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GENERALIZED VECTOR VARIATIONAL-LIKE INEQUALITIES WITH CORRESPONDING NON-SMOOTH VECTOR OPTIMIZATION PROBLEMS

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ABSTRACT. In [1], Mishra and Wang established relationships between vector variational-like inequality problems and non-smooth vector optimization problems under non-smooth invexity in finite-dimensional spaces. In this paper, we generalize recent results of Mishra and Wang to infinite-dimensional case.

1. INTRODUCTION

In [3], Yang discussed relationships between a solution of a vector variational inequality and a Pareto solution or a properly efficient solution of a vector optimization problem. He also showed that a vector variational inequality is a necessary and sufficient optimality condition for an efficient solution of the vector pseudolinear optimization problem in finite-dimensional spaces.

In 2006, Mishra and Wang [1] established relationships between vector vatiational-like inequality problems and non-smooth vector optimization problems under non-smooth invexity in finite-dimensional spaces. They also identified the critical points, the weakly efficient points and the solutions of the non-smooth weak vector variational-like inequality problems under non-smooth pseudo-invexity assumptions in finite-dimensional spaces.

This paper deals with the infinite-dimensional case of Mishra and Wang's results in [1].

2. PRELIMINARIES

Throughout this paper, X and Y are normed vector spaces, K is a nonempty

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subset of X and $\eta : K \times K \to K$ is a function. Two norms $\|\cdot\|_X$ in X and $\|\cdot\|_Y$ in Y are used as $\|\cdot\|$ together. We denote $\langle\cdot,\cdot\rangle$ the canonical duality in the product $K^* \times K$, that is $\langle w^*, x\rangle = w^*(x)$ for $x \in K$ and $w^* \in K^*$, the topological dual of K.

Definition 2.1. A function $f : K \to Y$ is said to be *Lipschitz near* $x \in K$ if for some $\alpha > 0$

$$\|f(y) - f(z)\| \le \alpha\|y - z\|$$

for y, $z \in N_x$, the neighborhood of x, and *locally Lipschitz on* K if it is Lipschitz near any point x of K.

Definition 2.2. If a function $f : K \to Y$ is Lipschitz near $x \in K$, then the generalized derivative (in the sense of Clarke) of f at $x \in K$ into the direction $v \in K$ is given by

$$f^\circ(x,v) := \overline{\lim_{\substack{y\to x\\ y\in K\\ t\downarrow 0}}} \frac{\|f(y+tv)\| - \|f(y)\|}{t},$$

and the *Clarke's generalized gradient* of f at $x \in K$ is defined as

$$\partial f(x) := \{w^* \in K^* : f^\circ(x,v) \ge \langle w^*, v\rangle \text{ for all } v \in K\}.$$

Remark 2.1. For any $v \in K$,

$$f^\circ(x,v) = \sup_{w^*\in\partial f(x)} \langle w^*, v\rangle.$$

Definition 2.3. A set K is said to be *invex at* $u \in K$ with respect to η if $u + t\cdot\eta(x,u) \in K$ for $x \in K$, $t \in [0,1]$. K is said to be *invex with respect to* η if it is invex at every point of K.

Definition 2.4. Let K be a nonempty closed and invex subset of X and $f : K \to Y$ be a non-differentiable function.

(i) f is *strictly-invex with respect to* η if

$$\|f(x)\| - \|f(u)\| > \langle w^*, \eta(x,u)\rangle$$

for x, $u \in K$ and $w^* \in \partial f(x)$.

(ii) f is *invex with respect to* η if

$$\|f(x)\| - \|f(u)\| \ge \langle w^*, \eta(x,u)\rangle$$

for x, $u \in K$ and $w^* \in \partial f(x)$.

(iii) f is *pseudo-invex with respect to* η if

$$\|f(x)\| - \|f(u)\| < 0 \quad \text{implies} \quad \langle w^*, \eta(x,u)\rangle < 0$$

for x, $u \in K$ and $w^* \in \partial f(x)$.

Definition 2.5. Let K be an open subset of X and $f : K \to Y$ be a function.

(i) A point $\bar{x} \in K$ is called an *efficient* (*Pareto*) *solution*, if there exists no $y \in K$ such that

$$\|f(y)\| \leq \|f(\bar{x})\|,$$

(ii) A point $\bar{x} \in K$ is called a *weakly efficient* (*Pareto*) *solution*, if there exists no $y \in K$ such that

$$\|f(y)\| < \|f(\bar{x})\|.$$

In this paper, we consider the following three problems;

(1) Non-smooth vector optimization problem (VOP) of finding min $\|f(x)\|$ subject to $x \in K$.

(2) Vector variational-like inequality problem (VVLIP) for non-smooth case of finding a point $y \in K$ such that there exists no $x \in K$ satisfying

$$\langle w^*, \eta(x,y)\rangle \leq 0 \quad \text{for} \quad w^* \in \partial f(y).$$

(3) Weak vector variational-like inequality problem (WVVLIP) for non-smooth case of finding a point $y \in K$ such that there exists no $x \in K$ satisfying

$$\langle w^*, \eta(x,y)\rangle < 0 \quad \text{for} \quad w^* \in \partial f(y).$$

3. Main Results

In this section, we generalize the results of Mishra and Wang [1] and extend the results given by Ruiz-Carzon et al. [2] to the non-smooth case.

Theorem 3.1. *Let a function $f : K(\subset X) \to Y$ be locally Lipschitz on K and invex with respect to η. If $y \in K$ solves VVLIP, then it is an efficient solution to the non-smooth VOP.*

Proof. If y is not an efficient solution to the non-smooth VOP, then there exists an $x \in K$ such that $\|f(x)\| - \|f(y)\| \leq 0$. Since f is invex with respect to η,

$$\langle w^*, \eta(x,y)\rangle \leq 0 \quad \text{for} \quad w^* \in \partial f(y),$$

which shows that y is not a solution of VVLIP. □

Theorem 3.2. *Let a function $\eta : K \times K \to Y$ satisfy $\eta(x,y) + \eta(y,x) = \bar{0}$, the zero vector of X, for x, $y \in K$. Let a function $f : K(\subset X) \to Y$ be locally Lipschitz on*

K and strictly-invex with respect to η. If $y \in K$ is a weakly efficient solution for VOP, then it solves VVLIP.

Proof. If y does not solve VVLIP, then there exists an $x \in K$ such that

$$\langle w^*, \eta(x,y)\rangle \leq 0 \quad \text{for} \quad w^* \in \partial f(y).$$

Since f is strictly-invex with respect to η,

$$\|f(y)\| - \|f(x)\| > \langle w^*, \eta(y,x)\rangle \quad \text{for} \quad x, y \in K \text{ and } w^* \in \partial f(y).$$

So by the condition on η, we have

$$\|f(x)\| - \|f(y)\| < \langle w^*, \eta(x,y)\rangle \leq 0 \quad \text{for} \quad w^* \in \partial f(y).$$

Hence

$$\|f(x)\| - \|f(y)\| < 0,$$

which means that y is not a weakly efficient solution for VOP. □

Remark 3.1. Since an efficient solution is weakly efficient, if we replace a weakly efficient solution with an efficient solution in the condition of Theorem 3.2, we have the same result.

Theorem 3.3. *Let K be an invex subset of X. If $y \in K$ is a weakly efficient solution for VOP, then y solves WVVLIP.*

Proof. Let $y \in K$ be a weakly efficient solution for VOP. Then there exists no $x \in K$ such that

$$\|f(y + t\eta(x,y)\| - \|f(y)\| < 0, \quad 0 < t < 1$$

from the invexity of K. Hence

$$f^\circ(y, \eta(x,y)) = \overline{\lim_{t \downarrow 0}} \frac{\|f(y + t\eta(x,y)\| - \|f(y)\|}{t} \leq 0.$$

Consequently, there exists no $x \in K$ such that

$$\langle w^*, \eta(x,y)\rangle < 0 \quad \text{for} \quad w^* \in \partial f(y),$$

which means that y solves the WVVLIP. □

Theorem 3.4. *Let $f : K(\subset X) \to Y$ be locally Lipschitz on K and pseudo-invex with respect to η. If $y(\in K)$ solves the WVVLIP, then it is a weakly efficient solution to VOP.*

Proof. If y is not a weakly efficient solution to VOP, then there exists an $x \in K$ such that $\|f(x)\| < \|f(y)\|$. Hence

$$\langle w^*, \eta(x,y)\rangle < 0 \quad \text{for} \quad w^* \in \partial f(y)$$

from the pseudo-invexity of f with respect to η. So y is not a solution to WVVLIP. □

Theorem 3.5. *Let $f : K(\subset X) \to Y$ be locally Lipschitz on K and strictly-invex with respect to η. If $y(\in K)$ is a weakly efficient solution to VOP, then it is also an efficient solution to VOP.*

Proof. Suppose that y is not an efficient solution to VOP, then there exists an $x \in K$ such that $\|f(x)\| \le \|f(y)\|$. Since the non-smooth function f is strictly-invex with respect to η, we have

$$0 \ge \|f(x)\| - \|f(y)\| > \langle w^*, \eta(x,y)\rangle \quad \text{for} \quad w^* \in \partial f(y).$$

Consequently, y does not solve WVVLIP. Hence y is not a weakly solution to VOP by Theorem 3.4. □

Remark 3.2. Our results generalize and extend the results of Mishra and Wang [1] to the infinite-dimensional case.

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