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Estimation for the Double Rayleigh Distribution Based on Multiply Type-II Censored Samples

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Abstract

In this paper, we derive the approximate maximum likelihood estimators of the scale parameter and the location parameter in a double Rayleigh distribution based on multiply Type-II censored samples. We compare the proposed estimators in the sense of the mean squared error for various censored samples.

Keywords: Approximate maximum likelihood estimator; multiply Type-II censored sample; double Rayleigh distribution.

1. Introduction

The cumulative distribution function(cdf) and the probability density function(pdf) of the random variable having the double Rayleigh distribution are given by

$$F(x) = \frac{1}{2}\left[1 + \mathrm{sgn}(x)\left\{1 - \exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)\right\}\right] \tag{1.1}$$

and

$$f(x) = \frac{|x-\theta|}{2\sigma^2}\exp\left[-\frac{(x-\theta)^2}{2\sigma^2}\right], \tag{1.2}$$

where θ and σ are the location and the scale parameters, respectively and

$$\mathrm{sgn}(x) = \begin{cases} 1, & x \geq \theta, \\ -1, & x < \theta. \end{cases}$$

The Rayleigh distribution is very useful in communication engineering and is a special case of a two-parameter Weibull distribution. Figure 1.1 shows that the distribution becomes more and more flattened as σ increases.

Dattatreya Rao and Narasimham (1989) obtained the best linear unbiased estimates (BLUEs) for the location and scale parameters with various shape parameter values in the double Weibull distribution for complete and censored samples. Vasudeva Rao *et al.* (1991) obtained the first two moments and product moments of the absolute value of the order statistics for the double exponential distribution and the double Weibull

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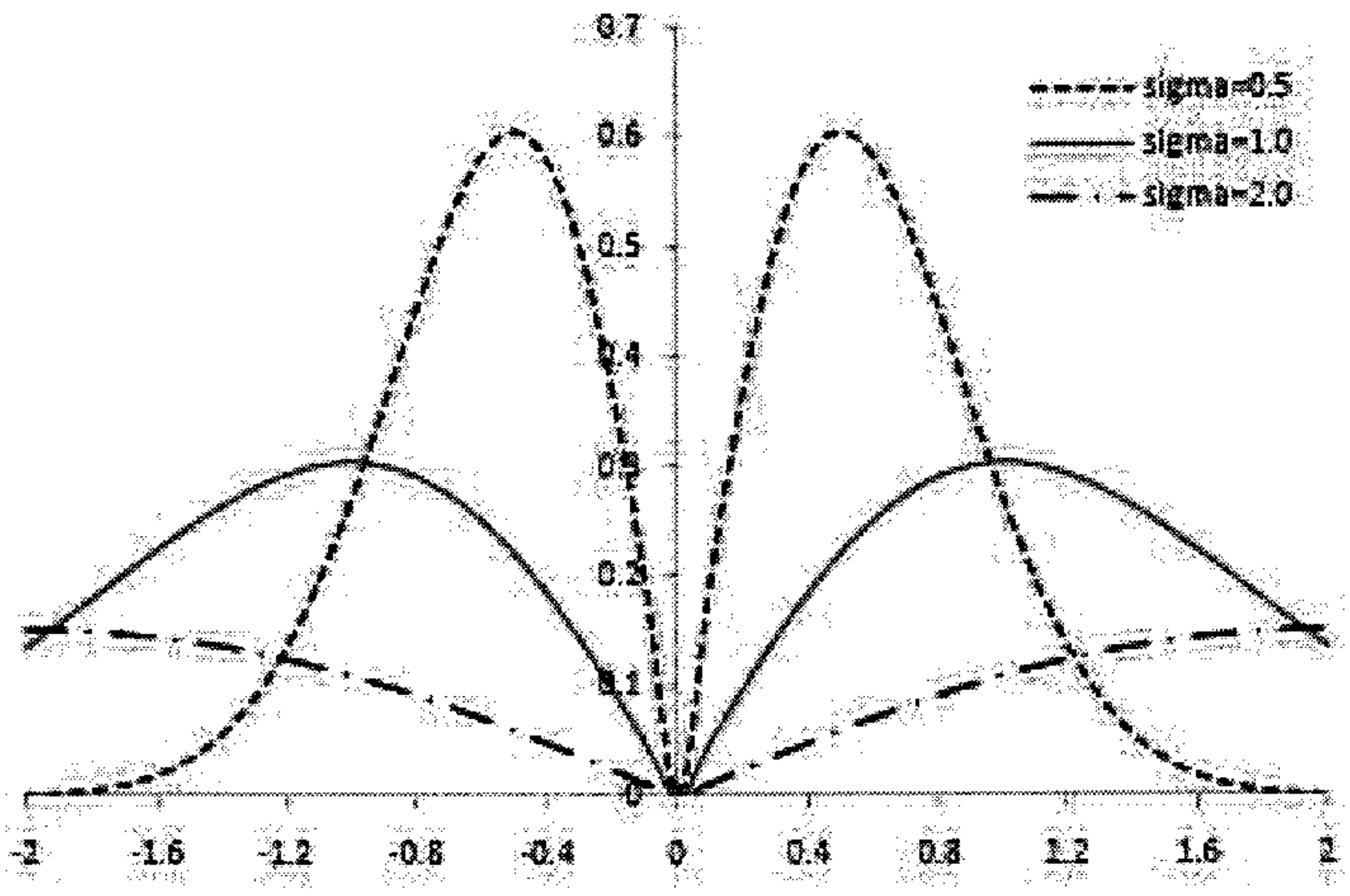

Figure 1.1: The *pdf* of double Rayleigh distribution with $\theta = 0.0$ and $\sigma = 0.5, 1.0, 2.0$.

distribution. They also obtained the optimum unbiased absolute estimator of the scale parameter. Son and Woo (2007) defined a skew-symmetric double Rayleigh distribution and derived an approximate maximum likelihood estimator(AMLE) and a moment estimator of a skewed parameter in a skew-symmetric double Rayleigh distribution.

In many life test studies, it is common that the lifetimes of test units may not be able to record exactly. The most common censoring schemes are Type-I and Type-II censoring, but the conventional Type-I and Type-II censoring schemes do not have flexibility. Multiply Type-II censoring is a generalization of Type-II censoring.

In most cases of censored and truncated samples, the explicit estimators might be not obtained by the maximum likelihood method. So we need another method for the purpose of providing the explicit estimators. The approximated maximum likelihood estimating method was first developed by Balakrishnan (1989) for the purpose of providing the explicit estimators of the scale parameter in Rayleigh distribution. Kang (1996) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples. Balakrishnan *et al.* (2004) discussed point and interval estimation for the extreme value distribution under progressively Type-II censoring. Kang *et al.* (2005) derived the AMLEs of the scale parameter in the two-parameter double exponential distribution based on Type-II censored samples. Han and Kang (2006) derived AMLEs of the scale parameter and the location parameter in the two-parameter Rayleigh distribution under multiply Type-II censoring by the approximate maximum likelihood estimation method.

Recently, Seo and Kang (2007) proposed the AMLEs of the scale parameter and the location parameter in the Rayleigh distribution under progressive Type-II censoring by the approximate maximum likelihood estimation methods.

In this paper, we derive the AMLEs of the scale parameter σ and the location parameter θ under multiply Type-II censored sample. We also compare the proposed estimators in the sense of the mean squared error(MSE) through Monte Carlo simulation for various censored samples.

2. Approximate Maximum Likelihood Estimators

We assume that n items are put on a life test, but only a_1^{th}, $a_2^{th}, \dots, a_s^{th}$ failures are observed, the rest are unobserved or missing, where a_1, $a_2, \dots, a_s$ are considered to be fixed. If this censoring arises, the scheme is known as multiply Type-II censoring scheme.

Let us assume that the following multiply Type-II censored sample from a sample of size n is

$$X_{a_1:n} \le X_{a_2:n} \le \cdots \le X_{a_s:n}, \tag{2.1}$$

where $1 \le a_1 < a_2 < \cdots < a_s \le n$.

Let $a_0 = 0$, $a_{s+1} = n+1$, $F(x_{a_0:n}) = 0$, $F(x_{a_{s+1}:n}) = 1$, then the likelihood function based on the multiply Type-II censored sample (2.1) can be written as

$$L = n! \prod_{j=1}^{s} f(x_{a_j:n}) \prod_{j=1}^{s+1} \frac{\{F(x_{a_j:n}) - F(x_{a_{j-1}:n})\}^{a_j - a_{j-1} - 1}}{(a_j - a_{j-1} - 1)!}. \tag{2.2}$$

The random variable $Z_{i:n} = (X_{i:n} - \theta)/\sigma$ has a standard double Rayleigh distribution with *pdf* and *cdf* :

$$f(z) = \frac{|z|}{2} \exp\left(-\frac{z^2}{2}\right)$$

and

$$F(x) = \begin{cases} 1 - \dfrac{1}{2}\exp\left(-\dfrac{z^2}{2}\right), & z \ge 0, \\ \dfrac{1}{2}\exp\left(-\dfrac{z^2}{2}\right), & z < 0. \end{cases}$$

From the equation (2.2), we obtain the likelihood equations as follows:

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} &= -\frac{1}{\sigma}\Bigg\{ 2s + (a_1 - 1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} - (n - a_s)\frac{f(Z_{a_s:n})}{1 - F(Z_{a_s:n})} Z_{a_s:n} - \sum_{j=1}^{s} Z_{a_j:n}^2 \\ &\quad + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\frac{f(Z_{a_j:n})Z_{a_j:n} - f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \Bigg\} \\ &= 0 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \frac{\partial \ln L}{\partial \theta} &= -\frac{1}{\sigma}\Bigg\{ (a_1 - 1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} - (n - a_s)\frac{f(Z_{a_s:n})}{1 - F(Z_{a_s:n})} + \sum_{j=1}^{s} \frac{1}{Z_{a_j:n}} - \sum_{j=1}^{s} Z_{a_j:n}^2 \\ &\quad + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\frac{f(Z_{a_j:n}) - f(Z_{a_{j-1}:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \Bigg\} \\ &= 0. \end{aligned} \tag{2.4}$$

Since these likelihood equations are very complicated, the equations (2.3) and (2.4) do not admit explicit solution for σ and θ. So we need some approximate likelihood equations which can be given explicit solutions.

Let

$$\xi_i = F^{-1}(p_i) = \begin{cases} [-2\ln\{2(1-p_i)\}]^{\frac{1}{2}}, & p_i \geq 0.5, \\ -\{-2\ln(2p_i)\}^{\frac{1}{2}}, & p_i < 0.5, \end{cases}$$

where $p_i = i/(n+1)$, $q_i = 1 - p_i$.

First, we can approximate these functions by

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} \simeq \alpha_1 + \beta_1 Z_{a_1:n}, \tag{2.5}$$

$$\frac{f(Z_{a_s:n})}{1-F(Z_{a_s:n})} Z_{a_s:n} \simeq \kappa_1 + \delta_1 Z_{a_s:n}, \tag{2.6}$$

$$\frac{f(Z_{a_j:n})Z_{a_j:n} - f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n}, \tag{2.7}$$

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} \simeq \alpha_2 + \beta_2 Z_{a_1:n}, \tag{2.8}$$

$$\frac{f(Z_{a_s:n})}{1-F(Z_{a_s:n})} \simeq \kappa_2 + \delta_2 Z_{a_s:n}, \tag{2.9}$$

$$\frac{1}{Z_{a_j:n}} \simeq \frac{2}{\xi_{a_j}} - \frac{1}{\xi_{a_j}} Z_{a_j:n}, \tag{2.10}$$

$$\frac{f(Z_{a_j:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{2j} + \beta_{2j} Z_{a_j:n} + \gamma_{2j} Z_{a_{j-1}:n}, \tag{2.11}$$

$$\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{3j} + \beta_{3j} Z_{a_j:n} + \gamma_{3j} Z_{a_{j-1}:n} \tag{2.12}$$

and

$$\frac{f(Z_{a_j:n}) - f(Z_{a_{j-1}:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{4j} + \beta_{4j} Z_{a_j:n} + \gamma_{4j} Z_{a_{j-1}:n}, \tag{2.13}$$

where

$$\alpha_1 = \begin{cases} -\frac{q_{a_1}}{p_{a_1}} \xi_{a_1}^2 \left(1 - \xi_{a_1}^2 - \frac{q_{a_1}}{p_{a_1}} \xi_{a_1}^2\right), & p_{a_1} \geq 0.5, \\ \xi_{a_1}^2, & p_{a_1} < 0.5, \end{cases}$$

$$\beta_1 = \begin{cases} \frac{q_{a_1}}{p_{a_1}} \xi_{a_1} \left(2 - \xi_{a_1}^2 - \frac{q_{a_1}}{p_{a_1}} \xi_{a_1}^2\right), & p_{a_1} \geq 0.5, \\ -2\xi_{a_1}, & p_{a_1} < 0.5, \end{cases}$$

$$\kappa_1 = \begin{cases} -\xi_{a_s}^2, & p_{a_s} \geq 0.5, \\ \frac{p_{a_s}}{q_{a_s}} \xi_{a_s}^2 \left(1 - \xi_{a_s}^2 - \frac{p_{a_s}}{q_{a_s}} \xi_{a_s}^2\right), & p_{a_s} < 0.5, \end{cases}$$

$$\delta_1 = \begin{cases} 2\xi_{a_s}, & p_{a_s} \geq 0.5, \\ -\dfrac{p_{a_s}}{q_{a_s}}\xi_{a_s}\left(2-\xi_{a_s}^2-\dfrac{p_{a_s}}{q_{a_s}}\xi_{a_s}^2\right), & p_{a_s} < 0.5, \end{cases}$$

$$\alpha_2 = \begin{cases} \dfrac{q_{a_1}}{p_{a_1}^2}\xi_{a_1}^3, & p_{a_1} \geq 0.5, \\ 0, & p_{a_1} < 0.5, \end{cases}$$

$$\beta_2 = \begin{cases} \dfrac{q_{a_1}}{p_{a_1}}\left(1-\xi_{a_1}^2-\dfrac{q_{a_1}}{p_{a_1}}\xi_{a_1}^2\right), & p_{a_1} \geq 0.5, \\ -1, & p_{a_1} < 0.5, \end{cases}$$

$$\kappa_2 = \begin{cases} 0, & p_{a_s} \geq 0.5, \\ -\dfrac{p_{a_s}}{q_{a_s}}\xi_{a_s}^3\left(1+\dfrac{p_{a_s}}{q_{a_s}}\right), & p_{a_s} < 0.5, \end{cases}$$

$$\delta_2 = \begin{cases} 1, & p_{a_s} \geq 0.5, \\ -\dfrac{p_{a_s}}{q_{a_s}}\left(1-\xi_{a_s}^2-\dfrac{p_{a_s}}{q_{a_s}}\xi_{a_s}^2\right), & p_{a_s} < 0.5, \end{cases}$$

$$\alpha_{1j} = \begin{cases} K^2 - \dfrac{\left(1-\xi_{a_j}^2\right)\xi_{a_j}^2 q_{a_j} - \left(1-\xi_{a_{j-1}}^2\right)\xi_{a_{j-1}}^2 q_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}, & p_{a_{j-1}} \geq 0.5, \\ K^2 - \dfrac{\left(1-\xi_{a_j}^2\right)\xi_{a_j}^2 q_{a_j} + \left(1-\xi_{a_{j-1}}^2\right)\xi_{a_{j-1}}^2 p_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}, & p_{a_{j-1}} < 0.5 \leq p_{a_j}, \\ K^2 + \dfrac{\left(1-\xi_{a_j}^2\right)\xi_{a_j}^2 p_{a_j} - \left(1-\xi_{a_{j-1}}^2\right)\xi_{a_{j-1}}^2 p_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}, & p_{a_j} < 0.5, \end{cases}$$

$$\beta_{1j} = \begin{cases} \dfrac{\xi_{a_j} q_{a_j}}{p_{a_j}-p_{a_{j-1}}}\left(2-K-\xi_{a_j}^2\right), & p_{a_{j-1}} \geq 0.5, \\ \dfrac{\xi_{a_j} q_{a_j}}{p_{a_j}-p_{a_{j-1}}}\left(2-K-\xi_{a_j}^2\right), & p_{a_{j-1}} < 0.5 \leq p_{a_j}, \\ -\dfrac{\xi_{a_j} p_{a_j}}{p_{a_j}-p_{a_{j-1}}}\left(2-K-\xi_{a_j}^2\right), & p_{a_j} < 0.5, \end{cases}$$

$$\gamma_{1j} = \begin{cases} -\dfrac{\xi_{a_{j-1}} q_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\left(2-K-\xi_{a_{j-1}}^2\right), & p_{a_{j-1}} \geq 0.5, \\ \dfrac{\xi_{a_{j-1}} p_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\left(2-K-\xi_{a_{j-1}}^2\right), & p_{a_{j-1}} < 0.5 \leq p_{a_j}, \\ \dfrac{\xi_{a_{j-1}} p_{a_{j-1}}}{p_{a_j}-p_{a_{j-1}}}\left(2-K-\xi_{a_{j-1}}^2\right), & p_{a_j} < 0.5, \end{cases}$$

$$\alpha_{2j} = \begin{cases} \dfrac{\xi_{a_j} q_{a_j}}{p_{a_j}-p_{a_{j-1}}}\left(K+\xi_{a_j}^2\right), & p_{a_{j-1}} \geq 0.5, \\ \dfrac{\xi_{a_j} q_{a_j}}{p_{a_j}-p_{a_{j-1}}}\left(K+\xi_{a_j}^2\right), & p_{a_{j-1}} < 0.5 \leq p_{a_j}, \\ -\dfrac{\xi_{a_j} p_{a_j}}{p_{a_j}-p_{a_{j-1}}}\left(K+\xi_{a_j}^2\right), & p_{a_j} < 0.5, \end{cases}$$

$$\beta_{2j} = \begin{cases} \dfrac{q_{a_j}}{p_{a_j} - p_{a_{j-1}}}\left(1 - \xi_{a_j}^2 - \dfrac{\xi_{a_j}^2 q_{a_j}}{p_{a_j} - p_{a_{j-1}}}\right), & p_{a_{j-1}} \geq 0.5, \\ \dfrac{q_{a_j}}{p_{a_j} - p_{a_{j-1}}}\left(1 - \xi_{a_j}^2 - \dfrac{\xi_{a_j}^2 q_{a_j}}{p_{a_j} - p_{a_{j-1}}}\right), & p_{a_{j-1}} < 0.5 \leq p_{a_j}, \\ -\dfrac{p_{a_j}}{p_{a_j} - p_{a_{j-1}}}\left(1 - \xi_{a_j}^2 + \dfrac{\xi_{a_j}^2 p_{a_j}}{p_{a_j} - p_{a_{j-1}}}\right), & p_{a_j} < 0.5, \end{cases}$$

$$\gamma_{2j} = \begin{cases} \dfrac{\xi_{a_j} q_{a_j} \xi_{a_{j-1}} q_{a_{j-1}}}{(p_{a_j} - p_{a_{j-1}})^2}, & p_{a_{j-1}} \geq 0.5, \\ -\dfrac{\xi_{a_j} q_{a_j} \xi_{a_{j-1}} p_{a_{j-1}}}{(p_{a_j} - p_{a_{j-1}})^2}, & p_{a_{j-1}} < 0.5 \leq p_{a_j}, \\ \dfrac{\xi_{a_j} p_{a_j} \xi_{a_{j-1}} p_{a_{j-1}}}{(p_{a_j} - p_{a_{j-1}})^2}, & p_{a_j} < 0.5, \end{cases}$$

$$\alpha_{3j} = \begin{cases} \dfrac{\xi_{a_{j-1}} q_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\left(K + \xi_{a_{j-1}}^2\right), & p_{a_{j-1}} \geq 0.5, \\ -\dfrac{\xi_{a_{j-1}} p_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\left(K + \xi_{a_{j-1}}^2\right), & p_{a_{j-1}} < 0.5 \leq p_{a_j}, \\ -\dfrac{\xi_{a_{j-1}} p_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\left(K + \xi_{a_{j-1}}^2\right), & p_{a_j} < 0.5, \end{cases}$$

$$\beta_{3j} = -\gamma_{2j}$$

$$\gamma_{3j} = \begin{cases} \dfrac{q_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\left(1 - \xi_{a_{j-1}}^2 + \dfrac{\xi_{a_{j-1}}^2 q_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\right), & p_{a_{j-1}} \geq 0.5, \\ -\dfrac{p_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\left(1 - \xi_{a_{j-1}}^2 - \dfrac{\xi_{a_{j-1}}^2 p_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\right), & p_{a_{j-1}} < 0.5 \leq p_{a_j}, \\ -\dfrac{p_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\left(1 - \xi_{a_{j-1}}^2 - \dfrac{\xi_{a_{j-1}}^2 p_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}\right), & p_{a_j} < 0.5, \end{cases}$$

$$K = \begin{cases} \dfrac{\xi_{a_j}^2 q_{a_j} - \xi_{a_{j-1}}^2 q_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}, & p_{a_{j-1}} \geq 0.5, \\ \dfrac{\xi_{a_j}^2 q_{a_j} + \xi_{a_{j-1}}^2 p_{a_{j-1}}}{p_{a_j} - p_{a_{j-1}}}, & p_{a_{j-1}} < 0.5 \leq p_{a_j}, \\ \dfrac{\xi_{a_{j-1}}^2 p_{a_{j-1}} - \xi_{a_j}^2 p_{a_j}}{p_{a_j} - p_{a_{j-1}}}, & p_{a_j} < 0.5, \end{cases}$$

$$\alpha_{4j} = \alpha_{2j} - \alpha_{3j}, \qquad \beta_{4j} = \beta_{2j} - \beta_{3j}, \qquad \gamma_{4j} = \gamma_{2j} - \gamma_{3j}.$$

Now making use of the approximate expressions in (2.5), (2.6), (2.7), (2.8), (2.9), (2.10) and (2.13), we may approximate the likelihood equations (2.3) and (2.4) as follows:

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &= -\frac{1}{\sigma}\left\{2s + (a_1-1)(\alpha_1+\beta_1 Z_{a_1:n}) - (n-a_s)(\kappa_1+\delta_1 Z_{a_s:n}) - \sum_{j=1}^{s} Z_{a_j:n}^2 \right.\\ &\quad \left. + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left(\alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n}\right)\right\} \qquad (2.14)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta} &= -\frac{1}{\sigma}\left\{(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n}) - (n-a_s)(\kappa_2+\delta_2 Z_{a_s:n}) + \sum_{j=1}^{s}\frac{2}{\xi_{a_j}} - \sum_{j=1}^{s}\frac{1}{\xi_{a_j}^2} Z_{a_j:n}^2 \right.\\ &\quad \left. - \sum_{j=1}^{s} Z_{a_j:n}^2 + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left(\alpha_{4j} + \beta_{4j} Z_{a_j:n} + \gamma_{4j} Z_{a_{j-1}:n}\right)\right\} \qquad (2.15)\\ &= 0.\end{aligned}$$

Upon solving the equations (2.14) and (2.15) for σ and θ, we derive AMLEs of σ and θ as follows:

$$\hat{\sigma}_1 = \frac{-B_1 + \sqrt{{B_1}^2 - 4A_1C_1}}{2A_1} \qquad (2.16)$$

and

$$\hat{\theta}_1 = M_1\hat{\sigma}_1 + M_2, \qquad (2.17)$$

where

$$\begin{aligned}A &= (a_1-1)\alpha_2 - (n-a_s)\kappa_2 + \sum_{j=1}^{s}\frac{2}{\xi_{a_j}} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\alpha_{4j},\\ B &= (a_1-1)\beta_2 X_{a_1:n} - (n-a_s)\delta_2 X_{a_s:n} - \sum_{j=1}^{s}\frac{1}{\xi_{a_j}^2}X_{a_j:n} - \sum_{j=1}^{s} X_{a_j:n}\\ &\quad + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_{4j}X_{a_j:n} + \gamma_{4j}X_{a_{j-1}:n}),\\ C &= (a_1-1)\beta_2 - (n-a_s)\delta_2 - \sum_{j=1}^{s}\frac{1}{\xi_{a_j}^2} - s + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_{4j} + \gamma_{4j}),\\ M_1 &= A/C, \qquad M_2 = B/C,\\ A_1 &= (2 - M_1^2)s + (a_1-1)(\alpha_1 - \beta_1 M_1) - (n-a_s)(\kappa_1 - \delta_1 M_1)\\ &\quad + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\alpha_{1j} - \beta_{1j}M_1 - \gamma_{1j}M_1),\\ B_1 &= (a_1-1)\beta_1(X_{a_1:n} - M_2) - (n-a_s)\delta_1(X_{a_s:n} - M_2) + 2M_1\sum_{j=1}^{s}(X_{a_j:n} - M_2)\\ &\quad + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left\{\beta_{1j}(X_{a_j:n} - M_2) + \gamma_{1j}(X_{a_{j-1}:n} - M_2)\right\},\end{aligned}$$

$$C_1 = -\sum_{j=1}^{s}(X_{a_j:n} - M_2)^2.$$

Second, making use of the approximate expressions in (2.7), (2.8), (2.9), (2.10) and (2.13), we may approximate the likelihood equation (2.3) as follows:

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &= -\frac{1}{\sigma}\Big\{2s + (a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})Z_{a_1:n} - (n-a_s)(\kappa_2+\delta_2 Z_{a_s:n})Z_{a_s:n} \\ &\quad -\sum_{j=1}^{s} Z_{a_j:n}^2 + \sum_{j=2}^{s}(a_j-a_{j-1}-1)\left(\alpha_{1j}+\beta_{1j}Z_{a_j:n}+\gamma_{1j}Z_{a_{j-1}:n}\right)\Big\} \\ &= 0.\end{aligned} \tag{2.18}$$

Upon solving the equations (2.18) and (2.15) for σ and θ, we derive another AMLEs of σ and θ as follows:

$$\hat{\sigma}_2 = \frac{-B_2+\sqrt{{B_2}^2-4A_2C_2}}{2A_2} \tag{2.19}$$

and

$$\hat{\theta}_2 = M_1\hat{\sigma}_2 + M_2, \tag{2.20}$$

where

$$\begin{aligned}A_2 &= (2-M_1^2)s - (a_1-1)M_1(\alpha_2-\beta_2 M_1) + (n-a_s)M_1(\kappa_2-\delta_2 M_1) \\ &\quad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}-\beta_{1j}M_1-\gamma_{1j}M_1), \\ B_2 &= (a_1-1)(X_{a_1:n}-M_2)(\alpha_2-2\beta_2 M_1) - (n-a_s)(X_{a_s:n}-M_2)(\kappa_2-2\delta_2 M_1) \\ &\quad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left\{\beta_{1j}(X_{a_j:n}-M_2)+\gamma_{1j}(X_{a_{j-1}:n}-M_2)\right\} \\ &\quad +2M_1\sum_{j=1}^{s}(X_{a_j:n}-M_2), \\ C_2 &= (a_1-1)\beta_2(X_{a_1:n}-M_2)^2 - (n-a_s)\delta_2(X_{a_s:n}-M_2)^2 - \sum_{j=1}^{s}(X_{a_j:n}-M_2)^2.\end{aligned}$$

Third, making use of the approximate expressions in (2.5), (2.6), (2.10), (2.11), (2.12) and (2.13), we may approximate the likelihood equation (2.3) as follows:

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &= -\frac{1}{\sigma}\Big[2s + (a_1-1)(\alpha_1+\beta_1 Z_{a_1:n}) - (n-a_s)(\kappa_1+\delta_1 Z_{a_s:n}) - \sum_{j=1}^{s} Z_{a_j:n}^2 \\ &\quad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left\{(\alpha_{2j}+\beta_{2j}Z_{a_j:n}+\gamma_{2j}Z_{a_{j-1}:n})Z_{a_j:n}\right. \\ &\quad \left.-(\alpha_{3j}+\beta_{3j}Z_{a_j:n}+\gamma_{3j}Z_{a_{j-1}:n})Z_{a_{j-1}:n}\right\}\Big] \\ &= 0.\end{aligned} \tag{2.21}$$

Upon solving the equations (2.21) and (2.15) for σ and θ, we derive the other AMLEs of σ and θ as follows:

$$\hat{\sigma}_3 = \frac{-B_3 + \sqrt{B_3{}^2 - 4A_3C_3}}{2A_3} \tag{2.22}$$

and

$$\hat{\theta}_3 = M_1\hat{\sigma}_3 + M_2, \tag{2.23}$$

where

$$\begin{aligned}
A_3 =& (2 - M_1^2)s + (a_1 - 1)(\alpha_1 - \beta_1 M_1) - (n - a_s)(\kappa_1 - \delta_1 M_1)\\
&+ \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_{4j}M_1^2 + \gamma_{4j}M_1^2 - \alpha_{4j}M_1),\\
B_3 =& (a_1 - 1)\beta_1(X_{a_1:n} - M_2) - (n - a_s)\delta_1(X_{a_s:n} - M_2)\\
&+ 2M_1\sum_{j=1}^{s}(X_{a_j:n} - M_2) + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left\{(\alpha_{2j} - 2M_1\beta_{2j})(X_{a_j:n} - M_2)\right.\\
&\left. - 2\gamma_{2j}M_1(X_{a_j:n} - M_2)(X_{a_{j-1}:n} - M_2) - (\alpha_{3j} - 2M_1\gamma_{3j})(X_{a_{j-1}:n} - M_2)\right\},\\
C_3 =& -\sum_{j=1}^{s}(X_{a_j:n} - M_2)^2 + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\left\{\beta_2(X_{a_j:n} - M_2)^2 + 2\gamma_{2j}(X_{a_j:n} - M_2)\right.\\
&\left. - \gamma_{3j}(X_{a_{j-1}:n} - M_2)^2\right\}.
\end{aligned}$$

Fourth, making use of the approximate expressions in (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13), we may approximate the likelihood equation (2.3) as follows:

$$\begin{aligned}
\frac{\partial \ln L}{\partial \sigma} =& -\frac{1}{\sigma}\Bigg[2s + (a_1 - 1)(\alpha_2 + \beta_2 Z_{a_1:n})Z_{a_1:n} - (n - a_s)(\kappa_2 + \delta_2 Z_{a_s:n})Z_{a_s:n}\\
&- \sum_{j=1}^{s} Z_{a_j:n}^2 + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\big\{(\alpha_{2j} + \beta_{2j}Z_{a_j:n} + \gamma_{2j}Z_{a_{j-1}:n})Z_{a_j:n}\\
&- (\alpha_{3j} + \beta_{3j}Z_{a_j:n} + \gamma_{3j}Z_{a_{j-1}:n})Z_{a_{j-1}:n}\big\}\Bigg] \qquad (2.24)\\
=& 0.
\end{aligned}$$

Upon solving the equations (2.24) and (2.15) for σ and θ, we derive the AMLEs of σ and θ as follows:

$$\hat{\sigma}_4 = \frac{-B_4 + \sqrt{B_4{}^2 - 4A_4C_4}}{2A_4} \tag{2.25}$$

and

$$\hat{\theta}_4 = M_1\hat{\sigma}_4 + M_2, \tag{2.26}$$

Table 3.1: The relative mean squared errors for the estimators of the scale parameter σ.

n	m	a_j	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\sigma}_3$	$\hat{\sigma}_4$
20	0	1∼20	0.013528	0.013528	0.013528	0.013528
	1	1∼19	0.014557	0.014444	0.014557	0.014444
		2∼20	0.014552	0.014422	0.014552	0.014422
	2	1∼18	0.015683	0.015414	0.015683	0.015414
		3∼20	0.015805	0.015519	0.015805	0.015519
		2∼19	0.015641	0.015378	0.015641	0.015378
	3	1∼17	0.017551	0.017079	0.017551	0.017079
		4∼20	0.017588	0.017101	0.017588	0.017101
		2∼18	0.016862	0.016427	0.016862	0.016427
		3∼19	0.016948	0.016519	0.016948	0.016519
	4	2∼17	0.018869	0.018213	0.018869	0.018213
		4∼19	0.018825	0.018181	0.018825	0.018181
		3∼18	0.018233	0.017628	0.018233	0.017628
		2∼4 7∼14 16∼20	0.014862	0.014693	0.014429	0.014452
	5	3∼17	0.020344	0.019499	0.020344	0.019499
		4∼18	0.020202	0.019362	0.020202	0.019362
		2∼6 10∼19	0.015649	0.015392	0.015862	0.015866
	6	4∼17	0.022448	0.021339	0.022448	0.021339
		1 2 6∼9 12∼15 17∼20	0.014467	0.014467	0.013886	0.013886
50	0	1∼50	0.005243	0.005243	0.005243	0.005243
	1	1∼49	0.005376	0.005371	0.005376	0.005371
		2∼50	0.005376	0.005369	0.005376	0.005369
	2	1∼48	0.005497	0.005484	0.005497	0.005484
		3∼50	0.005523	0.005511	0.005523	0.005511
		2∼49	0.005507	0.005495	0.005507	0.005495
	3	1∼47	0.005629	0.005607	0.005629	0.005607
		4∼50	0.005660	0.005639	0.005660	0.005639
		2∼48	0.005628	0.005608	0.005628	0.005608
		3∼49	0.005656	0.005638	0.005656	0.005638
	4	2∼47	0.005762	0.005732	0.005762	0.005732
		4∼49	0.005794	0.005765	0.005794	0.005765
		3∼48	0.005776	0.005751	0.005776	0.005751
		2∼24 27∼44 46∼50	0.005469	0.005455	0.005595	0.005578
	5	3∼47	0.005910	0.005874	0.005910	0.005874
		4∼48	0.005913	0.005877	0.005913	0.005877
		2∼16 19∼28 31∼50	0.005388	0.005380	0.006131	0.006151
	6	4∼47	0.006048	0.006001	0.006048	0.006001
		1 2 6∼9 12∼15 17∼50	0.005311	0.005311	0.005363	0.005363

where

$$\begin{aligned}
A_4 &= (2-M_1^2)s-(a_1-1)M_1(\alpha_2-\beta_2M_1)+(n-a_s)M_1(\kappa_2-\delta_2M_1)\\
&\quad+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{4j}M_1^2+\gamma_{4j}M_1^2-\alpha_{4j}M_1),\\
B_4 &= (a_1-1)(X_{a_1:n}-M_2)(\alpha_2-2\beta_2M_1)-(n-a_s)(X_{a_s:n}-M_2)(\kappa_2-2\delta_2M_1)\\
&\quad+2M_1\sum_{j=1}^{s}(X_{a_j:n}-M_2)+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\{(\alpha_{2j}-2M_1\beta_{2j})(X_{a_j:n}-M_2)\\
&\quad-2\gamma_{2j}M_1(X_{a_j:n}-M_2)(X_{a_{j-1}:n}-M_2)-(\alpha_{3j}-2M_1\gamma_{3j})(X_{a_{j-1}:n}-M_2)\},\\
C_4 &= (a_1-1)\beta_2(X_{a_1:n}-M_2)^2-(n-a_s)\delta_2(X_{a_s:n}-M_2)^2+C_3.
\end{aligned}$$

Table 3.2: The relative mean squared errors for the estimators of the location parameter θ.

n	m	a_j	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$
20	0	1∼20	0.182513	0.182513	0.182513	0.182513
	1	1∼19	0.183914	0.183898	0.183914	0.183898
		2∼20	0.183905	0.183888	0.183905	0.183888
	2	1∼18	0.186026	0.185963	0.186026	0.185963
		3∼20	0.186192	0.186129	0.186192	0.186129
		2∼19	0.185148	0.185148	0.185148	0.185148
	3	1∼17	0.189617	0.189486	0.189617	0.189486
		4∼20	0.189513	0.189379	0.189513	0.189379
		2∼18	0.187072	0.187057	0.187072	0.187057
		3∼19	0.187246	0.187231	0.187246	0.187231
	4	2∼17	0.190432	0.190380	0.190432	0.190380
		4∼19	0.190345	0.190291	0.190345	0.190291
		3∼18	0.188941	0.188941	0.188941	0.188941
		2∼4 7∼14 16∼20	0.185963	0.185941	0.185956	0.185946
	5	3∼17	0.192005	0.191994	0.192005	0.191994
		4∼18	0.191766	0.191755	0.191766	0.191755
		2∼6 10∼19	0.185401	0.185401	0.185408	0.185408
	6	4∼17	0.194468	0.194468	0.194468	0.194468
		1 2 6∼9 12∼15 17∼20	0.088537	0.088537	0.088521	0.088521
50	0	1∼50	0.101869	0.101869	0.101869	0.101869
	1	1∼49	0.102058	0.102057	0.102058	0.102057
		2∼50	0.102078	0.102077	0.102078	0.102077
	2	1∼48	0.102321	0.102318	0.102321	0.102318
		3∼50	0.102352	0.102349	0.102352	0.102349
		2∼49	0.102258	0.102258	0.102258	0.102258
	3	1∼47	0.102632	0.102627	0.102632	0.102627
		4∼50	0.102642	0.102636	0.102642	0.102636
		2∼48	0.102511	0.102511	0.102511	0.102511
		3∼49	0.102523	0.102522	0.102523	0.102522
	4	2∼47	0.102812	0.102810	0.102812	0.102810
		4∼49	0.102803	0.102801	0.102803	0.102801
		3∼48	0.102766	0.102766	0.102766	0.102766
		2∼24 27∼44 46∼50	0.040619	0.040618	0.040618	0.040617
	5	3∼47	0.103054	0.103054	0.103054	0.103054
		4∼48	0.103035	0.103034	0.103035	0.103034
		2∼16 19∼28 31∼50	0.102234	0.102233	0.102251	0.102250
	6	4∼47	0.103312	0.103312	0.103312	0.103312
		1 2 6∼9 12∼15 17∼50	0.102053	0.102053	0.102051	0.102051

3. The Simulated Results

The mean squared errors of the proposed estimators are simulated by Monte Carlo simulation method. The simulation procedure is repeated 10,000 times for the sample size $n = 20, 50$ and various choices of censoring ($m = n - s$ is the number of unobserved or missing data) under multiply Type-II censored samples. These values are given in Table 3.1 and Table 3.2.

From Table 3.1, when the censoring is left or right, or double, the estimators $\hat{\sigma}_2$ and $\hat{\sigma}_4$ are more efficient than the other estimators $\hat{\sigma}_1$ and $\hat{\sigma}_3$ of the scale parameter σ in the sense of the MSE. For complete sample, the MSEs of four estimators are same.

From Table 3.2, the estimators $\hat{\theta}_2$ and $\hat{\theta}_4$ that use the estimators $\hat{\sigma}_2$ and $\hat{\sigma}_4$ are more efficient than the order estimators of the location parameter θ in the sense of the MSE. For complete sample, the MSEs of four estimators are also same.

As expected, the MSE of all estimators decreases as sample size n increases. For fixed sample size, the MSE increases generally as the number of unobserved or missing data m increases.

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