

A SYSTEM OF FIRST-ORDER IMPULSIVE FUZZY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we introduce a new system of first-order impulsive fuzzy differential equations. By using Banach fixed point theorem, we obtain some new existence and uniqueness theorems of solutions for this system of first-order impulsive fuzzy differential equations in the metric space of normal fuzzy convex sets with distance given by maximum of the Hausdorff distance between level sets.

1. Introduction

The differential equations with applications in Banach spaces and fuzzy differential equations under various initial and boundary conditions have been studied by many authors. In 1972, Chang and Zadeh [5] first introduced the concept of fuzzy derivative. Afterwards, the framework for the study of fuzzy differential equations has also been developed and the basic properties of solutions of fuzzy differential equations is available (see, for example, [1]-[4], [10], [12], [17] [18]-[20], [22], [29], [31], [33], [35], [36] and the references therein).

Recently, Nieto [29] proved a version of the classical Peano existence theorem for initial value problems for a fuzzy differential equation in the metric space of normal fuzzy convex sets with the distance given by the maximum of the Hausdorff distance between level sets. The results of Nieto [29] complements the existence and uniqueness result of Kaleva [18]. Further, Georgiou and Kougias [10] studied the following second-order problem:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & \forall t \in [t_0, T], \\ x(t_0) = k_1, \quad x'(t_0) = k_2, \end{cases}$$

with $f : [t_0, T] \times E^n \times E^n \rightarrow E^n$ continuous and k_1, k_2 real constants. Georgiou et al. [11] considered n th-order fuzzy differential equations with initial value

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conditions and proved the existence and uniqueness of solution for nonlinearities satisfying a Lipschitz condition. Huang and Lan [15] proved some new existence theorems of solution for a new class of first-order implicit ordinary differential equations in Banach spaces by using the directional Lipschitzian condition and Krasnoselskii's fixed point theorem.

Very recently, by using Banach fixed point theorem, Lan and Huang [24] obtained some new existence and uniqueness theorems of solutions for a class of initial value problems of nonlinear first order implicit fuzzy differential equations in the metric space of normal fuzzy convex sets E^n with distance given by maximum of the Hausdorff distance between level sets:

$$\begin{cases} x'(t) = f(t, x(t), \lambda x'(t)), \\ x(t_0) = x_0. \end{cases}$$

On the other hand, the theory of impulsive differential equations or implicit impulsive integro-differential equations has been emerging as an important area of investigation in recent years and has been developed very rapidly due to the fact that such equations find a wide range of applications modeling adequately many real processes observed in physics, chemistry, biology and engineering (see, for example, [14], [15], [25], [34] and the references therein). Correspondingly, applications of the theory of impulsive differential equations to different areas were considered by many authors and some basic results on impulsive differential equations have been obtained (see, for example, [6], [9], [16], [26]-[28], [32], [37]-[41] and the references therein). Furthermore, some basic results on impulsive fuzzy differential equations have also been studied by several authors, see [7], [13], [21], [23], [30], but the theory still remains to be developed.

Inspired and motivated by the above works, in this paper, by using Banach contraction mapping principle theorem, we obtain some new existence and uniqueness theorems of solutions for the following systems of first-order impulsive fuzzy differential equations in the metric space of normal fuzzy convex sets with distance given by maximum of the Hausdorff distance between level sets: Find $(x, y) : J \times J \rightarrow E^n \times E^n$ such that

$$(1.1) \quad \begin{cases} x'(t) = f(t, x(t), y'(t)), & t \neq t_k, \\ y'(t) = g(t, \lambda y(t)), & t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & (k = 1, 2, \dots, m), \\ \Delta y|_{t=t_k} = \hat{I}_k(y(t_k)), & (k = 1, 2, \dots, m), \\ x(t_0) = x_0, & y(t_0) = y_0, \end{cases}$$

where $J = [t_0, t_0 + a] \subset R = (-\infty, +\infty)$ is a compact interval, $f : J \times E^n \times E^n \rightarrow E^n$ and $g : J \times E^n \rightarrow E^n$ are continuous, E^n is the family of all fuzzy sets $u : R^n \rightarrow [0, 1]$, $x_0, y_0 \in E^n$, $I_k, \hat{I}_k \in C[E^n, E^n]$ ($k = 1, 2, \dots, m$), and $\lambda \geq 0$ is a constant.

If $\lambda = 0$, then the problem (1.1) becomes to finding $(x, y) : J \times J \rightarrow E^n \times E^n$ such that

$$(1.2) \quad \begin{cases} x'(t) = f(t, x(t), y'(t)), & t \neq t_k, \\ y'(t) = g(t), & t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & (k = 1, 2, \dots, m), \\ \Delta y|_{t=t_k} = \hat{I}_k(y(t_k)), & (k = 1, 2, \dots, m), \\ x(t_0) = x_0, & y(t_0) = y_0, \end{cases}$$

2. Preliminaries

Let $\mathcal{P}_k(R^n)$ denote the family of non-empty compact, convex subsets of R^n . If $\alpha, \beta \in R$ and $A, B \in \mathcal{P}_k(R^n)$

$$\alpha(A + B) = \alpha A + \alpha B,$$

$$\alpha(\beta A) = (\alpha\beta)A, \quad 1 \cdot A = A$$

and, if $\alpha, \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$. For $A, B \in \mathcal{P}_k(R^n)$, the Hausdorff metric is defined as

$$d(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}.$$

A fuzzy set in R^n is a function with domain R^n and values in $[0, 1]$, i.e., an element of $[0, 1]^{R^n}$ (see [42]). Let $u, v \in [0, 1]^{R^n}$. Then we have (see [42])

(a) u is contained in v denoted by $u \leq v$ if and only if $u(x) \leq v(x)$ for all $x \in R^n$;

(b) $u \wedge v \in [0, 1]^{R^n}$ by $(u \wedge v)(x) = \min\{u(x), v(x)\}$ for all $x \in R^n$ (intersection);

(c) $u \vee v \in [0, 1]^{R^n}$ by $(u \vee v)(x) = \max\{u(x), v(x)\}$ for all $x \in R^n$ (union);

(d) $u^c \in [0, 1]^{R^n}$ by $u^c(x) = 1 - u(x)$ for all $x \in R^n$.

Denote by $E^n = \{u : R^n \rightarrow [0, 1] \text{ such that } u \text{ satisfies (i) to (iv) mentioned below}\}$:

(1) u is normal, that is, there exists an $x_0 \in R$ such that $u(x_0) = 1$;

(2) u is fuzzy convex, that is, for $x, y \in R^n$ and $0 \leq \nu \leq 1$,

$$u(\nu x + (1 - \nu)y) \geq \min\{u(x), u(y)\};$$

(3) u is upper semicontinuous;

(4) $[u]^0 = \{x \in R^n : u(x) > 0\}$ is compact.

Thus, if $u \in E^n$, then it follows from (1)-(4) that, for each $\alpha \in (0, 1]$, the α -level set

$$[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$$

is a nonempty compact convex subset of R^n , that is, $[u]^\alpha \in \mathcal{P}_k(R^n)$ for all $0 \leq \alpha \leq 1$. Further, define $D : E^n \times E^n \rightarrow [0, +\infty)$ as

$$D(u, v) = \sup\{d([u]^\alpha, [v]^\alpha) : \alpha \in [0, 1]\}.$$

It is well known that D is a metric in E^n and (E^n, D) is a complete metric space. Moreover, if $u, v, w \in E^n$ and $\lambda > 0$, then the addition and (positive) scalar multiplication in E^n are defined in terms of the α -level sets by

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [\lambda \cdot u]^\alpha = \lambda[u]^\alpha, \quad \forall \alpha \in [0, 1]$$

and D has a linear structure in the sense that

$$D(u + w, v + w) = D(u, v), \quad D(\lambda u, \lambda v) = \lambda D(u, v).$$

Note that (E^n, D) is not a vector space but it can be embedded isomorphically as a cone in a Banach space (see [20]).

Let $J = [t_0, t_0 + a]$ with $a > 0$ and $x, y \in E^n$. A mapping $F : J \rightarrow E^n$ is differentiable at $t \in J$ if there exists a $F'(t) \in E^n$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{F(t+h) - F(t)}{h}$$

and

$$\lim_{h \rightarrow 0^+} \frac{F(t) - F(t-h)}{h}$$

exist and are equal to $F'(t)$. Here the limits are taken in the metric space (E^n, D) . At the endpoints of J , we consider the one-sided derivatives.

Let $F : J \rightarrow E^n$. Then the integral of F over J denoted by $\int_J F(t)dt$, is defined levelwise by the equation

$$\begin{aligned} & \left[\int_J F(t)dt \right]^\alpha \\ &= \int_J F_\alpha(t)dt \\ &= \left\{ \int_J F(t)dt \mid F : J \rightarrow E^n \text{ is a measurable selection for } F_\alpha \right\}. \end{aligned}$$

We say that a mapping $F : J \rightarrow E^n$ is strongly measurable if, for all $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha : J \rightarrow \mathcal{P}_k(E^n)$ is defined by $F_\alpha(t) = [F(t)]^\alpha$. Moreover, the following results (see [17]) will be useful in what follows.

Lemma 2.1. *If $F : J \rightarrow E^n$ is continuous, then it is integrable and the function*

$$G(t) = \int_{t_0}^t F(s)ds, \quad \forall t \in J$$

is differentiable and $G'(t) = F(t)$. Furthermore,

$$F(t) - F(t_0) = \int_{t_0}^t F'(s)ds.$$

3. Main Results

In this section, we are in a position to prove our main results concerning with the solutions of the first-order fuzzy differential equation systems (1.1) and (1.2).

Throughout this paper, let $J = [t_0, t_0 + a]$ (where $a > 0$), $t_0 < t_1 < \dots < t_m < t_0 + a < +\infty$, $J_0 = [t_0, t_1]$, $J_1 = (t_1, t_2]$, \dots , $J_k = (t_k, t_{k+1}]$, \dots , $J_m = (t_m, t_0 + a]$ and

$$PC^1(J, E^n) = \{x : x \text{ is a map from } J \text{ into } E^n \text{ such that } x(t) \text{ is continuously differential on } (t_k, t_{k+1}), \text{ left continuous at } t_k, \text{ and } x(t_k^+), x'(t_k^-), x'(t_k^+) \text{ exists, } k = 1, 2, \dots, m\},$$

where $x(t_k^+)$ represents the right limits of $x(t)$ at $t = t_k$, and $x'(t_k^-)$ and $x'(t_k^+)$ represent the left and right derivatives of $x(t)$ at $t = t_k$, respectively. For all $x \in PC^1(J, E^n)$, by virtue of the mean value theorem,

$$x(t_k) - x(t_k - h) \in h\overline{co}\{x'(t) : t_k - h < t < t_k\} \quad (h > 0),$$

it is easy to see that the left derivative $x'_-(t_k)$ exists and

$$x'_-(t_k) = \lim_{h \rightarrow 0^+} h^{-1}[x(t_k) - x(t_k - h)] = x'(t_k^-).$$

In the sequel, $x'(t_k)$ is understood as $x'_-(t_k)$. Further, we define $H(x, y)$ by

$$(3.1) \quad H(x, y) = \sup_{t \in J} \{D(x(t), y(t)) + D(x'(t), y'(t))\}$$

for all $x, y \in PC^1(J, E^n)$, where $\Gamma > 0$ is a constant. Then, by using the same method as in [17], it is clear that $(PC^1(J, E^n), H)$ is a complete metric space.

By using Lemma 2.1 and Lemma 2.1 of [25], it is easy to prove the following lemma.

Lemma 3.1. *Assume that $f, g : J \times E^n \times E^n \rightarrow E^n$ is continuous. Then a mapping $(x, y) : J \times J \rightarrow E^n \times E^n$ is a solution of the problem (1.1) in $PC^1(J, E^n)$ if and only if x satisfies the following impulsive integral equation*

$$\begin{cases} x(t) = x_0 e^{-M(t-t_0)} + \int_{t_0}^t e^{-M(t-s)} [f(s, x(s), y'(s)) + Mx(s)] ds \\ \quad + \sum_{t_0 < t_k < t} e^{-M(t-t_k)} I_k(x(t_k)), \\ y(t) = y_0 e^{-M(t-t_0)} + \int_{t_0}^t e^{-M(t-s)} [g(s, \lambda y(s)) + My(s)] ds \\ \quad + \sum_{t_0 < t_k < t} e^{-M(t-t_k)} \hat{I}_k(y(t_k)), \end{cases}$$

where $M > 0$ is a constant.

Theorem 3.1. *Suppose that $f : J \times E^n \times E^n \rightarrow E^n$ and $g : J \times E^n \rightarrow E^n$ is continuous. If, for all $x_i, y_i, z_i : J \rightarrow E^n$ ($i = 1, 2$), there exist nonnegative*

constants ρ, ϱ, ϖ and b_i ($i = 1, 2$) such that

$$(3.2) \quad \begin{cases} D(f(t, x_1(t), y_1(t)), f(t, x_2(t), y_2(t))) \\ \leq \rho D(x_1(t), x_2(t)) + \varrho D(y_1(t), y_2(t)), \\ D(g(t, z_1(t)), g(t, z_2(t))) \leq \varpi D(z_1(t), z_2(t)), \quad \forall t \in J, \end{cases}$$

$$(3.3) \quad \begin{cases} D(I_k(x_1(t)), I_k(x_2(t))) \leq b_1 D(x_1(t), x_2(t)), \\ D(\hat{I}_k(y_1(t)), \hat{I}_k(y_2(t))) \leq b_2 D(y_1(t), y_2(t)), \quad \forall k = 1, 2, \dots, m. \end{cases}$$

Then problem (1.1) has a unique solution on J .

Proof. Let $(\tau_1, \bar{x}(t), \bar{y}(t)) \in J \times E^n \times E^n$ be arbitrary and $\delta > 0$ be a constant, $\sigma = \max\{mb_1 + h(\rho + M) + M(1 - mb_1 - h(\rho + M)) + \rho, mb_2 + h(\lambda\varpi + M) + M(1 - mb_2 - h(\lambda\varrho + M)) + \lambda\varrho, h\varrho + \varrho(1 - hM)\} < 1$, where $h = \frac{e^{M\delta} - 1}{M}$ and $\lambda > 0$ is a constant. We will first show that the initial value problem

$$(3.4) \quad \begin{cases} x'(t) = f(t, x(t), y'(t)), \\ y'(t) = g(t, \lambda y(t)), \quad \forall t \in J_1, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \\ \Delta y|_{t=t_k} = \hat{I}_k(y(t_k)), \quad (k = 1, 2, \dots, m), \\ x(\tau_1) = \bar{x}, \quad y(\tau_1) = \bar{y}, \end{cases}$$

has a unique solution on $J_1 = [\tau_1, \tau_1 + \delta]$. For any $x, y \in PC^1(J, E^n)$, define (Ax, Gy) on $J_1 \times J_1$ by the equation

$$(3.5) \quad \begin{cases} A(x(t)) = \bar{x}e^{-M(t-\tau_1)} + \int_{\tau_1}^t e^{-M(t-s)} [f(s, x(s), y'(s)) + Mx(s)] ds \\ \quad + \sum_{\tau_1 < t_k < t} e^{-M(t-t_k)} I_k(x(t_k)), \\ G(y(t)) = \bar{y}e^{-M(t-\tau_1)} + \int_{\tau_1}^t e^{-M(t-s)} [g(s, \lambda y(s)) + My(s)] ds \\ \quad + \sum_{\tau_1 < t_k < t} e^{-M(t-t_k)} \hat{I}_k(y(t_k)). \end{cases}$$

Now, define $\|\cdot\|_1$ on $PC^1(J, E^n) \times PC^1(J, E^n)$ by

$$\|(x, y)\|_1 = \|x\| + \|y\|, \quad \forall (x, y) \in PC^1(J, E^n) \times PC^1(J, E^n).$$

It is easy to see that $(PC^1(J, E^n) \times PC^1(J, E^n), \|\cdot\|_1)$ is a Banach space (see [8]). In the sequel, we prove that $F : PC^1(J, E^n) \times PC^1(J, E^n) \rightarrow PC^1(J, E^n) \times PC^1(J, E^n)$ is a contraction mapping. In deed, for any given $(x, y) \in PC^1(J, E^n) \times PC^1(J, E^n)$ and $t \neq t_k, k = 1, 2, \dots, m$, it follows from (3.5) that

$$(3.6) \quad (Ax)'(t) = -MA(x(t)) + Mx(t) + f(t, x(t), y'(t)),$$

$$(3.7) \quad (Gy)'(t) = -MG(y(t)) + My(t) + g(t, \lambda y(t)),$$

and so $F(x, y) = (Ax, Gy) \in PC^1(J, E^n) \times PC^1(J, E^n)$, i.e., F is a mapping from $PC^1(J, E^n) \times PC^1(J, E^n)$ into $PC^1(J, E^n) \times PC^1(J, E^n)$. By virtue of (3.2) to (3.4) and (3.6), for any $(x_1, y_1), (x_2, y_2) \in PC^1(J, E^n) \times PC^1(J, E^n)$,

$$\begin{aligned}
 & D(A(x_1(t)), A(x_2(t))) \\
 & \leq \int_{\tau_1}^{\tau_1+\delta} e^{-M(t-s)} [D(f(s, x_1(t), y_1'(s)), f(s, x_2(s), y_2'(s))) \\
 & \quad + MD(x_1(s), x_2(s))] ds + \sum_{\tau_1 < t_k < t} e^{-M(t-t_k)} D(I_k(x_1(t_k)), I_k(x_2(t_k))) \\
 (3.8) \quad & = e^{-Mt} \int_{\tau_1}^{\tau_1+\delta} e^{Ms} [D(f(s, x_1(t), y_1'(s)), f(s, x_2(s), y_2'(s))) \\
 & \quad + MD(x_1(s), x_2(s))] ds + \sum_{\tau_1 < t_k < t} e^{-M(t-t_k)} D(I_k(x_1(t_k)), I_k(x_2(t_k))) \\
 & \leq e^{-Mt} \int_{\tau_1}^{\tau_1+\delta} e^{Ms} [(\rho + M)D(x_1(s), x_2(s)) + \varrho D(y_1'(s), y_2'(s))] ds \\
 & \quad + B(t),
 \end{aligned}$$

where

$$\begin{aligned}
 B(t) &= \sum_{\tau_1 < t_k < t} b_1 e^{-M(t-t_k)} D(x_1(t_k), x_2(t_k)) \\
 &\leq b_1 D(x_1(t), x_2(t)) \sum_{t_0 < t_k < t} e^{-M(t-t_k)},
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 B(t) &\leq b_1 D(x_1(t), x_2(t)) \sum_{t_0 < t_k < t} e^{-M(t-t_k)} \\
 &= b_1 D(x_1(t), x_2(t)) \sum_{t_0 < t_k < t} e^{-M(t-t_k)},
 \end{aligned}$$

and so

$$(3.9) \quad \sup_{t \in J} e^{-\Gamma t} B(t) \leq b_1 D(x_1(t), x_2(t)) \max_{1 \leq k \leq m} \{C_k\},$$

where, for all $1 \leq k \leq m$,

$$\begin{aligned}
 C_k &= \sup_{t \in J_k} \sum_{t_0 < t_j < t} e^{-M(t-t_j)} = \sup_{t \in J_k} [\sum_{j=1}^{k-1} e^{-M(t-t_j)} + e^{-M(t-t_k)}] \\
 &\leq D_k + E_k
 \end{aligned}$$

and

$$D_k = \sup_{t \in J_k} \sum_{j=1}^{k-1} e^{-M(t-t_j)}, \quad E_k = \sup_{t \in J_k} e^{-M(t-t_k)}.$$

For all $1 \leq j \leq k-1$, setting $\nu_j(t) = e^{-M(t-t_j)}$ and $\delta_* = \min\{t_{k+1} - t_k | 1 \leq k \leq m\}$, we have

$$\nu_j(t) \leq e^{-M\delta_*}, \quad \forall t \in J_k = (t_k, t_{k+1}],$$

and so

$$D_k \leq \sum_{j=1}^{k-1} e^{-M\delta_*} \leq \sum_{j=1}^{m-1} e^{-M\delta_*} = (m-1)e^{-M\delta_*} < m-1, \quad \forall 1 \leq k \leq m.$$

Now we consider E_k : take $\nu_k(t) = e^{-M(t-t_k)}$, $t \in J_k$. Since $t - t_k \geq 0$ and $-M(t - t_k) \leq 0$, we know that $\nu_k(t) \leq 1$, i.e., $E_k \leq 1$. Hence, $E_k \leq 1$ and $C_k \leq m$ for all $1 \leq k \leq m$. It follows from (3.9) that

$$(3.10) \quad \sup_{t \in J} B(t) \leq mb_1 D(x_1(t), x_2(t)).$$

It follows from (3.8) and (3.10) that

$$(3.11) \quad \begin{aligned} & D(A(x_1(t)), A(x_2(t))) \\ & \leq e^{-Mt} \int_{\tau_1}^{\tau_1+\delta} e^{Ms} [(\rho_1 + M)D(x_1(s), x_2(s)) + \varrho_1 D(y'_1(s), y'_2(s))] ds \\ & \quad + mb_1 D(x_1(t), x_2(t)) \\ & \leq \frac{e^{-M(t-\tau_1)}}{M} (e^{M\delta} - 1) [(\rho + M)D(x_1(t), x_2(t)) + \varrho D(y'_1(t), y'_2(t))] ds \\ & \quad + mb_1 D(x_1(t), x_2(t)) \\ & \leq [mb_1 + h(\rho + M)] D(x_1(t), x_2(t)) + h\varrho D(y'_1(t), y'_2(t)), \end{aligned}$$

where $h = \frac{e^{M\delta} - 1}{M}$. Further, by (3.6) and the above proof, now we know

$$(3.12) \quad \begin{aligned} & D((Ax_1)'(t), (Ax_2)'(t)) \\ & = D(-MA(x_1(t)) + Mx_1(t) + f(t, x_1(t), y'_1(t)), \\ & \quad - MA(x_2(t)) + Mx_2(t) + f(t, x_2(t), y'_2(t))) \\ & = -MD(A(x_1(t)), A(x_2(t))) \\ & \quad + MD(x_1(t), x_2(t)) + D(f(t, x_1(t), y'_1(t)), f(t, x_2(t), y'_2(t))) \\ & \leq -M\{[mb_1 + h(\rho + M)]D(x_1(t), x_2(t)) + h\varrho D(y'_1(t), y'_2(t))\} \\ & \quad + MD(x_1(t), x_2(t)) + \rho D(x_1(t), x_2(t)) + \varrho D(y'_1(t), y'_2(t)) \\ & = [M(1 - mb_1 - h(\rho + M)) + \rho] D(x_1(t), x_2(t)) \\ & \quad + \varrho(1 - hM) D(y'_1(t), y'_2(t)). \end{aligned}$$

Similarly, by (3.2), (3.3), (3.5) and (3.7), we know

$$\begin{aligned}
 & D(G(y_1(t)), G(y_2(t))) \\
 & \leq \int_{\tau_1}^{\tau_1+\delta} e^{-M(t-s)} [D(g(s, \lambda y_1(s)), g(s, \lambda y_2(s))) \\
 & \quad + MD(y_1(s), y_2(s))] ds + \sum_{\tau_1 < t_k < t} e^{-M(t-t_k)} D(\hat{I}_k(y_1(t_k)), \hat{I}_k(y_2(t_k))) \\
 (3.13) \quad & \leq e^{-Mt} \int_{\tau_1}^{\tau_1+\delta} e^{Ms} (\lambda\varpi + M) D(y_1(s), y_2(s)) ds + mb_2 D(y_1(t), y_2(t)) \\
 & \leq \frac{e^{-M(t-\tau_1)}}{M} (e^{M\delta} - 1) (\lambda\varpi + M) D(y_1(t), y_2(t)) + mb_2 D(y_1(t), y_2(t)) \\
 & \leq [mb_2 + h(\lambda\varpi + M)] D(y_1(t), y_2(t))
 \end{aligned}$$

and

$$\begin{aligned}
 & D((Gy_1)'(t), (Gy_2)'(t)) \\
 & = D(-MG(y_1(t)) + My_1(t) + g(t, \lambda y_1(t)), \\
 & \quad - MG(y_2(t)) + My_2(t) + g(t, \lambda y_2(t))) \\
 (3.14) \quad & = -MD(G(y_1(t)), G(y_2(t))) \\
 & \quad + MD(y_1(t), y_2(t)) + D(g(t, \lambda y_1(t)), g(t, \lambda y_2(t))) \\
 & \leq -M[mb_2 + h(\lambda\varpi + M)] D(y_1(t), y_2(t)) \\
 & \quad + MD(x_1(t), x_2(t)) + \lambda\varrho D(y_1(t), y_2(t)) \\
 & = [M(1 - mb_2 - h(\lambda\varrho + M)) + \lambda\varrho] D(y_1(t), y_2(t)).
 \end{aligned}$$

It follows from (3.11)-(3.14) that

$$\begin{aligned}
 & H(F(x_1, y_1), F(x_2, y_2)) \\
 & = \sup_{t \in J_1} \{D(F(x_1(t), y_1(t)), F(x_2(t), y_2(t))) + D((F(x_1, y_1))'(t), (F(x_2, y_2))'(t))\} \\
 & = \sup_{t \in J_1} \{D((A(x_1(t)), G(y_1(t))), (A(x_2(t)), G(y_2(t)))) \\
 & \quad + D(((Ax_1)'(t), (Gy_1)'(t)), ((Ax_2)'(t), (Gy_2)'(t)))\} \\
 & \leq D(A(x_1(t)), A(x_2(t))) + D(G(y_1(t)), G(y_2(t))) \\
 & \quad + D((Ax_1)'(t), (Ax_2)'(t)) + D((Gy_1)'(t), (Gy_2)'(t)) \\
 & \leq [mb_1 + h(\rho + M)] D(x_1(t), x_2(t)) + h\varrho D(y_1'(t), y_2'(t)) \\
 & \quad + [mb_2 + h(\lambda\varpi + M)] D(y_1(t), y_2(t)) \\
 & \quad + [M(1 - mb_1 - h(\rho + M)) + \rho] D(x_1(t), x_2(t)) + \varrho(1 - hM) D(y_1'(t), y_2'(t)) \\
 & \quad + [M(1 - mb_2 - h(\lambda\varrho + M)) + \lambda\varrho] D(y_1(t), y_2(t)) \\
 & \leq \sigma H((x_1, y_1), (x_2, y_2)),
 \end{aligned}$$

where $\sigma = \max\{mb_1 + h(\rho + M) + M(1 - mb_1 - h(\rho + M)) + \rho, mb_2 + h(\lambda\varpi + M) + M(1 - mb_2 - h(\lambda\varrho + M)) + \lambda\varrho, h\varrho + \varrho(1 - hM)\}$.

Therefore, by Banach fixed point theorem, F has a unique fixed point, which by Lemma 3.1 is the desired solution to the problem (1.1).

Express J as a union of a finite family of intervals J_k with the length of each interval less than δ . The preceding paragraph guarantees the existence of a unique solution to problem (1.1) on each interval J_k . Piecing these solutions together gives us the unique solution on the whole interval J . This completes the proof. \square

Remark 3.1. If $\varrho = 0$ in (3.2), then we can obtain the corresponding results from the problem (1.1).

Theorem 3.2. Let $f : J \times E^n \times E^n \rightarrow E^n$ and $g : J \times E^n \rightarrow E^n$ be continuous mappings. Assume that for all $x_i, y_i, z_i : J \rightarrow E^n$ ($i = 1, 2$), there exist nonnegative constants ρ, ϱ, ϖ and b_i ($i = 1, 2$) such that, for all $t \in J$, $\alpha \in [0, 1]$ and $k = 1, 2, \dots, m$,

$$\begin{cases} d([f(t, x_1(t), y_1(t))]^\alpha, [f(t, x_2(t), y_2(t))]^\alpha) \\ \leq \rho d([x_1(t)]^\alpha, [x_2(t)]^\alpha) + \varrho d([y_1(t)]^\alpha, [y_2(t)]^\alpha), \\ d([g(t, z_1(t))]^\alpha, [g(t, z_2(t))]^\alpha) \leq \varpi d([z_1(t)]^\alpha, [z_2(t)]^\alpha), \\ \\ \begin{cases} d([I_k(x_1(t))]^\alpha, [I_k(x_2(t))]^\alpha) \leq b_1 d([x_1(t)]^\alpha, [x_2(t)]^\alpha), \\ d([\hat{I}_k(y_1(t))]^\alpha, [\hat{I}_k(y_2(t))]^\alpha) \leq b_2 d([y_1(t)]^\alpha, [y_2(t)]^\alpha), \end{cases} \end{cases}$$

Then the problem (1.1) has a unique solution on J .

Proof. In fact, we have

$$\begin{aligned} & D(f(t, x_1(t), y_1(t)), f(t, x_2(t), y_2(t))) \\ &= \sup\{d([f(t, x_1(t), y_1(t))]^\alpha, [f(t, x_2(t), y_2(t))]^\alpha) : \alpha \in [0, 1]\} \\ &\leq \rho \sup\{d([x_1(t)]^\alpha, [x_2(t)]^\alpha) : \alpha \in [0, 1]\} \\ &\quad + \varrho \sup\{d([y_1(t)]^\alpha, [y_2(t)]^\alpha) : \alpha \in [0, 1]\} \\ &= \rho D(x_1(t), x_2(t)) + \varrho D(y_1(t), y_2(t)), \\ \\ D(g(t, z_1(t)), g(t, z_2(t))) &= \sup\{d([g(t, z_1(t))]^\alpha, [g(t, z_2(t))]^\alpha) : \alpha \in [0, 1]\} \\ &\leq \varpi \sup\{d([z_1(t)]^\alpha, [z_2(t)]^\alpha) : \alpha \in [0, 1]\} \\ &= \varpi D(z_1(t), z_2(t)), \\ \\ D(I_k(x_1(t)), I_k(x_2(t))) &= \sup\{d([I_k(x_1(t))]^\alpha, [I_k(x_2(t))]^\alpha) : \alpha \in [0, 1]\} \\ &\leq b_1 \sup\{d([x_1(t)]^\alpha, [x_2(t)]^\alpha) : \alpha \in [0, 1]\} \\ &= b_1 D(x_1(t), x_2(t)), \\ \\ D(\hat{I}_k(y_1(t)), \hat{I}_k(y_2(t))) &= \sup\{d([\hat{I}_k(y_1(t))]^\alpha, [\hat{I}_k(y_2(t))]^\alpha) : \alpha \in [0, 1]\} \\ &\leq b_2 \sup\{d([y_1(t)]^\alpha, [y_2(t)]^\alpha) : \alpha \in [0, 1]\} \\ &= b_2 D(y_1(t), y_2(t)). \end{aligned}$$

Thus, by Theorem 3.1 we know that problem (1.1) has a unique solution on J . This completes the proof. \square

Remark 3.2. Using the same method as Theorems 3.1 and 3.2, we can consider initial value problems (1.2) and get the corresponding conclusions.

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