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Noninformative Priors for the Coefficient of Variation in Two Inverse Gaussian Distributions

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Abstract

In this paper, we develop the noninformative priors when the parameter of interest is the common coefficient of variation in two inverse Gaussian distributions. We want to develop the first and second order probability matching priors. But we prove that the second order probability matching prior does not exist. It turns out that the one-at-a-time and two group reference priors satisfy the first order matching criterion but Jeffreys' prior does not. The Bayesian credible intervals based on the one-at-a-time reference prior meet the frequentist target coverage probabilities much better than that of Jeffreys' prior. Some simulations are given.

Keywords: Coefficient of variation; inverse Gaussian distribution; probability matching prior; reference prior.

1. Introduction

The inverse Gaussian distribution is a very versatile and flexible probabilistic model for positive right-skewed data and has potentially useful applications in a wide variety of fields such as biology, economics, reliability theory, life testing and social sciences as discussed in Folks and Chhikara (1979), Chhikara and Folks (1989), Whitmore (1979), Seshadri (1999) and Mudholkar and Natarajan (2002). Tweedie (1957a, 1957b) established many important statistical properties of the inverse Gaussian distribution and discussed the similarity between statistical methods based on the inverse Gaussian distribution and those based on the normal theory. The inverse Gaussian distribution IG(μ, λ) is given by

$$f(x|\mu, \lambda) = \sqrt{\frac{\lambda}{2\pi}} x^{-\frac{3}{2}} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \quad x > 0, \tag{1.1}$$

where $\mu > 0$ is the mean parameter and $\lambda > 0$ is the scale parameter.

Consider that $X_1, \ldots, X_{n_1}$ are independent and identically distributed random variables according to the inverse Gaussian IG$(\mu_1, \mu_1/\gamma^2)$ and $Y_1, \ldots, Y_{n_2}$ are independent and identically distributed random variables according to the IG$(\mu_2, \mu_2/\gamma^2)$, where μ_1

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and μ_2 are the mean parameters and γ is the common coefficient of variation. Then the likelihood function of μ_1, μ_2 and γ given $\mathbf{x} = (x_1, \ldots, x_{n_1})$ and $\mathbf{y} = (y_1, \ldots, y_{n_2})$ is

$$L(\mu_1, \mu_2, \gamma|\mathbf{x}, \mathbf{y}) \propto \mu_1^{\frac{n_1}{2}} \mu_2^{\frac{n_2}{2}} \gamma^{-(n_1+n_2)} \exp\left\{ -\sum_{i=1}^{n_1} \frac{(x_i - \mu_1)^2}{2\mu_1\gamma^2 x_i} - \sum_{i=1}^{n_2} \frac{(y_i - \mu_2)^2}{2\mu_2\gamma^2 y_i} \right\}, \quad (1.2)$$

where $\mu_1 > 0$, $\mu_2 > 0$ and $\gamma > 0$. The parameter of interest is $\theta_1 = \gamma$, the common coefficient of variation.

The present paper focuses on developing noninformative priors for θ_1. We consider Bayesian priors such that the resulting credible intervals for θ_1 have coverage probabilities equivalent to their frequentist counterparts. Although this matching can be justified only asymptotically, our simulation results indicate that this is indeed achieved for small or moderate sample sizes as well.

This matching idea goes back to Welch and Peers (1963). Interest in such priors revived with the work of Stein (1985) and Tibshirani (1989). Among others, we may cite the work of Mukerjee and Dey (1993), DiCiccio and Stern (1994), Datta and Ghosh (1995a), Datta and Ghosh (1995b, 1996), Mukerjee and Ghosh (1997).

On the other hand, Ghosh and Mukerjee (1992) and Berger and Bernardo (1989, 1992) extended Bernardo's (1979) reference prior approach, giving a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems. Quite often reference priors satisfy the matching criterion.

The coefficient of variation has been widely used as a measure of precision and repeatability of data in medical studies. For example, in toxicology, the coefficient of variation can be used as a measure of precision within and between laboratories, or among replicates for each treatment concentration. For the coefficient of variation in single inverse Gaussian distribution, Hsieh (1990) derived the likelihood ratio test and obtained the confidence bounds. Choi and Kim (2004) derived the likelihood ratio, Lagrange Multiplier and Wald tests for testing of the homogeneity of coefficients of variation in inverse Gaussian populations. They concluded that the likelihood ratio test is most powerful in the case of small to moderate samples in their simulation results. However there is a little work in this problem from the viewpoint of Bayesian framework. It is well known that the role of the objective priors such as the probability matching prior or the reference prior in the presence of nuisance parameters is very important in Bayesian inference. Kang *et al.* (2004) developed the noninformative priors for the ratio of parameters in inverse Gaussian distribution. They showed that the second order matching prior does not exist and the one-at-a-time reference prior satisfying the first order matching criterion meets very well the target coverage probabilities than Jeffreys' prior.

The outline of the remaining sections is as follows. In section 2, we consider the first order and second order probability matching priors for the common coefficient of variation in two inverse Gaussian distributions. We reveal that the second order matching prior does not exist. It turns out that the one-at-a-time and two group reference priors satisfy the first order matching criterion but Jeffreys' prior does not. We provide that the propriety of the posterior distribution for the first order matching prior and the reference priors in section 3. In section 4, simulated frequentist coverage probabilities under the proposed priors are given.

2. The Noninformative Priors

2.1. The probability matching priors

For a prior π, let $\theta_1^{1-\alpha}(\pi;\mathbf{X})$ denote the $(1-\alpha)^{th}$ percentile of the posterior distribution of θ_1, that is,

$$P^{\pi}[\theta_1 \leq \theta_1^{1-\alpha}(\pi;\mathbf{X})|\mathbf{X}] = 1-\alpha, \tag{2.1}$$

where $\boldsymbol{\theta} = (\theta_1,\dots,\theta_t)^T$ and θ_1 is the parameter of interest. We want to find priors π for which

$$P[\theta_1 \leq \theta_1^{1-\alpha}(\pi;\mathbf{X})|\boldsymbol{\theta}] = 1-\alpha+o(n^{-u}), \tag{2.2}$$

for some $u>0$, as n goes to infinity. Priors π satisfying (2.2) are called matching priors. If $u=1/2$, then π is referred to as a first order matching prior, while if $u=1$, π is referred to as a second order matching prior.

In order to find such matching priors π, let

$$\theta_1=\gamma, \qquad \theta_2=\mu_1(2+\gamma^2)^{-1} \quad \text{and} \quad \theta_3=\mu_2(2+\gamma^2)^{-1}.$$

With this parameterization, the likelihood function of $(\theta_1,\theta_2,\theta_3)$ for the model (1.2) is given by

$$\begin{aligned} L(\theta_1,\theta_2,\theta_3|\mathbf{x},\mathbf{y}) \propto\ & \theta_1^{-(n_1+n_2)}\theta_2^{\frac{n_1}{2}}\theta_3^{\frac{n_2}{2}}(2+\theta_1^2)^{\frac{n_1+n_2}{2}} \\ & \times \exp\left[-\sum_{i=1}^{n_1}\frac{\{x_i-\theta_2(2+\theta_1^2)\}^2}{2\theta_1^2\theta_2(2+\theta_1^2)x_i}-\sum_{i=1}^{n_2}\frac{\{y_i-\theta_3(2+\theta_1^2)\}^2}{2\theta_1^2\theta_3(2+\theta_1^2)y_i}\right]. \end{aligned} \tag{2.3}$$

Based on the likelihood function (2.3), the Fisher information matrix is given by

$$\mathbf{I} = \begin{pmatrix} \dfrac{4(n_1+n_2)}{\theta_1^2(2+\theta_1^2)} & 0 & 0 \\ 0 & \dfrac{n_1(2+\theta_1^2)}{2\theta_1^2\theta_2^2} & 0 \\ 0 & 0 & \dfrac{n_2(2+\theta_1^2)}{2\theta_1^2\theta_3^2} \end{pmatrix}.$$

From the above Fisher information matrix $\mathbf{I}$, θ_1 is orthogonal to θ_2 and θ_3 in the sense of Cox and Reid (1987). Following Tibshirani (1989), the class of the first order probability matching prior is characterized by

$$\pi_m^{(1)}(\theta_1,\theta_2,\theta_3) \propto \theta_1^{-1}(2+\theta_1^2)^{-\frac{1}{2}}d(\theta_2,\theta_3), \tag{2.4}$$

where $d(\theta_2,\theta_3)>0$ is an arbitrary function differentiable in its arguments.

The class of first order probability matching prior given in (2.4) is so broad, so one can narrow down this prior to the second order probability matching prior as given in Mukerjee and Ghosh (1997).

The second order probability matching prior is of the form (2.4) and also the function $d(\cdot)$ must satisfy an additional differential equation (*cf.* (2.10) of Mukerjee and Ghosh (1997)), namely,

$$\frac{1}{6}d(\theta_2,\theta_3)\frac{\partial}{\partial\theta_1}\left(I_{11}^{-\frac{3}{2}}L_{1,1,1}\right)+\sum_{v=2}^{3}\frac{\partial}{\partial\theta_v}\left\{I_{11}^{-\frac{1}{2}}L_{11v}I^{vv}d(\theta_2,\theta_3)\right\}=0, \qquad (2.5)$$

where

$$\begin{aligned}
L_{1,1,1} &= E\left[\left(\frac{\partial\log L}{\partial\theta_1}\right)^3\right]=\frac{8(n_1+n_2)(8+6\theta_1^2+3\theta_1^4)}{\theta_1^3(2+\theta_1^2)^3},\\
L_{112} &= E\left(\frac{\partial^3\log L}{\partial\theta_1^2\partial\theta_2}\right)=\frac{4n_1}{(2+\theta_1^2)^2\theta_2},\\
L_{113} &= E\left(\frac{\partial^3\log L}{\partial\theta_1^2\partial\theta_3}\right)=\frac{4n_2}{(2+\theta_1^2)^2\theta_3},\\
I_{11} &= \frac{4(n_1+n_2)}{\theta_1^2(2+\theta_1^2)},\qquad I^{22}=\frac{2\theta_1^2\theta_2^2}{n_1(2+\theta_1^2)}\quad\text{and}\quad I^{33}=\frac{2\theta_1^2\theta_3^2}{n_2(2+\theta_1^2)}.
\end{aligned}$$

Then (2.5) simplifies to

$$\frac{6+\theta_1^2}{8}d(\theta_2,\theta_3)+\frac{\partial}{\partial\theta_2}\{\theta_2 d(\theta_2,\theta_3)\}+\frac{\partial}{\partial\theta_3}\{\theta_3 d(\theta_2,\theta_3)\}=0. \qquad (2.6)$$

Note that the first term depends only on $(\theta_1,\theta_2,\theta_3)$ and the second and third only on (θ_2,θ_3). Hence there can be no solution to (2.6) unless the first term is zero. Therefore the second order matching prior does not exist.

2.2. The reference priors

Reference priors introduced by Bernardo (1979) and extended further by Berger and Bernardo (1989, 1992) have become very popular over the years for the development of noninformative priors. In this Section, we derive the reference priors for different groups of ordering of $(\theta_1,\theta_2,\theta_3)$. Then due to the orthogonality of the parameters, following Datta and Ghosh (1995b), choosing rectangular compacts for each θ_1, θ_2 and θ_3 when θ_1 is the parameter of interest, the reference priors are given as follow.

If θ_1 is the parameter of interest, then the reference prior distributions for different groups of ordering of $(\theta_1,\theta_2,\theta_3)$ are:

Group ordering	Reference prior
$\{(\theta_1,\theta_2,\theta_3)\}$,	$\pi_1(\theta_1,\theta_2,\theta_3)\propto\theta_1^{-3}(2+\theta_1^2)^{\frac{1}{2}}\theta_2^{-1}\theta_3^{-1}$,
$\{\theta_1,(\theta_2,\theta_3)\},\{\theta_1,\theta_2,\theta_3\},\{\theta_1,\theta_3,\theta_2\}$,	$\pi_2(\theta_1,\theta_2,\theta_3)\propto\theta_1^{-1}(2+\theta_1^2)^{-\frac{1}{2}}\theta_2^{-1}\theta_3^{-1}$.

Remark 2.1 The one-at-a-time and two group reference priors π_2 satisfy the first order matching criterion but Jeffreys' prior π_1 does not.

Notice that the matching prior (2.4) includes many different matching priors because of the arbitrary selection of the function d. However all functions are not permissible in the construction of priors. For instance, we consider any function of the form $\theta_2^a\theta_3^b$. If a and b are positive integer, then the posterior distribution under function of the form $\theta_2^a\theta_3^b$ is proper. But the condition of propriety in this form strongly depend on a and b. Moreover there does not seem to be any improvement in the coverage probabilities with this posterior distribution. So we have chosen d to be $\theta_2^{-1}\theta_3^{-1}$. The resulting prior is given by

$$\pi_m^{(1)}(\theta_1,\theta_2,\theta_3) \propto \theta_1^{-1}(2+\theta_1^2)^{-\frac{1}{2}}\theta_2^{-1}\theta_3^{-1}. \tag{2.7}$$

Because of this, the matching prior is the one-at-a-time reference prior.

3. Implementation of the Bayesian Procedure

We investigate the propriety of posterior for general priors which include the reference priors and the first order probability matching prior (2.7). We consider the class of priors

$$\pi_g(\theta_1,\theta_2,\theta_3) \propto \theta_1^{-a}\theta_2^{-b}\theta_3^{-c}(2+\theta_1^2)^d, \tag{3.1}$$

where $a, b, c > 0$ and $|d| > 0$. The following theorem shows the propriety of posterior under the prior (3.1).

Theorem 3.1 The posterior distribution of $(\theta_1,\theta_2,\theta_3)$ under the prior (3.1) is proper if $a - 2d - 1 > 0$, $n_1 - 2b + 2 > 0$ and $n_2 - 2c + 2 > 0$.

Proof: Under the prior (3.1), the joint posterior for θ_1, θ_2 and θ_3 given $\mathbf{x}$ and $\mathbf{y}$ is

$$\begin{aligned}\pi(\theta_1,\theta_2,\theta_3|\mathbf{x},\mathbf{y}) \quad \propto \quad & \theta_1^{-(n_1+n_2)-a}\theta_2^{\frac{n_1}{2}-b}\theta_3^{\frac{n_2}{2}-c}(2+\theta_1^2)^{\frac{n_1+n_2}{2}+d} \\ & \times \exp\left[-\sum_{i=1}^{n_1}\frac{\{x_i-\theta_2(2+\theta_1^2)\}^2}{2\theta_1^2\theta_2(2+\theta_1^2)x_i} - \sum_{i=1}^{n_2}\frac{\{y_i-\theta_3(2+\theta_1^2)\}^2}{2\theta_1^2\theta_3(2+\theta_1^2)y_i}\right].\end{aligned}$$

Let $\theta_1 = \gamma$, $\theta_2 = \mu_1(2+\gamma^2)^{-1}$ and $\theta_3 = \mu_2(2+\gamma^2)^{-1}$. Then the posterior is given by

$$\begin{aligned}\pi(\mu_1,\mu_2,\gamma|\mathbf{x},\mathbf{y}) \quad \propto \quad & \mu_1^{\frac{n_1}{2}-b}\mu_2^{\frac{n_2}{2}-c}\gamma^{-(n_1+n_2+a)}(2+\gamma^2)^{b+c+d-2} \\ & \times \exp\left\{-\sum_{i=1}^{n_1}\frac{(x_i-\mu_1)^2}{2\mu_1\gamma^2x_i} - \sum_{i=1}^{n_2}\frac{(y_i-\mu_2)^2}{2\mu_2\gamma^2y_i}\right\},\end{aligned} \tag{3.2}$$

For γ, we have the following facts:

$$\begin{aligned}
&(2+\gamma^2)^{b+c+d-2} \le 3^{b+c+d-2}, && \text{for } 0<\gamma\le 1 \text{ and } b+c+d-2>0,\\
&(2+\gamma^2)^{b+c+d-2} \le 2^{b+c+d-2}, && \text{for } 0<\gamma\le 1 \text{ and } b+c+d-2<0,\\
&(2+\gamma^2)^{b+c+d-2} \le \gamma^{2(b+c+d-2)}3^{b+c+d-2}, && \text{for } 1<\gamma\le \infty \text{ and } b+c+d-2>0,\\
&(2+\gamma^2)^{b+c+d-2} \le 3^{b+c+d-2}, && \text{for } 1<\gamma\le \infty \text{ and } b+c+d-2<0.
\end{aligned}$$

Thus we only need to show that the following function is finite.

$$\begin{aligned}\pi'(\mu_1,\mu_2,\gamma|\mathbf{x},\mathbf{y}) \quad \propto \quad & \mu_1^{\frac{n_1}{2}-b}\mu_2^{\frac{n_2}{2}-c}\gamma^{-(n_1+n_2+a)+2(b+c+d-2)} \\ & \times \exp\left\{-\sum_{i=1}^{n_1}\frac{(x_i-\mu_1)^2}{2\mu_1\gamma^2x_i} - \sum_{i=1}^{n_2}\frac{(y_i-\mu_2)^2}{2\mu_2\gamma^2y_i}\right\}.\end{aligned} \tag{3.3}$$

Integrating with respect to γ in posterior (3.3), we have

$$\pi'(\mu_1,\mu_2|\mathbf{x},\mathbf{y}) \propto \mu_1^{\frac{n_1}{2}-b}\mu_2^{\frac{n_2}{2}-c} \times\left\{\mu_1\left(s_1+\frac{n_1\left(\frac{\bar{x}}{\mu_1}-1\right)^2}{\bar{x}}\right)+\mu_2\left(s_2+\frac{n_2\left(\frac{\bar{y}}{\mu_2}-1\right)^2}{\bar{y}}\right)\right\}^{-\frac{k}{2}}, \quad (3.4)$$

where $s_1=\sum_{i=1}^{n_1}(1/x_i-1/\bar{x})$, $\bar{x}=\sum_{i=1}^{n_1}x_i/n_1$, $s_2=\sum_{i=1}^{n_2}(1/y_i-1/\bar{y})$, $\bar{y}=\sum_{i=1}^{n_2}y_i/n_2$ and $k=n_1+n_2+a-2(b+c+d)+3$, if $k>0$. For (3.4), substituting $t_1=\mu_1^{-1}$ and $t_2=\mu_2^{-1}$, then

$$\pi'(t_1,t_2|\mathbf{x},\mathbf{y}) \propto t_1^{\frac{n_2+a-2(c+d)-1}{2}}t_2^{\frac{n_1+a-2(b+d)-1}{2}} \times\left\{t_2\left(s_1+\frac{n_1(\bar{x}t_1-1)^2}{2\bar{x}}\right)+t_1\left(s_2+\frac{n_2(\bar{y}t_2-1)^2}{2\bar{y}}\right)\right\}^{-\frac{k}{2}}. \quad (3.5)$$

For $0<t_1\le 1$ and $0<t_2\le 1$,

$$\int_0^1\int_0^1 \pi'(t_1,t_2|\mathbf{x},\mathbf{y})d\mu_1 d\mu_2 \le \int_0^1\int_0^1 t_1^{\frac{n_2+a-2(c+d)-1}{2}}t_2^{\frac{n_1+a-2(b+d)-1}{2}} \times(t_2s_1+t_1s_2)^{-\frac{k}{2}}dt_1dt_2<\infty,$$

if $a-2d-1>0$, $n_1-2b+2>0$ and $n_2+a-2(c+d)+1>0$. For $1<t_1<\infty$ and $1<t_2<\infty$,

$$\int_1^\infty\int_1^\infty \pi'(t_1,t_2|\mathbf{x},\mathbf{y})d\mu_1 d\mu_2 \le \int_1^\infty\int_1^\infty t_1^{\frac{n_2+a-2(c+d)-1}{2}}t_2^{\frac{n_1+a-2(b+d)-1}{2}} \times\left\{\frac{n_1t_2(\bar{x}t_1-1)^2}{2\bar{x}}+\frac{n_2t_1(\bar{y}t_2-1)^2}{2\bar{y}}\right\}^{-\frac{k}{2}}dt_1dt_2<\infty,$$

if $n_1-2b+2>0$ and $n_2-2c+2>0$. For $0<t_1\le 1$ and $1<t_2<\infty$,

$$\int_1^\infty\int_0^1 \pi'(t_1,t_2|\mathbf{x},\mathbf{y})d\mu_1 d\mu_2 \le \int_1^\infty\int_0^1 t_1^{\frac{n_2+a-2(c+d)-1}{2}}t_2^{\frac{n_1+a-2(b+d)-1}{2}} \times(t_2s_1)^{-\frac{k}{2}}dt_1dt_2<\infty,$$

if $n_2+a-2(c+d)+1>0$ and $n_2-2c+2>0$. This completes the proof. □

Theorem 3.2 The marginal posterior density of θ_1 under the prior (3.1) is given by

$$\pi(\theta_1|\mathbf{x},\mathbf{y}) \propto \frac{(2+\theta_1^2)^{\frac{n_1+n_2+2d}{2}}}{\theta_1^{n_1+n_2+a}}\text{BesselK}\left[\frac{n_1-2b+2}{2},\frac{\left(\sum_{i=1}^{n_1}x_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n_1}x_i^{-1}\right)^{\frac{1}{2}}}{\theta_1^2}\right]$$

$$\times \text{BesselK}\left[\frac{n_2-2c+2}{2}, \frac{\left(\sum_{i=1}^{n_2} y_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n_2} y_i^{-1}\right)^{\frac{1}{2}}}{\theta_1^2}\right],$$

where BesselK$[\cdot,\cdot]$ is the modified Bessel function of the second order.

Actually the normalizing constant for the marginal density of θ_1 requires one dimensional integration. Therefore we can have the marginal posterior density of θ_1 and the marginal moment of θ_1.

4. Numerical Studies and Discussion

We evaluate the frequentist coverage probability by investigating the credible interval of the marginal posterior density of θ_1 under the noninformative prior π given in section 3 for several configurations (μ_1, μ_2, γ) and (n_1, n_2). That is to say, the frequentist coverage of a $(1-\alpha)$ posterior quantile should be close to $(1-\alpha)$. This is done numerically. Tables 4.1, 4.2 and 4.3 give numerical values of the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for the proposed priors. The computation of these numerical values is based on the following algorithm for any fixed true (μ_1, μ_2, γ) and any prespecified probability α. Here α is 0.05 (0.95).

Let $\theta_1^\pi(\alpha|\mathbf{X},\mathbf{Y})$ be the posterior α-quantile of θ_1 given $\mathbf{X}$ and $\mathbf{Y}$. That is to say, $F(\theta_1^\pi(\alpha|\mathbf{X},\mathbf{Y})|\mathbf{X},\mathbf{Y}) = \alpha$, where $F(\cdot|\mathbf{X},\mathbf{Y})$ is the marginal posterior distribution function of θ_1. Then the frequentist coverage probability of this one sided credible interval of θ_1 is

$$P_{(\mu_1,\mu_2,\gamma)}(\alpha;\theta_1) = P_{(\mu_1,\mu_2,\gamma)}(0 < \theta_1 \le \theta_1^\pi(\alpha|\mathbf{X},\mathbf{Y})).$$

The estimated $P_{(\mu_1,\mu_2,\gamma)}(\alpha;\theta_1)$ when $\alpha = 0.05\,(0.95)$ is shown in Tables 4.1, 4.2 and 4.3. In particular, for fixed (μ_1, μ_2, γ), we take 10,000 independent random samples of $\mathbf{X}$ and $\mathbf{Y}$ from the model (1.2). For the cases presented in Tables 4.1, 4.2 and 4.3, we see that the one-at-a-time reference prior π_2 meets very well the target coverage probabilities for small values of n_1 and n_2. Also the results of tables are not much sensitive to the change of the values of (μ_1, μ_2) under small values of γ. Thus we can recommend to use the one-at-a-time reference prior when using the matching criterion. Note that Jeffreys' prior does not satisfy the first order matching criterion.

It appears that when γ is large (the case of $\theta_1 = 10$), the values of the freqentist coverage probabilities are far from target probabilities. The poor performance of all the priors for certain regions of the parameter space is not very surprising. Gleser and Hwang (1987, Theorem 1) show that based on any sample of arbitrary but fixed size, any confidence interval for θ of finite expected length has coverage probability equal to zero. In our case, this poor performance happens when θ_1 is large, that is, the case that the means are smaller than the standard deviations.

Table 4.1: Frequentist Coverage Probabilities of 0.05 (0.95) Posterior Quantiles for θ_1

γ	μ_1	μ_2	n_1	n_2	π_1		π_2	
0.1	0.1	0.1	5	5	0.018	(0.865)	0.050	(0.952)
			5	10	0.024	(0.885)	0.049	(0.949)
			10	10	0.025	(0.903)	0.049	(0.954)
			10	15	0.029	(0.906)	0.051	(0.950)
		1	5	5	0.018	(0.861)	0.049	(0.952)
			5	10	0.023	(0.885)	0.053	(0.950)
			10	10	0.028	(0.901)	0.052	(0.949)
			10	15	0.027	(0.913)	0.046	(0.952)
		10	5	5	0.021	(0.860)	0.051	(0.950)
			5	10	0.024	(0.892)	0.051	(0.950)
			10	10	0.027	(0.901)	0.052	(0.948)
			10	15	0.027	(0.909)	0.049	(0.950)
	1	0.1	5	5	0.018	(0.864)	0.049	(0.952)
			5	10	0.021	(0.884)	0.046	(0.947)
			10	10	0.025	(0.900)	0.050	(0.949)
			10	15	0.025	(0.910)	0.048	(0.950)
		1	5	5	0.018	(0.862)	0.047	(0.953)
			5	10	0.024	(0.890)	0.051	(0.951)
			10	10	0.024	(0.899)	0.046	(0.947)
			10	15	0.027	(0.911)	0.049	(0.951)
		10	5	5	0.020	(0.863)	0.050	(0.953)
			5	10	0.025	(0.892)	0.052	(0.952)
			10	10	0.026	(0.902)	0.051	(0.952)
			10	15	0.027	(0.905)	0.050	(0.946)
	10	0.1	5	5	0.019	(0.867)	0.048	(0.952)
			5	10	0.023	(0.882)	0.050	(0.946)
			10	10	0.028	(0.904)	0.049	(0.952)
			10	15	0.027	(0.909)	0.047	(0.951)
		1	5	5	0.020	(0.863)	0.050	(0.949)
			5	10	0.025	(0.888)	0.055	(0.949)
			10	10	0.026	(0.907)	0.051	(0.953)
			10	15	0.029	(0.905)	0.054	(0.949)
		10	5	5	0.020	(0.862)	0.051	(0.952)
			5	10	0.023	(0.891)	0.052	(0.950)
			10	10	0.026	(0.900)	0.049	(0.953)
			10	15	0.027	(0.912)	0.050	(0.952)

Table 4.2: Frequentist Coverage Probabilities of 0.05 (0.95) Posterior Quantiles for θ_1

γ	μ_1	μ_2	n_1	n_2	π_1		π_2	
1	0.1	0.1	5	5	0.023	(0.882)	0.051	(0.954)
			5	10	0.029	(0.896)	0.050	(0.949)
			10	10	0.032	(0.913)	0.051	(0.954)
			10	15	0.032	(0.918)	0.048	(0.955)
		1	5	5	0.025	(0.885)	0.053	(0.957)
			5	10	0.030	(0.903)	0.052	(0.955)
			10	10	0.031	(0.909)	0.056	(0.952)
			10	15	0.030	(0.913)	0.050	(0.950)
		10	5	5	0.022	(0.881)	0.048	(0.956)
			5	10	0.029	(0.900)	0.049	(0.952)
			10	10	0.031	(0.910)	0.052	(0.952)
			10	15	0.032	(0.915)	0.051	(0.953)
	1	0.1	5	5	0.024	(0.882)	0.050	(0.956)
			5	10	0.029	(0.905)	0.049	(0.955)
			10	10	0.030	(0.908)	0.052	(0.952)
			10	15	0.030	(0.916)	0.048	(0.952)
		1	5	5	0.025	(0.880)	0.051	(0.955)
			5	10	0.029	(0.903)	0.050	(0.953)
			10	10	0.033	(0.909)	0.054	(0.951)
			10	15	0.033	(0.917)	0.052	(0.953)
		10	5	5	0.025	(0.879)	0.053	(0.955)
			5	10	0.033	(0.900)	0.055	(0.954)
			10	10	0.032	(0.914)	0.053	(0.954)
			10	15	0.029	(0.916)	0.049	(0.952)
	10	0.1	5	5	0.026	(0.882)	0.052	(0.956)
			5	10	0.028	(0.901)	0.052	(0.956)
			10	10	0.029	(0.910)	0.050	(0.951)
			10	15	0.031	(0.922)	0.051	(0.955)
		1	5	5	0.026	(0.881)	0.053	(0.954)
			5	10	0.030	(0.898)	0.054	(0.950)
			10	10	0.032	(0.907)	0.051	(0.950)
			10	15	0.032	(0.913)	0.050	(0.950)
		10	5	5	0.025	(0.876)	0.049	(0.951)
			5	10	0.030	(0.894)	0.055	(0.948)
			10	10	0.033	(0.907)	0.054	(0.948)
			10	15	0.031	(0.913)	0.050	(0.949)

Table 4.3: Frequentist Coverage Probabilities of 0.05 (0.95) Posterior Quantiles for θ_1

γ	μ_1	μ_2	n_1	n_2	π_1		π_2	
10	0.1	0.1	5	5	0.001	(0.846)	0.001	(0.910)
			5	10	0.004	(0.897)	0.004	(0.932)
			10	10	0.006	(0.911)	0.006	(0.936)
			10	15	0.007	(0.930)	0.007	(0.949)
		1	5	5	0.002	(0.850)	0.002	(0.906)
			5	10	0.004	(0.892)	0.004	(0.927)
			10	10	0.005	(0.910)	0.005	(0.935)
			10	15	0.006	(0.927)	0.006	(0.947)
		10	5	5	0.002	(0.851)	0.002	(0.909)
			5	10	0.004	(0.890)	0.004	(0.922)
			10	10	0.006	(0.914)	0.006	(0.940)
			10	15	0.007	(0.924)	0.007	(0.943)
	1	0.1	5	5	0.003	(0.844)	0.003	(0.903)
			5	10	0.004	(0.893)	0.004	(0.928)
			10	10	0.006	(0.919)	0.006	(0.941)
			10	15	0.007	(0.929)	0.007	(0.946)
		1	5	5	0.002	(0.851)	0.002	(0.909)
			5	10	0.003	(0.894)	0.004	(0.930)
			10	10	0.004	(0.911)	0.004	(0.936)
			10	15	0.006	(0.926)	0.006	(0.947)
		10	5	5	0.002	(0.844)	0.003	(0.903)
			5	10	0.004	(0.893)	0.004	(0.928)
			10	10	0.005	(0.913)	0.005	(0.939)
			10	15	0.008	(0.924)	0.008	(0.942)
	10	0.1	5	5	0.002	(0.851)	0.003	(0.910)
			5	10	0.003	(0.899)	0.003	(0.932)
			10	10	0.005	(0.910)	0.005	(0.936)
			10	15	0.007	(0.926)	0.007	(0.943)
		1	5	5	0.002	(0.844)	0.003	(0.902)
			5	10	0.003	(0.892)	0.003	(0.929)
			10	10	0.005	(0.908)	0.006	(0.933)
			10	15	0.007	(0.926)	0.007	(0.944)
		10	5	5	0.002	(0.847)	0.002	(0.907)
			5	10	0.004	(0.892)	0.004	(0.928)
			10	10	0.006	(0.915)	0.006	(0.939)
			10	15	0.008	(0.929)	0.008	(0.948)

5. Conclusion

In the inverse Gaussian distributions, we have found the first order matching priors and the reference priors for the common coefficient of variation. We have proved that the second order matching prior does not exist. And the one-at-a-time reference prior possess good frequentist properties in the sense that the coverage probabilities of credible intervals for the common coefficient of variation based on this prior match their frequentist counterparts very closely even for small sample sizes. Also Jeffreys' prior does not satisfy the first order matching criterion. From our simulation results, we recommend to use the one-at-a-time reference prior for the Bayesian inference of the common coefficient of variation.

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