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A GENERAL FIXED POINT THEOREM IN FUZZY METRIC SPACES VIA AN IMPLICIT FUNCTION

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ABSTRACT. We employ the notion of implicit functions to prove a general common fixed point theorem in fuzzy metric spaces besides adopting the idea of R-weak commutativity of type (P) in fuzzy setting. In process, several previously known results are deduced as special cases to our main result.

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1. Introduction

There do exist many situations wherein the distances between the points are rather inexact than being a single nonnegative real number which led to the introduction of probabilistic metric spaces and continues to be a subject of interest for the researchers of this domain. But if uncertainty is due to fuzziness rather than randomness, then in this situation concept of fuzzy metric spaces is relatively more suitable. Inspired from these observations, Deng [3], Erceg [4] and Kramosil and Michalek [13] introduced the notion of fuzzy metric spaces by generalizing the concept of the probabilistic metric spaces to the fuzzy situations.

On the other hand, Kaleva and Seikkala [11] generalized the notion of metric spaces by setting the distance between the points to be non-negative fuzzy numbers where triangle inequality is realized by defining an ordering in the set of fuzzy numbers. This natural way of defining fuzzy metric spaces has been exploited by several researchers of this domain especially metric fixed point theorists and by now there exists considerable literature on fixed point theorems in fuzzy metric spaces which includes [5, 7-12, 15, 18, 21]. In [24], Xia and Guo also redefined the fuzzy metric spaces using fuzzy scalers instead of fuzzy numbers or real numbers along with some results on completeness of fuzzy metric spaces.

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The approach of Xia and Guo [24] is more natural and soothing and as per our expectation, it will inspire further developments in near future.

In recent years, Popa [17] used implicit functions rather than contraction conditions to prove fixed point theorems in metric spaces whose strength lies in its unifying power as an implicit function can cover several contraction conditions at the same time which includes known as well as unknown contraction conditions. This fact is evident from examples furnished in Popa [17].

In 1999, Vasuki [22] proved the following theorem for a pair of R-weakly commuting mappings satisfying a Boyd and Wong [1] type contraction condition which is a fuzzy version of a result due to Pant [14, Theorem 1].

Theorem 1. *Let $(X, M, \star)$ be a complete fuzzy metric space and A and S be R-weakly commuting self mappings of X satisfying $A(X) \subset S(X)$ and $M(Ax, Ay, t) \geq r(M(Sx, Sy, t))$ for all $x, y \in X$, where $r : [0,1] \to [0,1]$ is continuous function such that $r(s) > s$ for each $0 < s < 1$. Suppose that one of A and S is continuous. Then A and S have a unique common fixed point in X.*

Here it may be pointed out that Theorem 1 has been further extended for two pairs of R-weakly commuting mappings by Chugh and Kumar [2] and Singh and Jain [20].

In this paper, we introduce a suitable implicit function to prove fixed point theorems in fuzzy metric spaces and also furnish several examples enjoying the format of our implicit function. We are not aware of any fixed point theorem proved via implicit functions in fuzzy metric spaces. In process, several previously known results due to Chugh and Kumar [2], Imdad and Ali [10], Singh and Jain [20] and Vasuki [22] can be deduced as a special case. Moreover, adopting R-weak commutativity of type (A_f), type (Ag) to fuzzy setting and to introduce R-weak commutativity of type (P) which are to be used to prove our results in this paper.

Our improvement in this paper is four fold which includes:

(i) relaxing the continuity requirement of all maps completely,

(ii) minimizing the commutativity requirement of the maps to the point of coincidence,

(iii) replacing the completeness requirement of the space by four alternative natural conditions,

(iv) replacing contraction condition with a suitable implicit function to prove our results.

2. Preliminaries

In what follows, we collect relevant definitions, results and examples to make our presentation as self-contained as possible.

Definition 1([25]). A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2([19]). A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $\{[0,1], \star\}$ is an Abelian topological monoid with unit 1 such that $a \star b \leq c \star d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0,1]$.

Definition 3([13]). The triplet $(X, M, \star)$ is a fuzzy metric space if X is an arbitrary set, $\star$ is a continuous t-norm, and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, .) : [0, \infty) \to [0,1]$ is left continuous for all $x, y, z \in X$ and $s, t > 0$.

In recent years, George and Veeramani [7] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [13] and defined Hausdorff topology of metric spaces which is later proved to be metrizable. They also showed that every metric induces a fuzzy metric and furnished the following example (in sense of George and Veeramani [7]).

Example 1. Every metric space induces a fuzzy metric space. Let (X, d) be a metric space. Define $a \star b = ab$ and $M(x, y, t) = \dfrac{kt^n}{kt^n + md(x, y)}$, $k, m, n, t \in \Re^+$. Then $(X, M, \star)$ is a fuzzy metric space. If we put $k = m = n = 1$, we get

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

The fuzzy metric induced by the metric d is referred as standard fuzzy metric.

Definition 4(cf. [9]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ converges to $x \in X$ if

$$\lim_{n \to \infty} M(x_n, x, t) = 1 \text{ for each } t > 0.$$

Recently, there is some debate on existing definitions of Cauchy sequences which are available in [21, 23] wherein the Cauchy sequences defined by Grabiec [9] are labeled as G-Cauchy sequences. But in order to prove our results, we adopt the definition of Cauchy sequence the sense of Vasuki and Veeramani [23].

Definition 5(cf. [9]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ is called G-Cauchy if $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and each $p > 0$. Moreover, $(X, M, \star)$ is called G-complete if every G-Cauchy sequence in X converges in X.

Definition 6. A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, \star)$ is said to be

(i) weakly commuting (cf. [22]) if $M(fgx, gfx, t) \geq M(fx, gx, t)$,

(ii) R-weakly commuting (cf. [22]) if there exists some $R > 0$ such that $M(fgx, gfx, t) \geq M(fx, gx, t/R)$,
(iii) R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that $M(fgx, ggx, t) \geq M(fx, gx, t/R)$,
(iv) R-weakly commuting mappings of type (Ag) if there exists some $R > 0$ such that $M(gfx, ffx, t) \geq M(fx, gx, t/R)$,
(v) R-weakly commuting mappings of type (P) if there exists some $R > 0$ such that $M(ffx, ggx, t) \geq M(fx, gx, t/R)$, for all $x \in X$ and $t > 0$.

Notice that Definition 6(iii) and Definition 6(iv) are inspired by Pathak et al. [16] whereas Definition 6(v) seems to be unreported.

Example 2(cf. [22]). Let $X = \Re$, the set of real numbers. Define $a \star b = ab$ and

$$M(x,y,t) = \begin{cases} \left(e^{\frac{|x-y|}{t}}\right)^{-1} & \text{for all } x, y \in X \text{ and } t > 0 \\ 0 & \text{for all } x, y \in X \text{and } t = 0. \end{cases}$$

Then it is well known that $(X, M, \star)$ is a fuzzy metric space (cf. [22]). Define $fx = 2x - 1$ and $gx = x^2$. Then by a straightforward calculation, one can show that

$$\begin{aligned} M(fgx, gfx, t) &= \left(e^{\frac{2|x-1|^2}{t}}\right)^{-1} \\ &= M(fx, gx, t/2) \end{aligned}$$

which shows that the pair (f, g) is R-weakly commuting for $R = 2$. Note that the pair (f, g) is not weakly commuting due to strict increasing property of exponential function.

However, various kinds of above mentioned 'R-weak commutativity' notions are independent of one another and none implies the other. The earlier example can be utilized to demonstrate this inter independence.

To demonstrate the independence of 'R-weak commutativity' with 'R-weak commutativity' of type (A_f), notice that

$$\begin{aligned} M(fgx, ggx, t) &= \left(e^{\frac{|x^4-2x^2+1|}{t}}\right)^{-1} = \left(e^{\frac{R(x-1)^2}{t}\frac{(x+1)^2}{R}}\right)^{-1} \\ &< \left(e^{\frac{R|x-1|^2}{t}}\right)^{-1} = M(fx, gx, t/R) \text{ when } x > 1 \end{aligned}$$

which shows that 'R-weak commutativity' does not imply 'R-weak commutativity' of type (A_f).

Secondly, in order to demonstrate the independence of 'R-weak commutativity' with 'R-weak commutativity' of type (P), note that

$$M(ffx, ggx, t) = \left(e^{\frac{|x^4-4x+3|}{t}}\right)^{-1} = \left(e^{\frac{R(x-1)^2}{t}\frac{(x^2+2x+3)}{R}}\right)^{-1}$$

$$< \left(e^{\frac{R|x-1|^2}{t}}\right)^{-1} = M(fx, gx, t/R) \text{ for } x > 1.$$

Finally for a change the pair (f, g) is R-weakly commuting of type (Ag) as

$$M(gfx, ffx, t) = \left(e\frac{|(2x-1)^2 - 4x + 3|}{t}\right)^{-1}$$
$$= \left(e\frac{4|x-1|^2}{t}\right)^{-1}$$
$$= M(fx, gx, t/4)$$

which shows that (f, g) is R-weakly commuting of type (Ag) for $R = 4$. This situation may also be utilized to interprete that an R-weakly commuting pair of type (Ag) need not be R-weakly commuting pair of type (A_f) or type (P). It is not difficult to find examples to establish the independence of one of these definitions from the others which shows that there exist situations to suit a definition but not the others.

However, the R-weak commutativity of type (A_f), type (Ag) and type (P) can together imply R-weak commutativity in a specific setting which can be described as follows.

Proposition 1. *Let (f, g) be a pair of self-mappings which is R-weakly commuting of type (A_f), type (Ag) and type (P) (at the same time) and $a \star b = min\{a, b\}$. Then the pair (f, g) is R-weakly commuting.*

Proof. It is straightforward to write

$$M(fgx, gfx, t) \geq M(fgx, ggx, t/3) \star M(ggx, ffx, t/3) \star M(ffx, gfx, t/3).$$

Now using the definitions of R-weakly commuting of type (A_f), type (Ag) and type (P) there exists constants $R_1, R_2, R_3 > 0$ satisfying

$$M(fgx, gfx, t) \geq M(fx, gx, t/3R_1) \star M(fx, gx, t/3R_2) \star M(fx, gx, t/3R_3)$$

implying thereby

$$M(fgx, gfx, t) \geq M(fx, gx, t/3R_i)$$

(for some $1 \leq i \leq 3$) which shows that the pair (f, g) is R-weakly commuting. □

3. Implicit functions

In this section, we define a suitable implicit function in fuzzy metric spaces to prove our results. Let Ψ denote the family of all continuous functions $F : [0,1]^4 \to \Re$ satisfying the following conditions:

F_1 : For every $u > 0, v \geq 0$ with $F(u, v, u, v) \geq 0$ or $F(u, v, v, u) \geq 0$, we have $u > v$.

F_2 : $F(u, u, 1, 1) < 0, \ \forall \ u > 0$.

Example 3. Define $F : [0,1]^4 \to \Re$ as

$$F(t_1, t_2, t_3, t_4) = t_1 - \phi(\min\{t_2, t_3, t_4\}),$$

where $\phi : [0,1] \to [0,1]$ is a continuous function such that $\phi(s) > s$ for $0 < s < 1$. Then

F_1 : $F(u, v, u, v) = u - \phi(\min\{v, u, v\}) \geq 0$. If $u \leq v$, then $u - \phi(u) \geq 0$ imply $u \geq \phi(u) > u$, a contradiction. Hence $u > v$.

F_2 : $F(u, u, 1, 1) = u - \phi(\min\{u, 1, 1\}) = u - \phi(u) < 0, \ \forall\ u > 0$.

Example 4. Define $F : [0,1]^4 \to \Re$ as

$$F(t_1, t_2, t_3, t_4) = t_1 - a \min\{t_2, t_3, t_4\}, \text{ where } a > 1.$$

F_1 : $F(u, v, u, v) = u - a \min\{v, u, v\} \geq 0$. If $u \leq v$, then $u \geq au > u$, a contradiction. Hence $u > v$.

F_2 : $F(u, u, 1, 1) = u - a \min\{u, 1, 1\} = u(1 - a) < 0, \ \forall\ u > 0$.

Example 5. Define $F : [0,1]^4 \to \Re$ as

$$F(t_1, t_2, t_3, t_4) = t_1 - at_2 - \min\{t_3, t_4\}, \text{ where } a > 0.$$

F_1 : $F(u, v, u, v) = u - av - \min\{u, v\} \geq 0$. If $u \leq v$, then $av \leq 0$, a contradiction. Hence $u > v$.

F_2 : $F(u, u, 1, 1) = u - au - \min\{1, 1\} = (1 - a)u - 1 < 0, \ \forall\ u > 0$.

Example 6. Define $F : [0,1]^4 \to \Re$ as

$$F(t_1, t_2, t_3, t_4) = t_1 - at_2 - bt_3 - ct_4,$$

where $a > 1, \ b, c \geq 0(\neq 1)$.

Example 7. Define $F : [0,1]^4 \to \Re$ as

$$F(t_1, t_2, t_3, t_4) = t_1 - at_2 - b(t_3 + t_4),$$

where $a > 1, \ b \geq 0(\neq 1)$.

Example 8. Define $F : [0,1]^4 \to \Re$ as

$$F(t_1, t_2, t_3, t_4) = t_1^3 - a\ t_2 t_3 t_4, \text{ where } a > 1.$$

Since verification of requirements (F_1 and F_2) for Examples 6-8 is straightforward, hence details are omitted.

4. Main results

Now we state and prove our main result as follows:

Theorem 2. *Let A, B, S and T be four self-mappings of a fuzzy metric space $(X, M, \star)$ satisfying the condition:*

$$F(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)) \geq 0 \qquad (1)$$

for all distinct $x, y \in X$ *and* $t > 0$, *where* $F \in \Psi$. *If* $A(X) \subset T(X)$, $B(X) \subset S(X)$ *and one of* $A(X), B(X), S(X)$ *or* $T(X)$ *is a complete subspace of* X. *Then*

(*a*) *the pair* (A, S) *has a point of coincidence, and*

(*b*) *the pair* (B, T) *has a point of coincidence.*

Moreover, if the pairs (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Then following arguments of Fisher [6], one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}.$$

The sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x, y_n \to y, t > 0$ implies $M(x_n, y_n, t) \to M(x, y, t)$.

Now making use of (1), we have

$$F(M(Ax_{2n}, Bx_{2n+1}, t), M(Sx_{2n}, Tx_{2n+1}, t), M(Sx_{2n}, Ax_{2n}, t), M(Bx_{2n+1}, Tx_{2n+1}, t)) \geq 0$$

$$\text{or } F(M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t)) \geq 0.$$

Hence in view of (F_1), we have

$$M(y_{2n}, y_{2n+1}, t) > M(y_{2n-1}, y_{2n}, t). \tag{2}$$

Thus $\{M(y_{2n}, y_{2n+1}, t), n \geq 0\}$ is an increasing sequence of positive real numbers in [0, 1] and therefore tends to a limit $l \leq 1$. We assert that $l = 1$. If not, (i.e. $l < 1$) then on letting $n \to \infty$ in (2) one gets $l > l$ a contradiction. Hence $l = 1$. Therefore for every $n \in N$, using analogous arguments one can also show that $\{M(y_{2n+1}, y_{2n+2}, t), n \geq 0\}$ is a sequence of positive real numbers in [0, 1] which converges to 1. Therefore for every $n \in N$

$$M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t) \text{ and } \lim_{n \to \infty} M(y_n, y_{n+1}, t) = 1.$$

Now for any positive integer p

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) \star \ldots \star M(y_{n+p-1}, y_{n+p}, t/p).$$

Since $\lim_{n \to \infty} M(y_n, y_{n+1}, t) = 1$ for $t > 0$, it follows that

$$\lim_{n \to \infty} M(y_n, y_{n+p}, t) \geq 1 \star 1 \star \ldots \star 1 = 1$$

which shows that $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that $S(X)$ is a complete subspace of X, then the subsequence $\{y_{2n+1}\}$ must converge in $S(X)$. Call this limit to be u and $v \in S^{-1}u$. Then $Sv = u$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of $\{y_{2n}\}$ being a subsequence of the convergent sequence $\{y_n\}$. If $Av \neq Sv$, then on setting $x = v$ and $y = x_{2n+1}$ in (1), one gets (for $t > 0$)

$$F(M(Av, Bx_{2n+1}, t), M(Sv, Tx_{2n+1}, t), M(Sv, Av, t), M(Bx_{2n+1}, Tx_{2n+1}, t)) \geq 0$$

which on letting $n \to \infty$ reduces to

$$F(M(Av,u,t), M(Sv,u,t), M(Sv,Av,t), M(u,u,t)) \geq 0$$
$$F(M(Av,Sv,t), 1, M(Sv,Av,t), 1) \geq 0$$

yielding thereby, $M(Av,Sv,t) > 1$, a contradiction. Hence, $Av = Sv$ which shows that the pair (A,S) has a point of coincidence.

As $A(X) \subset T(X)$ and $Av = u$ implies that $u \in T(X)$. Let $w \in T^{-1}u$, then $Tw = u$. Suppose that $Tw \neq Bw$. Again using (1), we have

$$F(M(Ax_{2n}, Bw, t), M(Sx_{2n}, Tw, t), M(Sx_{2n}, Ax_{2n}, t), M(Bw, Tw, t)) \geq 0$$

which on letting $n \to \infty$ reduces to

$$F(M(Tw,Bw,t), 1, 1, M(Tw,Bw,t)) \geq 0$$

implying thereby, $M(Tw,Bw,t) > 1$, a contradiction. Hence $Tw = Bw$. Thus we have $u = Av = Sv = Bw = Tw$ which amounts to say that both the pairs have point of coincidence. If one assumes $T(X)$ to be complete, then analogous arguments establish this claim.

The remaining two cases pertain essentially to the previous cases. Indeed, if $A(X)$ is complete then $u \in A(X) \subset T(X)$ and if $B(X)$ is complete then $u \in B(X) \subset S(X)$. Thus (a) and (b) are completely established.

Moreover, if the pairs (A,S) and (B,T) are weakly compatible at v and w respectively, then

$$Au = A(Sv) = S(Av) = Su \text{ and } Bu = B(Tw) = T(Bw) = Tu.$$

If $Au \neq u$, then for $t > 0$

$$F(M(Au,Bw,t), M(Su,Tw,t), M(Su,Au,t), M(Bw,Tw,t)) \geq 0$$

$$F(M(Au,u,t), M(Au,u,t), 1, 1) \geq 0$$

which contradicts (F_2). Hence $Au = u$. Similarly one can show that $Bu = u$. Thus u is a common fixed point of A, B, S and T. The uniqueness of common fixed point follows easily. Also u remains the unique common fixed point of both the pairs separately. This completes the proof. □

By setting $B = A$ and $T = S$, we have the following corollary for two maps.

Corollary 1. *Let A and S be two self-mappings of a fuzzy metric space $(X, M, \star)$ satisfying the condition:*

$$F(M(Ax,Ay,t), M(Sx,Sy,t), M(Sx,Ax,t), M(Ay,Sy,t)) \geq 0 \quad (3)$$

for all $x, y \in X$ and $t > 0$, where $F \in \Psi$. If $A(X) \subset S(X)$ and one of $A(X)$ and $S(X)$ is complete subspace of X. Then

(c) the pair (A,S) has a point of coincidence.

Moreover, if the pair (A,S) is weakly compatible, then A and S have a unique common fixed point.

Corollary 2. *The conclusions of Theorem 2 remain true if for all distinct $x, y \in X$ implicit function (1) is replaced by one of the following:*

(a_1) $M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\})$, *where* $\phi : [0, 1] \to [0, 1]$ *is a continuous function such that* $\phi(s) > s$ *for all* $0 < s < 1$.

(a_2) $M(Ax, By, t) \geq a \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\}$, *where* $a > 1$.

(a_3) $M(Ax, By, t) \geq aM(Sx, Ty, t) + \min\{M(Sx, Ax, t), M(By, Ty, t)\}$, *where* $a > 0$.

(a_4) $M(Ax, By, t) \geq aM(Sx, Ty, t) + bM(Sx, Ax, t) + cM(By, Ty, t)$, *where* $a > 1$ *and* $b, c \geq 0 (\neq 1)$.

(a_5) $M(Ax, By, t) \geq aM(Sx, Ty, t) + b[M(Sx, Ax, t) + M(By, Ty, t)]$, *where* $a > 1$ *and* $b \geq 0 (\neq 1)$.

(a_6) $M^3(Ax, By, t) \geq aM(Sx, Ty, t)M(Sx, Ax, t)(By, Ty, t)$, *where* $a > 1$.

Proof. The proof of the corollaries corresponding to contraction conditions $a_1 - a_6$ follows from Theorem 2 and Examples 3-8. □

Remark 1. Corollary corresponding to contraction condition (a_1) is a result due to Imdad and Ali [10] and generalized form of results contained in [2, 20, 22]. We also point out that some of above corollaries are new to the literature (e.g. Corollaries corresponding to $a_2 - a_6$).

Theorem 3. *Theorem 2 remains true if 'weak compatibility' property is replaced by any one of the following (retaining the rest of the hypotheses):*

(i) *R-weakly commuting property,*
(ii) *R-weakly commuting property of type* (A_f),
(iii) *R-weakly commuting property of type* (Ag),
(iv) *R-weakly commuting property of type* (P),
(v) *weakly commuting property.*

Proof. Since all the conditions of Theorem 2 are satisfied, therefore the existence of coincidence points for both the pairs is guaranteed. Let x be an arbitrary point of coincidence for the pair (A, S), then using R-weak commutativity one gets

$$M(ASx, SAx, t) \geq M(Ax, Sx, t/R) = 1$$

which amounts to say that $ASx = SAx$. Thus the pair (A, S) is weakly compatible. Similarly (B, T) commutes at all of its coincidence points. Now appealing Theorem 2, one concludes that A, B, S and T have a unique common fixed point.

In case (A, S) is R-weakly commuting pair of type (A_f), then

$$M(ASx, S^2x, t) \geq M(Ax, Sx, t/R) = 1$$

which amounts to say that $ASx = S^2x$. Now

$$M(ASx, SAx, t) \geq M(ASx, S^2x, t/2) \star M(S^2x, SAx, t/2) = 1 \star 1 = 1$$

yielding thereby $ASx = SAx$. Similarly if the pair (A, S) is R-weakly commuting mapping of type (Ag) or type (P) or weakly commuting, then the pair (A, S)

also commutes at the points of coincidence. Similarly, one can show that the pair (B, T) also commutes at the points of coincidence. Now in view of Theorem 2, in all four cases A, B, S and T have a unique common fixed point. This completes the proof. □

As an application of Theorem 2, we prove a common fixed point theorem for four finite families of mappings which runs as follows:

Theorem 4. *Let* $\{A_1, A_2, \ldots, A_m\}, \{B_1, B_2, \ldots, B_n\}, \{S_1, S_2, \ldots, S_p\}$ *and* $\{T_1, T_2, \ldots, T_q\}$ *be four finite families of self-mappings of a fuzzy metric space* $(X, M, \star)$ *such that* $A = A_1A_2\ldots A_m$, $B = B_1B_2\ldots B_n$, $S = S_1S_2\ldots S_p$ *and* $T = T_1T_2\ldots T_q$ *satisfy condition (1) with* $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$. *If one of* $A(X), B(X), S(X)$ *or* $T(X)$ *is a complete subspace of* X, *then*

(d) A and S have a point of coincidence,

(e) B and T have a point of coincidence.

Moreover, if $A_iA_j = A_jA_i$, $B_kB_l = B_lB_k$, $S_rS_s = S_sS_r$, $T_tT_u = T_uT_t$, $A_iS_r = S_rA_i$ *and* $B_kT_t = T_tB_k$ *for all* $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, n\}$, $r, s \in I_3 = \{1, 2, \ldots, p\}$ *and* $t, u \in I_4 = \{1, 2, \ldots, q\}$, *then (for all* $i \in I_1, k \in I_2, r \in I_3$ *and* $t \in I_4$*)* A_i, S_r, B_k *and* T_t *have a common fixed point.*

Proof. The proof follows from Theorem 3.3 of Imdad and Ali [10]. □

Corollary 3. *Let* A, B, S *and* T *be four self-mappings of a fuzzy metric space* $(X, M, \star)$ *such that* A^m, B^n, S^p *and* T^q *satisfy the condition (1). If one of* $A^m(X)$, $B^n(X)$, $S^p(X)$ *or* $T^q(X)$ *is a complete subspace of* X, *then* A, B, S *and* T *have a unique common fixed point provided* (A, S) *and* (B, T) *commute.*

The following example furnishes an instance where Corollary $3(a_1)$ is applicable but Theorem 1 (also theorem due to Chugh and Kumar [2]) cannot be used due to the absence of continuity requirement.

Example 9. Consider $X = [0, 1]$ equipped with the natural metric $d(x, y) = |x - y|$. Now for $t \in [0, \infty)$ define

$$M(x, y, t) = \begin{cases} 0, & \text{if } t = 0 \text{ and } x, y \in X \\ \dfrac{t}{t + |x - y|}, & \text{if } t > 0 \text{ and } x, y \in X. \end{cases}$$

Clearly $(X, M, \star)$ is a fuzzy metric on X where $\star$ is defined as $a \star b = ab$.

Define A, B, S and T on [0, 1] as

$$Ax = \begin{cases} 1, & if\ x \in [0, 1] \cap Q \\ 0, & if\ x \notin [0, 1] \cap Q, \end{cases} \qquad Bx = \begin{cases} 1, & if\ x \in [0, 1] \cap Q \\ \dfrac{1}{2}, & if\ x \notin [0, 1] \cap Q, \end{cases}$$

$$Sx = \begin{cases} \frac{1}{2}, & if\ 0 \le x < 1 \\ 1, & if\ x = 1, \end{cases} \quad \text{and } Tx = \begin{cases} \frac{1}{4}, & if\ 0 \le x < 1 \\ 1, & if\ x = 1. \end{cases}$$

Then $A^2(X) = \{1\} \subset \{\frac{1}{4}, 1\} = T^2(X)$ and $B^2(X) = \{1\} \subset \{\frac{1}{2}, 1\} = S^2(X)$. Define $\phi : [0,1] \to [0,1]$ as $\phi(0) = 0, \phi(1) = 1$ and $\phi(s) = \sqrt{s}$ for all $s \in (0,1)$. Then

$$\begin{aligned} 1 &= M(A^2x, B^2y, t) \\ &\ge \phi\Big(\min\{M(S^2x, T^2y, t), M(S^2x, A^2x, t), M(B^2y, T^2y, t)\}\Big) \end{aligned}$$

for all $t > 0$. Also the various componentwise commutativity conditions ensure the commutativity of the both pairs (A, S) and (B, T). Thus all the conditions of the Corollary 3 are satisfied and 1 is the common fixed point of A, B, S and T.

Here one needs to note that Theorem 1 (also theorem due to Chugh and Kumar [2]) cannot be used in the context of this example because if we take $x, y \notin Q$, then

$$\frac{t}{t+\frac{1}{2}} = M(Ax, By, t) \ge \phi\left(\min\left\{\frac{t}{t+\frac{1}{4}}, \frac{t}{t+\frac{1}{2}}, \frac{t}{t+\frac{1}{4}}\right\}\right)$$

which is not always true for $t > 0$ (*e.g.* $t = 0.5$). On the other hand all the four mappings are discontinuous which is not in lieu of the requirements of the Theorem 1 (also theorem due to Chugh and Kumar [2]).

Finally, we furnish an example to create a situation which demonstrates the utility of Theorem 4.

Example 10. Consider $(X, M, \star)$ as in Example 9. Define four finite families of maps as

$$A_nx = \begin{cases} 1, & if\ x \in [0,1] \cap Q \\ \frac{1}{4n}, & if\ x \notin [0,1] \cap Q, \end{cases} \qquad B_nx = \begin{cases} 1, & if\ x \in [0,1] \cap Q \\ \frac{1}{6n}, & if\ x \notin [0,1] \cap Q, \end{cases}$$

$$S_nx = \begin{cases} 1, & if\ x \in [0,1] \cap Q \\ \frac{1}{3n}, & if\ x \notin [0,1] \cap Q \end{cases} \quad \text{and } T_nx = \begin{cases} 1, & if\ x \in [0,1] \cap Q \\ \frac{1}{2n}, & if\ x \notin [0,1] \cap Q, \end{cases}$$

where $n = 1, 2, \ldots, 100$.

Evidently $A(X) = A_1A_2 \ldots A_{100}(X) = T_1T_2 \ldots T_{100}(X) = T(X)$ and $B(X) = B_1B_2 \ldots B_{100}(X) = S_1S_2 \ldots S_{100}(X) = S(X)$. Define $\phi : [0,1] \to [0,1]$ as in Example 9.

Considering the same implicit function as in Example 9, by routine calculations one can easily verify that the condition (1) is satisfied for all distinct

$x, y \in [0, 1]$. Also the various componentwise commutativity ensure the commutativity of both the pairs (A, S) and (B, T). Thus all the conditions of Theorem 4 are satisfied and 1 is the common fixed point of A, B, S and T. Notice that every member map of all the four families is discontinuous.

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REFERENCES

1. D. W. Boyd and J. S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458-462.
2. R. Chugh and S. Kumar, *Common fixed point theorem in fuzzy metric spaces*, Bull. Calcutta Math. Soc. **94**(1) (2002), 17-22.
3. Z. Deng, *Fuzzy pseudo-metric space*, J. Math. Anal. Appl. **86** (1982), 74-95.
4. M. A. Erceg, *Metric space in fuzzy set theory*, J. Math. Anal. Appl. **69** (1979), 205-230.
5. J. X. Fang, *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets Systems **46** (1992), 107-113.
6. B. Fisher, *Common fixed points of four mappings*, Bull. Inst. Math. Acad. Sinica **11** (1983), 103-113.
7. A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Systems **64** (1994), 395-399.
8. A. George and P. Veeramani, *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets Systems **90** (1997), 365-368.
9. M. Grabiec, *Fixed points in fuzzy metric space*, Fuzzy Sets Systems **27** (1988), 385-389.
10. M. Imdad and Javid Ali, *Some common fixed point theorems in fuzzy metric spaces*, Math. Commun. **11** (2006), 153-163.
11. O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets Systems **12** (1984), 215-229.
12. O. Kaleva, *The completion of fuzzy metric spaces*, J. Math. Anal. Appl. **109** (1985), 194-198.
13. I. Kramosil and J. Michalek, *Fuzzy metrics and statistical metric spaces*, Kybernetika **11** (1975), 336-344.
14. R. P. Pant, *Common fixed point for noncommuting mappings*, J. Math. Anal. Appl. **188**(2) (1994), 436-440.
15. R. P. Pant and K. Jha, *A remark on common fixed points of four mappings in a fuzzy metric space*, J. Fuzzy Math. **12**(2) (2004), 433-437.
16. H. K. Pathak, Y. J. Cho and S. M. Kang, *Remarks on R-weakly mappings and common fixed point theorems*, Bull. Korean Math. Soc. **34**(2) (1997), 247-257.
17. V. Popa, *A fixed point theorem for mapping in d-complete topological spaces*, Math. Moravica **3** (1999), 43-48.
18. A. Razani and M. Shirdarazdi, *Some results on fixed points in the fuzzy metric space*, J. Appl. Math. Comput. **20** (2006), 401-408.
19. B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), 313-334.
20. B. Singh and S. Jain, *Weak compatibility and fixed point theorems in fuzzy metric space*, Ganita **56** (2005), 167-176.
21. G. Song, *Comments on a common fixed point theorem in fuzzy metric space*, Fuzzy Sets Systems **135** (2003), 409-413.
22. R. Vasuki, *Common fixed points for R-weakly commuting maps in fuzzy metric spaces*, Indian J. Pure Appl. Math. **30**(4) (1999), 419-423.

23. R. Vasuki and P. Veeramani, *Fixed point theorems and Cauchy sequences in fuzzy metric spaces*, Fuzzy Sets Systems **135** (2003), 415-417.
24. Z. Q. Xia and F. F. Guo, *Fuzzy metric spaces*, J. Appl. Math. Comput. **16** (2004), 371-381.
25. L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338-353.

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