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STRONG CONVERGENCE OF AN IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF STRONG SUCCESSIVELY Φ-PSEUDOCONTRACTIVE MAPS

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Abstract. The aim of this paper is to prove convergence of implicit iteration process to a common fixed point for a finite family of strong successive Φ-pseudocontractive mappings. The results presented in this paper extend and improve the corresponding results of S. S. Chang [On the convergence of implicit iteration process with error for a finite family asymptotically nonexpansive mappings, J. Math. Anal. Appl. 313 (2006), 273–283], M. O. Osilike [Implicit iteration process for common fixed points of a finite finite family of strictly pseudocontractive maps, Appl. Math. Comput. 189(2) (2007), 1058–1065].

1. Introduction

Let K be a nonempty closed convex subset of E, $x_0 \in K$ be any given point and $\{T_i\}_{i=1}^N : K \to K$ be N mappings. Let $\{\alpha_n\}$ be a sequence in $[0,1]$, with $\limsup_{n\to\infty} \alpha_n < 1$, x_n defined by

$$(1.1)\qquad \begin{cases} x_1 = \alpha_1 x_0 + (1-\alpha_1)T_1 x_1, \\ x_2 = \alpha_2 x_1 + (1-\alpha_2)T_2 x_2, \\ \qquad \cdots, \\ x_N = \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N, \\ x_{N+1} = \alpha_{N+1} x_N + (1-\alpha_N)T_1^2 x_N, \\ \qquad\quad \cdots, \\ x_{2N+1} = \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1}, \\ \qquad\quad \cdots . \end{cases}$$

A family $\{T_i\}_{i=1}^N$ is called *uniformly Lipschitzian* with a Lipschitzian constant $L \geq 1$, i.e., there exists a constant $L \geq 1$ such that

$$\|T_i^n - T_i^n\| \leq L\,\|x-y\|, \quad \forall n \geq 1,\ x, y \in K,\ i = 1, \cdots, N.$$

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Definition 1.1. A family $\{T_i\}_{i=1}^N$ is called *generalized strongly successively Φ-pseudocontractive* if, for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|).$$

Definition 1.2. A Banach space E is said to *satisfy Opial's condition* if, for any $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, the following inequality holds:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \ (y \neq x).$$

Tool 1: Let E be a real reflexive Banach space which satisfies Opial's condition, K be a nonempty closed convex subset of E and $T : K \to K$ be an continuous pseudocontrative mapping, Then $I - T$ is demiclosed at zero.

Tool 2: Let E be a uniformly convex Banach space with the modulus of uniform convexity δ_E. Then $\delta_E : [0,2] \to [0,1]$ is continuous increasing, $\delta_E(0) = 0$ and $\delta_E(t) > 0$ for all $t > 0$, while

$$\|cu + (1-c)v\| \leq 1 - 2\min\{c, 1-c\}\delta_E(\|u-v\|)$$

whenever $0 \leq c \leq 1$ and $\|u\|, \|v\| \leq 1$.

Thereom 1.1. ([1, Theorem 1, p. 276]) *Let E be a real uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of E with $K + K \subset K$, $\{T_i\}_{i=1}^N : K \to K$ be asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K, $\{\alpha_n\}$ be a sequences in $[0,1]$, $\{h_n\}$ be a sequence defined by*

$$\|T_i^n x - T_i^n y\| \leq h_n \|x-y\|, \quad x, y \in K,$$

and $L = sup_{n\geq 1} h_n \geq 1$ satisfying the following conditions:
(a) $\sum_{n=1}^\infty \|u_n\| < \infty$;
(b) $\sum_{n=1}^\infty (h_n - 1) < \infty$;
(c) *there exist constants $\gamma_1, \gamma_2 \in (0,1)$ such that*

$$\gamma_1 \leq (1-\alpha_n) \leq \gamma_2, \quad \forall n \geq 1.$$

Then the implicit iterative sequence $\{x_n\}$ defined by (1.1) *converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$ in K.*

Theorem 1.2. ([3, Theorem 3.1, p. 354]) *Let K be a nonempty closed convex subset of a Banach space. Let $\{T_i : i \in N\}$ be N asymptotically quasi-nonexpansive self-mappings of K with $u_{in} \in [0,\infty)$ (i.e., $\|T_i^n x - x_i^*\| \leq (1 + u_{in})\|x - x_i^*\|$ for all $x \in K$ abd $x_i^* \in F(T)$) such that $\sum_{n=1}^\infty u_{in} < \infty$ for all $i \in N$ and $F = \bigcap F(T_i) \neq \emptyset$. Suppose that $x_0 \in K$ and $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0,1)$. Then the implicit iterative sequence $\{x_n\}$ defined by* (1.1) *converges strongly to a common fixed point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Theorem 1.3. ([3, Theorem 3.3, p. 355]) *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space. Let $\{T_i : i \in N\}$ be N*

uniformly L-Lipschitzian asymptotically quasi-nonexpansive self-mappings of K *with* $u_{in} \in [0, \infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *for all* $i \in N$ *and* $F = F \bigcap F(T_i) \neq \emptyset$. *Suppose that there exists one semicompact member* T *in* $\{T_i : i \in N\}$ *and* $x_0 \in K$ *and* $\{\alpha_n\} \subset (s, 1-s)$ *for some* $s \in (0,1)$. *Then the implicit iterative sequence* $\{x_n\}$ *defined by* (1.1) *converges strongly to a common fixed point in* F.

2. Main Results

Now, we are ready to give our main result in this paper.

Theorem 2.1. *Let* E *be a real uniformly convex Banach space with a Fréchet differentiable norm. Let* K *be a closed convex subset of* E *and* $\{T_i\}_{i=1}^{N}$ *be a finite family of uniformly Lipschitzian strong successively* Φ*-pseudocontractive self-mappings of* K *such that* $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence in* K *defined by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} x_n, \quad n \geq 1,$$

where $n = (k(n)-1)N + i(n)$. *If* $\{\alpha_n\}$ *is chosen so that* $\alpha_n \in (0,1)$ *with* $\limsup_{n\to\infty} \alpha_n < 1$, *then the sequence* $\{x_n\}$ *converges weakly to a common fixed point of the family* $\{T_i\}_{i=1}^{N}$.

Proof. For all $p \in F$, $\lim_{n\to\infty} \|x_n - p\|$ exists. Obverse that

$$\begin{aligned}
\|x_n - p\|^2 &= \alpha_n \langle x_{n-1} - p, j(x_n - p)\rangle + (1-\alpha_n)\langle T_{i(n)}^{k(n)} x_n - p, j(x_n - p)\rangle \\
&\leq \alpha_n \|x_{n-1} - p\| \, \|x_n - p\| + (1-\alpha_n)[\|x_n - p\|^2 - \Phi(\|x_n - p\|)] \\
&\leq \alpha_n \|x_{n-1} - p\| \, \|x_n - p\| + (1-\alpha_n) \|x_n - p\|^2,
\end{aligned}$$

which implies that $\|x_n - p\| \leq \|x_{n-1} - p\|$ and so $\lim_{n\to\infty} \|x_n - p\|$ exist. Thus we can assume that $\lim_{n\to\infty} \|x_n - p\| = d$ where $d \geq 0$ is some real number. Since $\{\|x_n - p\|\}$ is a convergent sequence and so $\{x_n\}$ is a bounded sequence in K.

$$\begin{aligned}
\|x_n - p\| &\leq \left\| x_n - p + \frac{1-\alpha_n}{2\alpha_n}(x_n - T_{i(n)}^{k(n)} x_n) \right\| \\
&= \left\| x_n - p + \frac{1-\alpha_n}{2}(x_{n-1} - T_{i(n)}^{k(n)} x_n) \right\| \\
&= \left\| \frac{x_{n-1} + x_n}{2} - p \right\| \\
&= \left\| \frac{1}{2}(x_{n-1} - p) + \frac{1}{2}(x_n - p) \right\| \\
&= \|x_{n-1} - p\| \left\| \frac{1}{2} + \frac{1}{2} \frac{x_n - p}{\|(x_{n-1} - p)\|} \right\| \\
&\leq \|x_{n-1} - p\| \left[1 - \delta\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - p\|} \right) \right],
\end{aligned}$$

which implies that $\|x_{n-1}-p\|\,\delta(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-p\|}) \leq \|x_{n-1}-p\| - \|x_n-p\|$ and hence $x_n - x_{n-1} \to 0$.

Again, since $x_n - x_{n-1} \to 0$, we have $T_{i(n)}^{k(n)} x_n - x_{n-1} \to 0$ and so

$$x_n - T_{i(n)}^{k(n)} x_n \to 0$$

and

$$x_n - x_{n+j} \to 0, \quad \forall j = 1, 2, \cdots .N$$

For any positive interger $n \geq N$, it can be written as $n = i(n)(modN)$, where $n = (k(n)-1)N + i(n)$ for $i(n) \in \{1, 2, \cdots, N\}$. Letting $\sigma_n = \left\|T_{i(n)}^{k(n)} x_n - x_{n-1}\right\|$. Then we have

$$\begin{aligned}
\|x_{n-1} - T_n x_n\| &\leq \left\|x_{n-1} - T_{i(n)}^{k(n)} x_n\right\| + \left\|T_{i(n)}^{k(n)} x_n - T_n x_n\right\| \\
&= \sigma_n + \left\|T_{i(n)}^{k(n)} x_n - T_{i(n)} x_n\right\| \\
&\leq \sigma_n + L\left\|T_{i(n)}^{k(n)-1} x_n - x_n\right\| \\
&\leq \sigma_n + L\left\{\left\|T_{i(n)}^{k(n)-1} x_n - T_{n-N}^{k(n)-1} x_{n-N}\right\|\right. \\
&\quad \left. + \left\|T_{n-N}^{k(n)-1} x_{n-N} - x_{(n-N)-1}\right\| + \left\|x_{(n-N)-1} - x_n\right\|\right\}.
\end{aligned}$$

Since, for each $n > N$, $n = (n-N)(modN)$ and $n = (k(n)-1)N + i(n)$, we have

$$n - N = ((k(n)-1)-1)N + i(n) = (k(n-N)-1)N + i(n-N),$$

i.e.,

$$k(n-N) = k(n) - 1, \quad i(n-N) = i(n).$$

Therefore, we have

$$\begin{aligned}
&\left\|T_{i(n)}^{k(n)-1} x_n - T_{n-N}^{k(n)-1} x_{n-N}\right\| \\
&= \left\|T_{i(n)}^{k(n)-1} x_n - T_{i(n)}^{k(n)-1} x_{n-N}\right\| \leq L\left\|x_n - x_{n-N}\right\|
\end{aligned}$$

and

$$\left\|T_{n-N}^{k(n)-1} x_{n-N} - x_{(n-N)-1}\right\| = \left\|T_{i(n-N)}^{k(n-N)} x_{n-N} - x_{(n-N)-1}\right\| = \sigma_{n-N}$$

Thus we have

$$\|x_{n-1} - T_n x_n\| \leq \sigma_n + L^2\left\|x_n - x_{n-N}\right\| + L\sigma_{n-N} + L\left\|x_{n-N-1} - x_n\right\|$$

and so

$$\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0.$$

It follows that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

Consequently, for each $j = 1, 2, \cdots, N$,

$$\begin{aligned}
&\|x_n - T_{n+j}x_n\| \\
&\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| + \|T_{n+j}x_{n+j} - T_{n+j}x_n\| \\
&\leq (1+L)\,\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \to 0 \quad (n \to \infty),
\end{aligned}$$

which implies that the sequence

$$\bigcup_{j=1}^{N}\{\|x_n - T_{n+j}x_n\|\} \to 0 \quad (n \to \infty).$$

Since $\{\|x_n - T_l x_n\|\}$ is a subsequence of $\bigcup_{j=1}^{N}\{\|x_n - T_{n+j}x_n\|\}$ for each $l = 1, 2, \cdots, N$, we have

$$\lim_{n\to\infty}\|x_n - T_l x_n\| = 0.$$

Since E is uniformly convex, every bounded subset of E is weakly compact. Since $\{x_n\}$ is a bounded sequence in K, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q \in K$. Hence we have

$$\lim_{n_k\to\infty}\|x_{n_k} - T_l x_{n_k}\| = 0, \quad \forall l = 1, 2, \cdots, N.$$

By Tool 1, we have that $(I - T_l)q = 0$, i.e., $q \in F(T_l)$. By the arbitrariness of $l \in \{1, 2, \cdots, N\}$, we know that $q \in F = \bigcap_{l=1}^{N} F(T_l)$.

Next, we prove that $\{x_n\}$ converges weakly to q. Suppose the contrary, then there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_1 \in K$ and $q_1 \neq q$. Then, by the same method as given above, we can also prove that $q_1 \in F = \bigcap_{l=1}^{N} F(T_l)$. Taking $p = q$ and $p = q_1$. Then, by the same method as given above, two limits exist, i.e., $\lim_{n\to\infty}\|x_{n_k} - q\| = d_1$ and $\lim_{n\to\infty}\|x_{n_k} - q_1\| = d_2$, where d_1 and d_2 are two nonnegative numbers. By virtue of Opial's condition of E, we have

$$\begin{aligned}
d_1 = \limsup_{n\to\infty}\|x_{n_k} - q\| &< \limsup_{n_k\to\infty}\|x_{n_k} - q_1\| \\
&= \limsup_{n_j\to\infty}\left\|x_{n_j} - q_1\right\| \\
&< \limsup_{n_j\to\infty}\left\|x_{n_j} - q\right\| \\
&= d_1,
\end{aligned}$$

which is a contradiction. Hence $q = q_1$, This imply that $\{x_n\}$ converges weakly to q. This completes the proof. □

Remark 3.1. Theorem 2.1 extends and improves the corresponding results (Theorems 1.1∼1.3) of Chang [1] and Osilike [4] in several ways.

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