalized version of Theorem 10 by switching the admissibility of domain of the map to the Klee approximability of range as follows [23]:

Theorem 11. *Let E be a t.v.s. and X a nonempty subset of E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ such that $F(X)$ is Klee approximable into X has a fixed point.*

Examples. We give some examples of Klee approximable sets as follows:

(1) If a subset X of E is admissible (in the sense of Klee), then every compact subset K of X is Klee approximable into E.

(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.

(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.

(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.

(5) Any compact subset K of an admissible convex subset X of a t.v.s. is Klee approximable into X.

(6) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.

Note that (6)⇒(5)⇒(4)⇒(3).

The following are recently obtained in 2007 [25], where $\mathfrak{B}^p$ should be replaced by $\mathfrak{B}$:

Corollary 12. *Let X be an almost convex admissible subset of a t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then F has a fixed point.*

Corollary 13. *Let X be an almost convex subset of a locally convex t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then F has a fixed point.*

One of the most simple known example is that every compact continuous selfmap on an almost convex subset in a Euclidean space has a fixed point. This generalizes the Brouwer fixed point theorem.

Moreover, since the class $\mathfrak{B}(X, X)$ contains a large number of special types of maps, we can apply Theorem 11 to them. Note that Corollary 13 is a far-reaching generalization of the Kakutani theorem.

7. Further results

There have appeared a number of papers by other authors concerned with generalizations, variations, or applications of our works.

The concept of compact maps has variants (not necessarily generalizations) in that of various types of condensing maps (pseudo-condensing or countably condensing maps or of Mönch type). It is well-known that the theory of such types of condensing maps reduces to that of compact maps. Therefore, our theorems might be applied to those types of condensing maps.

In the author's works in 1991-2007, as applications of some of our fixed point theorems, we obtained results on best approximations, variational or quasi-varia- tional inequalities, minimax inequalities, saddle points in nonconvex sets, openness of maps, existence of maximal elements, the Walras excess demand theorem, acyclic versions of the Nash equilibrium theorem, generalized equilibrium problems, quasi-equilibrium problems, generalized complementarity problems, and others. See [15] and MATHSCINET.

Finally, we note that this paper is based on invited talks given at the first Korea-Slovenia Combinatorial and Computational Mathematics Conference, Koper, Slovenia on June 21-23, 2007; Department of Mathematics, University of Novi Sad, Novi Sad, Servia on July 2, 2007; and the 9th International Conference on Nonlinear Functional Analysis and Its Applications, Masan and Jinju, Korea on July 25-29, 2007.

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