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CHARACTERIZATIONS OF THE WEIBULL DISTRIBUTION BY THE INDEPENDENCE OF THE UPPER RECORD VALUES

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ABSTRACT. This paper presents characterizations of the Weibull distribution by the independence of record values. We prove that $X \in WEI(\alpha)$, if and only if $\dfrac{X_{U(n+1)}}{X_{U(n+1)} + X_{U(n)}}$ and $X_{U(n+1)}$ for $n \geq 1$ are independent or $\dfrac{X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $X_{U(n+1)}$ for $n \geq 1$ are independent. And also we establish that $X \in WEI(\alpha)$, if and only if $\dfrac{X_{U(n+1)} - X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $X_{U(n+1)}$ for $n \geq 1$ are independent.

1. INTRODUCTION

Let $\{X_n, \ n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Suppose $Y_n = \max(\min)\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper(lower) record value of this sequence, if $Y_j > (<) \ Y_{j-1}$ for $j > 1$. By definition, X_1 is an upper as well as a lower record value.

The indices at which the upper record values occur are given by the record times $\{U(n), \ n \geq 1\}$, where $U(n) = \min\{j | j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$.

We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, \ n \geq 1\}$ of i.i.d. random variables.

A continuous random variable X is said to have the Weibull distribution with parameter $\alpha > 0$ if it has a cdf $F(x)$ of the form

$$F(x) = \begin{cases} 1 - e^{-x^{\alpha}}, & x > 0, \alpha > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

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A notation that designates that X has the cdf (1) is $X \in WEI(\alpha)$.

Some characterizations by the independence of the upper record values are known. Ahsanullah(1995) characterized that $F(x) = 1 - e^{-\frac{x}{\sigma}}$, $x > 0, \sigma > 0$, if and only if $X_{U(n)} - X_{U(m)}$ and $X_{U(m)}$, $0 < m < n$ are independent. Moreover Ahsanullah(1995) characterized if $F(x) = 1 - \left(\frac{\alpha}{x}\right)^{\beta}$, α, $\beta > 0$, then $\frac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$, $0 < m < n$ are independent. Also, Lee and Chang(2005) characterized that $F(x) = 1 - e^{-\frac{x}{\sigma}}$, $x > 0, \sigma > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n+1)}}$ and $X_{U(n)}$, $n \geq 1$ are independent and $F(x) = 1 - x^{-\theta}$, $x > 1, \theta > 0$, if and only if $\frac{X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$, $n \geq 1$ are independent. Above results are characterizied by the simple quotient form of the upper record values. By the expansion form of denominator, we can get the the Weibull distribution.

In this paper, we will give characterizations of the Weibull distribution by the independence of the upper record values.

2. Main Results

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for all $x > 0$. Then $F(x) = 1 - e^{-x^{\alpha}}$ for all $x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(n+1)}}{X_{U(n+1)} + X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.*

Proof. If $F(x) = 1 - e^{-x^{\alpha}}$ for all $x > 0$ and $\alpha > 0$, then the joint pdf $f_{n,n+1}(x, y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x, y) = \frac{\alpha^2 x^{\alpha n-1} y^{\alpha-1} e^{-y^{\alpha}}}{\Gamma(n)}$$

for all $0 < x < y$, $\alpha > 0$ and $n \geq 1$.

Consider the functions $V = \frac{X_{U(n+1)}}{X_{U(n+1)} + X_{U(n)}}$ and $W = X_{U(n+1)}$. It follows that $x_{U(n)} = \frac{(1-v)w}{v}$, $x_{U(n+1)} = w$ and $|J| = \frac{w}{v^2}$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{\alpha^2 (1-v)^{\alpha n-1} w^{\alpha(n+1)-1} e^{-w^{\alpha}}}{\Gamma(n) v^{\alpha n+1}} \tag{2}$$

for all $\frac{1}{2} < v < 1$, $w > 0$, $\alpha > 0$ and $n \geq 1$.

The marginal pdf $f_V(v)$ of V is given by

$$
\begin{aligned}
f_V(v) &= \int_0^\infty f_{V,W}(v,w)\,dw \\
&= \frac{\alpha^2\,(1-v)^{\alpha n-1}}{\Gamma(n)\,v^{\alpha n+1}} \int_0^\infty w^{\alpha(n+1)-1} e^{-w^\alpha}\,dw \qquad (3)\\
&= \frac{\alpha n\,(1-v)^{\alpha n-1}}{v^{\alpha n+1}}
\end{aligned}
$$

for all $\frac{1}{2} < v < 1$, $\alpha > 0$ and $n \geq 1$.

Also, the pdf $f_W(w)$ of W is given by

$$
\begin{aligned}
f_W(w) &= \frac{1}{\Gamma(n+1)}(R(w))^n f(w) \qquad (4)\\
&= \frac{\alpha\, w^{\alpha(n+1)-1} e^{-w^\alpha}}{\Gamma(n+1)}
\end{aligned}
$$

for all $w > 0$, $\alpha > 0$ and $n \geq 1$.

From (2), (3) and (4), we obtain $f_V(v)f_W(w) = f_{V,W}(v,w)$.

Hence $V = \dfrac{X_{U(n+1)}}{X_{U(n+1)} + X_{U(n)}}$ and $W = X_{U(n+1)}$ are independent for $n \geq 1$.

Now we will prove the sufficient condition. The joint pdf $f_{n,n+1}(x,y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x,y) = \frac{1}{\Gamma(n)}(R(x))^{n-1} r(x) f(y)$$

for all $0 < x < y$, $\alpha > 0$ and $n \geq 1$, where $R(x) = -ln(1 - F(x))$ and $r(x) = \dfrac{d}{dx}(R(x)) = \dfrac{f(x)}{1-F(x)}$.

Let us use the transformation $V = \dfrac{X_{U(n+1)}}{X_{U(n+1)} + X_{U(n)}}$ and $W = X_{U(n+1)}$. The Jacobian of the transformation is $|J| = \dfrac{w}{v^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{w}{\Gamma(n)\,v^2}\left(R\left(\frac{(1-v)w}{v}\right)\right)^{n-1} r\left(\frac{(1-v)w}{v}\right) f(w) \qquad (5)$$

for all $\frac{1}{2} < v < 1$, $w > 0$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{1}{\Gamma(n+1)}(R(w))^n f(w) \qquad (6)$$

for all $w > 0$ and $n \geq 1$.

From (5) and (6), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{n\,w}{v^2} \frac{\left(R\left(\frac{(1-v)w}{v}\right)\right)^{n-1} r\left(\frac{(1-v)w}{v}\right)}{(R(w))^n}$$

for all $\frac{1}{2} < v < 1$, $w > 0$ and $n \geq 1$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(- \left(\frac{R\left(\frac{(1-v)w}{v}\right)}{R(w)} \right)^n \right)$$

Since V and $W$are independent, the pdf $f_V(v)$ of V is a function of v only. Thus we must have

$$R\left(\frac{(1-v)w}{v}\right) = R\left(\frac{1-v}{v}\right) R(w) \tag{7}$$

for all $0 < \frac{1-v}{v} < 1$ and $w > 0$.

By the functional equations [see, Aczel (1996)], the only continous solution of (7) with the boundary condition $R(0) = 0$ is

$$R(x) = x^\alpha$$

for all $x > 0$ and $\alpha > 0$. Thus we have

$$F(x) = 1 - e^{-x^\alpha}$$

for all $x > 0$ and $\alpha > 0$.

This completes the proof. □

Remark 2.1. Let $\frac{X_{U(n+1)}}{X_{U(n+1)} + X_{U(n)}} = Y$. Then we can write $\frac{X_{U(n)}}{X_{U(n+1)} + X_{U(n)}} = 1-Y$. By the property of independence and the result of theorem 2.1, $1 - Y$ and $X_{U(n+1)}$ are independent. Hence we obtain that $F(x) = 1 - e^{-x^\alpha}$ for all $x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.

Remark 2.2. Let $\frac{X_{U(n+1)}}{X_{U(n+1)} + X_{U(n)}} = Y$. Then we can write $\frac{X_{U(n+1)} - X_{U(n)}}{X_{U(n+1)} + X_{U(n)}} = 2Y - 1$. By the property of independence and the result of theorem 2.1, $2Y - 1$ and $X_{U(n+1)}$ are independent. We obtain that $F(x) = 1 - e^{-x^\alpha}$ for all $x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(n+1)} - X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.

REFERENCES

1. J. Aczel: *Lectures on Functional Equations and Their Applications.* Academic Press, NY, 1966.
2. M. Ahsanuallah: *Record Statistics.* Nova Science Publishers, Inc, Commack NY, 1995.
3. J. Galambos & S. Kotz: *Characterization of Probability Distributions. Lecture Notes in Mathematics.* No. 675, Springer-Verlag, NY, 1978.
4. M.Y. Lee & S.K. Chang: Characterizations based on the independence of the exponential and Pareto distributions by record values. *J. Appl. Math. & Computing.* **18** (2005), no. 1-2, 497-503.

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