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Humanity mathematics education: revealing and clarifying ambiguities in mathematical concepts over the school mathematics curriculum

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This study discusses how the humanity mathematics education can be realized in practice. The essence of mathematical concept is gradually disclosed revealing the ambiguities in the concept currently accepted and clarifying them. Historical development of mathematical concepts has progressed as such, exemplified with the group-theoretical thought and continuous function. In learning of mathematical concepts, thus, students have to recognize, reveal and clarify the ambiguities that intuitive and context-dependent definitions in school mathematics have. We present the process of improvement of definitions of a tangent and a polygon in school mathematics as examples. In the process, students may recognize the limitations of their thoughts and reform them with feelings of humility and satisfaction. Therefore this learning process would contribute to cultivating students' minds as the humanity mathematics education pursues.

Ⅰ. Introduction

The theme 'school mathematics for humanity education' was gestated at the PME31 held in Seoul, South Korea. There, it was claimed that "the nature of mathematical knowledge demands strongly school mathematics to become a main subject for humanity education going beyond the practicality (Woo, 2007a, p.xxxiii)." Woo (2007b, p.87), in the plenary session, claimed that the humanity mathematics education enables students to be aware of the reality, located inside human being and all things, that dominates the world of phenomenon, by the knowledge of mathematics. He summarized it as 'the cultivation of mind'. Thus, it can be claimed that the humanity mathematics education contributes to changes of students' minds. In concrete, it purposes change (or improvement) of the cognitive framework with which students see, feel and interpret their experiences.

Mathematics we currently have is a cultural heritage of human beings and school mathematics is composed of basic concepts of it. What would it mean for a student to learn the basic concepts? Some students might use mathematics learnt at school in their collegiate time or in their lifetime

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doing, for example, stock exchange, banking or statistical research. Some students may become mathematicians or mathematics teachers. However, more than half of our students may live, at least apparently, without any relevance to the mathematics learnt at school. In this reality how can school mathematics be meaningfully taught to students? In Woo's (2007b) terms, how can we provide students wonderful and impressive experiences about mathematics to see the phenomena in the world in wonder with mathematical eyes? How can students' minds change while they learn mathematics?

We would like to look for the answer of this question in learning process rather than mathematical content. The process we discuss is not a short-term one for a single class but a long-term one which is gradually progressed over the whole mathematics curriculum from elementary to secondary school. In learning of school mathematics students have to recognize the ambiguities of the concepts they currently have and to improve the concepts by clarifying the ambiguities. Lakatos (1976) claims that in growing mathematical theories "growing concepts are the vehicles of progress and the most exciting developments come from exploring the boundary regions of concepts from stretching them and from differentiating formerly undifferentiated concepts (p. 140)." According to him, there is no theory that has not passed through such a period of growth.

Teaching and learning of mathematics should start from the concepts and thoughts students currently possess, which has been integrated and used as if 'common senses (Freudenthal, 1991)' for them. They need to be challenged with revealing the ambiguities in their concepts and encouraged to clarify them. From the experience of clarification they would feel improvements and satisfaction. Therefore, teachers have to stimulate students to draw out their own ideas, reflect on them, see limitations or flaws in their ideas (or concepts), and then improve them. We believe that it would lead to the cultivation of students' mind that the humanity mathematics education aims at.

This article is composed of five chapters. The theoretical backgrounds are followed in chapter 2. The development of mathematical concepts over the history of mathematics is discussed with some examples in chapter 3. In chapter 4, two learning procedures of mathematical concepts are proposed to exemplify our ideas on the humanity mathematics education.

II. Theoretical Backgrounds

From the beginning of human history, people have been trying to explain what knowledge is and how human beings get to know it (Jang, 2000). Some philosophers, assuming the absolute truth, maintain that human beings seek for the truth. Some other philosophers oppose those theories on account that human beings cannot ascertain what they know corresponds to the absolute truth. If they can ascertain it, they have already known the truth, thus do not have to pursue it. Rationalists emphasize human reason bestowing criteria of truth on it. They suppose innate reason upon which human beings recognize, discern, think and judge. Their views are criticized in that they regard the reason as absolute, given from the beginning. On the other hand, empiricists insist that a baby is born in a state of tabula rasa, and as he/she experience outer worlds his/her experience is carved on it. It is also criticized in that they regard human reason too passively. Rationalism, putting too much stress on the human innate reason, could disregard the growth of human cognition and education received from the moment a child is born whereas empiricism could over-degrade it. Hamlyn (1978) opposes to empiricism appealing that understanding must be added to the experience. To understand his/her experience human cognition must operate.

Kant (1998) claims that human beings a priori possess cognitive forms, through which they actively construct their knowledge. They cannot know the world itself (Ding an sich), but perceive and interpret it through their own cognitive forms. Piaget (1970), who applies Kant's epistemology into psychology, explicates how children adapt themselves to environments and organize their cognitive structures. He supposes cognitive functions that human beings innately have, i.e. adaptation and organization. The adaptation proceeds through assimilation and accommodation. When children confront cognitive disequilibrium, they try to assimilate the causes of the disequilibrium into their existent cognitive structures. If the causes conflict to their cognitive structures, they accommodate the structures for entering a cognitively equilibrium state.

From the theories discussed above, a common factor can be drawn out. They all assume a certain cognitive ability that human beings innately have. Human beings can construct knowledge from perception and interpretation of the world they live. We believe that, different from rationalists, this innate cognitive ability is developed and refined through applications of it into one's experience throughout one's whole life. Mathematics education also has to contribute to this development and refinement of the cognitive ability. This is what we intend with the humanity mathematics education.

Concerning the act of knowing, Polanyi (1962) supposes levels of the reality. According to him, there are logical gaps between the levels. The leap by which the logical gap is crossed indicates achieving discovery. Ascending the levels of the reality means improvements of the framework with which we see, feel, and interpret the world we dwell in. Our understandings of the world are in accordance with the indications and standards imposed by the framework (Polanyi, 1969). The framework is a particular form of mental existence. Thus, act of knowing means choosing alternative framework or modifying the current framework we dwell in, and involves a change in our way of being. Freudenthal and van Hiele also claim the existence of the levels and leaps in the development of mathematical thinking. Freudenthal (1983) expresses it with the alternation between phainomenon (or content) and nooumenon (or form). Van Hiele (1986) expounds it with five levels in geometrical thinking.

Education is basically a social activity carried out by teachers and learners. Therefore, social factors are not negligible in education. According to Vygotsky (1978), there are higher mental (psychological) functions in other people that a child has to recognize and develop through social interactions. Referring to the cognitive disequilibrium resulting a change of cognitive framework, Balacheff (1991) particularly emphasizes the social dimension of contradiction, looking upon the experience of contradiction as a starting point for developmental process. "Mathematics can no longer be learned by means of interactions with a physical environment, but requires the confrontation of the students' cognitive model with that of other students or of the teacher, in the context of a given mathematical activity (p.89)." Grounded on these theories, for the humanity mathematics education, we would emphasize social aspects of mathematics education as well as children's own cognitive abilities, structures and frameworks (in a word, children's minds) gradually developed level by level.

Polanyi (1969) identifies knowing as "a utilization of a framework for unfolding our understanding in accordance with the indications and standards imposed by the framework (p.134)." Therefore, development or change of a framework implies getting new standards for judgment (Oakeshott, 2001). Knowledge is true belief based on legitimate reason (Hamlyn, 1978). Accepting new knowledge means being persuaded by it, leaded to believe it and accepting it as a new standard (Polanyi, 1962). For this, students need to see problems, insufficiencies, inadequacies, flaws or inferiority (in a word, limitations) in the standards they currently possess. Realizing the limitations, expressed in various terms like cognitive conflicts (or disequilibrium, Piaget, 1970), problems (Polanyi, 1962), refutations (Lakatos, 1976) and counter examples (Balacheff, 1991), is highly emphasized as a starting point for the improvement. It is more plausible, as Vygotsky, Hamlyn and Balacheff assert, through the social activities with other people. In educational practice, it would be mainly with teachers who stimulate their students to reflect on their current standards, recognize its limitations and change their frameworks.

Students may not accept a new framework or modify present framework, unless they ascertain the excellence of the new one. The new framework, for instance, may resolve the conflict that the present framework confronts, or extend the boundary of it. The direction of change is not casual choice between two equally acceptable frameworks. Students need to feel themselves of improvement by changing their frameworks. Being educated may imply the improvement of way of thinking, viewing and interpreting, or way of being in Polanyi's (1969) terms. The humanity mathematics education pursues it.

Here, the improvements of students' cognitive frameworks need not assume a sort of realism or positivism. What we emphasize is a satisfaction about the change. From the experience of seeing limitations and modifying their framework, students may feel having better way of being and be satisfied with it. This satisfaction can be a motive and driving force for further learning. Polanyi (1962) describes this with intellectual passions and commitments, necessary for intellectual explorations. We believe that the humanity mathematics education should purpose this, a change of student's mind.

The improvement proceeds through sequential stages, step by step. The present state is never a completed one even though it is satisfying for the moment. A limitation of it would be revealed and improved some while later. The actual development of mathematical concepts has also been taken place in the history of mathematics in this manner.

III. Revealing and clarifying ambiguities in mathematical concepts in the history of mathematics

The view that the purpose of schooling is on bringing forth a change of learner's mind is pertinent to the characteristics of academic disciplines. The developmental processes of academic disciplines are endless processes of self-awareness, self-reforms and self-improvements. All the academic knowledge entered into the human history always has, more or less, incomplete and vague parts. In the history of academic disciplines, it is these incomplete or vague parts that impel continual pursuit of academic exploration. Academic progress has been made with the driving force of recognizing and aspiring of improvement to fulfill the incompleteness and to clarify the vagueness.

The history of mathematics has also been flowing down in this manner. The essence hidden in mathematical concepts or objects has been revealed gradually and the ambiguities of concepts, definitions and theories have been clarified step by step. These developmental processes have been pointed out by many mathematicians and mathematics educators such as Clairaut (1741, 1746), Toeplitz (1963), Klein (1948), Lakatos (1976), Freudenthal (1983, 1991), Brousseau (2002), and Branford (1908), for its importance and helpfulness in teaching and learning of mathematics. In this chapter, the ways of developmental process, revealing hidden essence and clarifying ambiguities, are discussed with examples.

1. Revealing hidden essence

Freudenthal (1983, 1991) says that mathematics is gradually systemized and organized through the mathematization process, which is a process of alternation of phainomenon and nooumenon. The nooumenon to organize phainomena at a level becomes an object to investigate (i.e. a phenomenon) at the next level. Applying this idea to geometry, van Hiele (1986) discerns geometrical thinking with five levels.

The ascending process of thinking levels can be understood as a process that the essence concealed with many folds of veils reveals itself gradually. When we concentrate ourselves on the examination of the things given to us, the figures (the essence hidden inside the things) reveal themselves to us. In the next level, when we examine the figures disclosed to us with a presumption that those would be the essence of things, the characteristics of figures (the essence hidden inside the figures) disclose themselves alluding that "the figures are not the final goal but the characteristics of figures, the essence, is hidden in there." And again, when we look through the characteristics of figures, the propositions hidden inside gradually reveal themselves to us as if saying "the characteristics of figures are not the essence." And then, when we focus on the propositions, a theoretic system of geometry gradually exhibits its features. This is how the essence hidden inside the things initially given in many folds reveals itself one after another.

The gradual process of revealing the essence is exemplified with the development of group concept. Group concept, a fundamental concept of modern algebra, has aided the advance of algebra as an example of the evolution of an algebraic structure and as a 'midwife' of modern algebra (Wussing, 1984) used to explore and clarify algebraic structures. The evolution of abstract group theory is generally asserted to arise at the end of the nineteenth century by pure abstraction from the concept of a permutation group derived from the theory of algebraic equations and Galois' theory. However, according to Wussing (1984), some mathematicians believe that the group idea is much older than generally thought. Poincaré said the history of group concept is as old as mathematics in view that the actual basis of the ancient Euclidean proofs was the concept and properties of a group. Miller, a group theorist, claims that the history of group theory coincides with the beginning of mathematics. Speiser also asserts that the modes of thought associated with geometric ornaments thousands years ago can be interpreted group theoretically. In the middle ages, there are also some instances that may be interpreted as pre-figurations of implicit group-theoretical thinking.

All the group-theoretical thinking before the eighteenth century was implicit and hidden. According to Wussing (1984), who sees the automorphisms of structures as hidden essence of group-theoretical thought, it is inherent not only in the theory of algebraic equation but also in other fields of mathematics. "Abstract group theory was the result of gradual process of abstraction from implicit and explicit group theoretical methods and concepts involving the interaction of its three (the theory of algebraic equations, number theory and geometry) historical roots (p.16) in the end of the eighteenth (the former two theories) and the beginning of the nineteenth (the latter) century (p.19)." The group-theoretical thought in geometry, though they were not linked to the contemporary development of the theory of permutation groups as in the theory of algebraic equation and in number theory, were in the study of geometric relations and in the concurrent consolidation of invariant theory (p.27). The explicit use of group theory in geometry was achieved by Klein in his Erlangen Program of 1872 to organize the geometric structures. The group-theoretical thoughts had been gradually explicitly recognized and constructed as a formal algebraic theory, and sooner or later, the group theory came to the center of modern mathematics being used as an important tool to analyze mathematical structures and the relationships between them.

2. Clarifying ambiguities

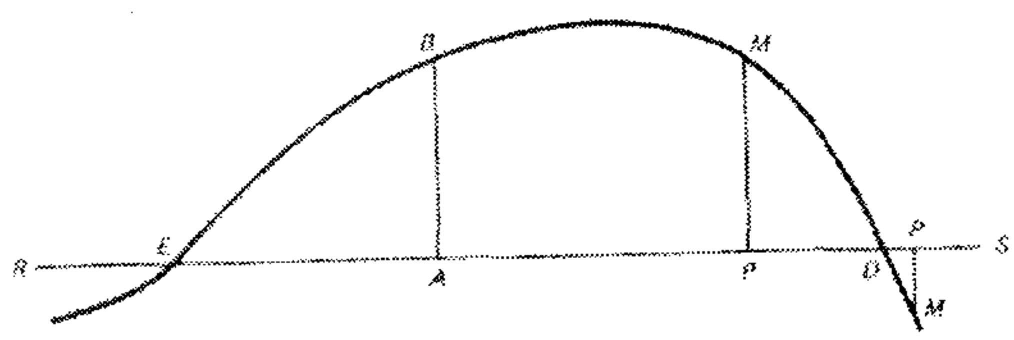
Brousseau (2002, pp. 58-60) identifies the development of mathematical concepts passing through three consecutive levels: 'protomathematical concept,' 'paramathematical concept' and 'mathematical concept'. The name 'protomathematical' is originally suggested by Chevallard, who distinguishes mathematical development with 'protomathematical concepts', 'paramathematical concepts' and 'actual mathematics' (Brousseau, 2002, p. 97).

At the level of mathematical concept, a concept is positioned at the level of a theoretical concept where ambiguities and errors are removed. This level can be stepped on by putting the concept dealt with under the control of a mathematical theory. Here, its exact definition is allowed in terms of structures in which it intervenes and of the properties that it satisfies. It is the final level. Protomathematical concept level and paramathematical concept level precede it. At the paramathematical level, the concepts used implicitly are recognized and served as a tool to name objects whose characteristics and properties are studied. Though the concepts are not organized or theorized, they are familiar and used very well, thus quite acceptable without any contradiction. This level is preceded by the level of protomathematical concept, where incomplete but coherent concepts are implicitly used for solving problems. Brousseau expressed such an evolution of a mathematical concept as a process to protect the concept, which has already been used as a tool for solving problems without recognition, from ambiguities and errors by focusing on, recognizing, clarifying, and removing the ambiguities and errors.

In the explication Brousseau proposes about the development of mathematical concepts, an important feature is seen. It is that the explicit theorization or even recognition of a mathematical concept is preceded with the use or familiarization of it. The initial state of the concept is quite ambiguous and possibly erroneous. However, starting from this level, a mathematical concept is gradually revealed, clarified and theorized. Examples of those concepts are easily found in the history of mathematics. The development of the concept of continuous function would be one of them.

A naive concept of continuity would be not cutting down or breaking down. A continuous line may be understood as a line which can be drawn in one stroke (i.e. drawn without lifting pencil from a paper) with no 'gap' or 'jump' in it. Euler's characterization of a continuous function in the Cartesian plane was similarly informal, rather intuitive and vague (Lakoff and Núñez, 1997). It was an application of the inferential structure of the everyday understanding of motion, flow, wholeness and generations of quantities to functions and variations using naive expressions such as 'approach indefinitely' and 'as little as one wishes' (Boyer, 1959; Jourdain, 1913; Núñez, Edwards & Matos, 1999), and it was also used by even eminent mathematicians like Newton and Leibniz in the 17th century (Boyer, 1959; Núñez, Edwards & Matos, 1999).

Euler defined the continuity as follows with an illustration of figure 1. "A continuous curved line is so defined, that its nature is expressed by a single definite function of x. But if a curved line is defined in such a way that its different parts BM, MD, DM, etc., are expressed by different functions of x, so that, then a part is defined by one function, a part MD is described by another function, we call curved lines of this kind discontinuous (Euler, referred by Bottazzini, 1986, p. 25)."



[Figure 1.] Euler's illustration of a continuous curve (Bottazzini, 1986, p. 25)

Afterward, mathematicians recognized that continuity in a curve did not depend upon its being expressible by means of a single equation in continuous functions, and the necessity of a new definition of continuity came up (Boyer, 1959). It was Cauchy who made the notion of continuity precisely mathematical and showed that this depends upon the concept of limit (Ibid.), pointing out the vagueness and uncertain character in the definition of Euler (Bottazzini, 1986). Though Cauchy defined the continuous function more clearly and built the theory of continuity upon a precise definition of the notion of limit, his concept was based on geometrical representation to believe that the continuity of a function was sufficient for the existence of a derivative (Boyer, 1959).

It was Weierstrass who noticed this erroneous concept and constructed a purely formal arithmetic basis for the continuity of a function, independent of all geometric intuition and of motion, flow and wholeness on which Euler, Newton and Leibnitz relied. He regarded the ideas of variable and limit as purely static interpreting a variable x as simply a letter designating any one of a collection of numerical values. Whereas Cauchy defined the limit as "when the values successively attributed to the same variable indefinitely approach a fixed value in such a way as to end by differing from it as little as one wishes, this latter is called limit of all the others (Cauchy, 1821, referred by Bottazzini, 1986, p. 103)," in the limit concept of Weierstrass, no idea of approaching is involved but only a static state exists.

Weierstass' definition of continuity deals better with complex and pathological cases (Lakoff and Núñez, 1997; Núñez, Edwards & Matos, 1999), where the intuitive informal definition of continuity does not hold. Building a new definition of continuity is also related to the re-establishment of arithmetic as the dominant theory via the huge program of the arithmetization of mathematics which went on from Cauchy to Weierstrass (Lakatos, 1976; Núñez, Edwards & Matos, 1999).

After Weierstrass, mathematicians noticed that the essence of the limit concept is on the concept of real number, and the fundamental theorems of limits could be proved rigorously and without recourse to geometry on the basis of a new definition of real number as Dedekind suggested (Boyer, 1959). To define the continuity of a function Bolzano and Cauchy had to have a concept of a function, and the independent variable of it was tacitly understood as one which could take on all values in an interval corresponding to the points of a line segment. Dedekind revealed this hidden assumption, arithmetized it beyond the geometric picture, and expressed formally with an ordered set (Ibid.).

Whence, the mathematical theory of continuity became based on the logically developed theories of number and sets of points.

We have examined that the development of mathematical concepts over the history of mathematics is a process of revealing the essence and clarifying ambiguities hidden in the concepts, with two examples. How can the gradual process of revealing and clarifying be applied to children's learning of mathematics?

Ⅳ. Revealing and clarifying ambiguities in mathematical concepts students have over the school mathematics curriculum

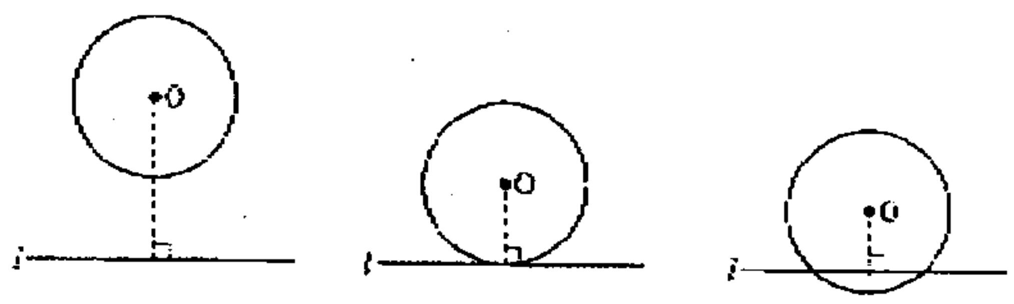
In school mathematics, to meet with children's eye-levels, mathematical concepts are introduced intuitively or context-dependently despite of its naivety and insufficient mathematical rigors. After then formalization and abstraction processes move on step by step revealing and clarifying ambiguities in the initial concepts and in the context-dependent characteristics. In the followings, the revealing and clarifying processes of definitions of a tangent and geometric figures over the whole school mathematics curriculum are instantiated.

1. Revealing ambiguities with counter examples

The 'logic of mathematical discovery' Lakatos (1976) proposes, indicates two directions for modification of definitions when counter examples turn up. The one is a modification to make the counter example not an example of the definition. The other is a modification to include the counter example as an example of the concept. There is certainly a significant difference between these two directions in that the former excludes the counter example whereas the latter includes it. However, they may be cognates in that a limitation of the present concept is revealed through the counter examples resulted an improvement of the concept. We name the former 'exclusive modification' and the latter 'inclusive modification'.

In school mathematics those counter examples make students troubled with a cognitive conflict (or cognitive disequilibrium in Piaget's terms). They have to reflect on the concept possessed at present to recognize its limitations and reorganize their cognitive structures with the improved concept. These processes can be experienced certainly not in a single class but in several classes at intervals over the school period. The conceptual improvement of definitions of a tangent of a curve that Yim and Park (2004) propose is an example.

In Korean school mathematics curriculum, 'a tangent of a curve' is firstly introduced in the context of circle at the 7th grade (the first year of middle school) as 'a line which meets at only one point with a curve (Definition 1)'. Looking at the pictures in Figure 2, students can compare three relationships between a circle and a line, and discover that the number of points at which a line meets a circle is different according to the relationships. Here, referring to the middle picture in Figure 2, the concept of a tangent of a curve may be formed as Definition 1.



[Figure 2.] Positional relationships between a circle and a line

At the 9th grade (the third year of middle school), students draw parabolic graphs learning quadratic functions. Here, they have two kinds of new experience about the concept of a tangent. First, as they learn to express a geometric figure with an algebraic expression, they can link a geometric tangent to an algebraic expression. A parabola (a graph of a quadratic function) can be represented with a quadratic expression, and a line with a linear expression. Finding out the meeting points of a curve and a line becomes solving two simultaneous equations, and the number of meeting points equals the number of roots of the equations. According to Definition 1, if a line is a tangent of a parabola, the simultaneous equations

$$\begin{cases} y = ax^2 + bx + c \\ y = mx + n \end{cases}$$

have only one solution. Solving the simultaneous equations transforms to solving a quadratic equation

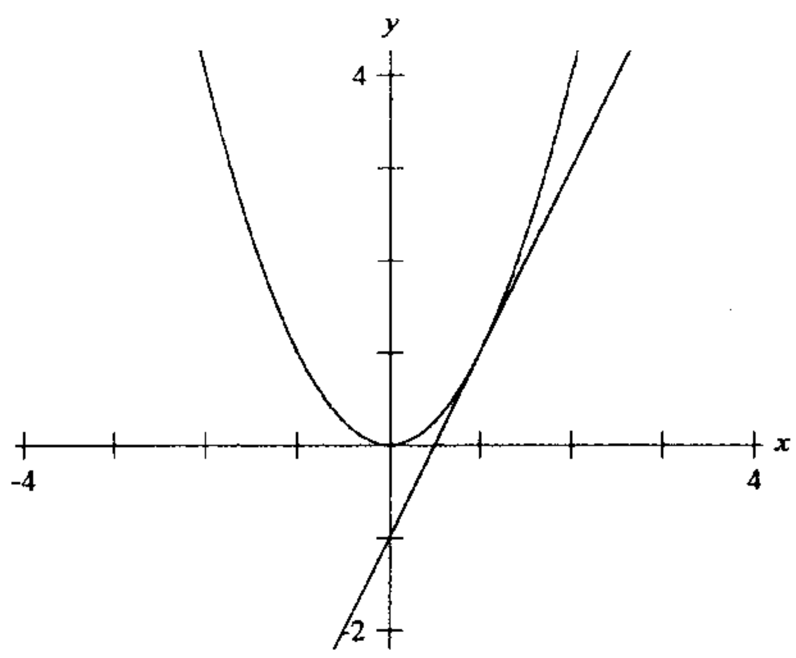
$$ax^2 + bx + c = mx + n,$$

and having only one solution means the determinant of the quadratic equation is 0, i.e.

$$D = (b - m)^2 - 4a(c - n) = 0.$$

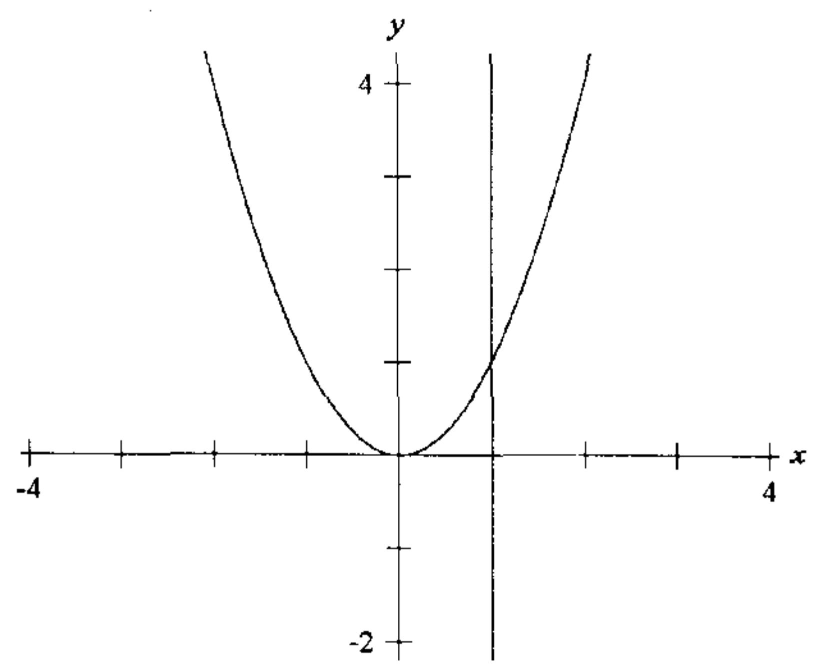
After this, students may apply this new link to the context of circle for 'recollected learning (Freudenthal, 1991)'. A circle can be expressed with a quadratic expression. Through a similar procedure of solving simultaneous equations, students may be able to understand that the determinant can be a standard for judging whether a line is a tangent of a circle or not. This conjecture holds in the context of circle, since the expression of a circle is also quadratic. But, in the other general curves it does not. In other words, linking tangent to the determinant is context-dependent (where a curve can be represented with a quadratic expression) concept although it is not recognized at the moment.

Here, students encounter an example of tangent which conflicts to the concept of tangent learnt at the 7th grade. The line in figure 3, a tangent of the curve $y = x^2$, meets at only one point with it. There is no conflict in this case.



[Figure 3.] A tangent of a quadratic function

But, the line, $x = 1$ at figure 4 meets at only one point with the same parabola, but does not touch the curve. In this case, whether the line $x = 1$ can be regarded as a tangent or not has to be discussed in class. During the discussion, the concepts of a tangent that students currently possess and its limitation would be revealed.



[Figure 4.] A line meeting at a point with a quadratic function

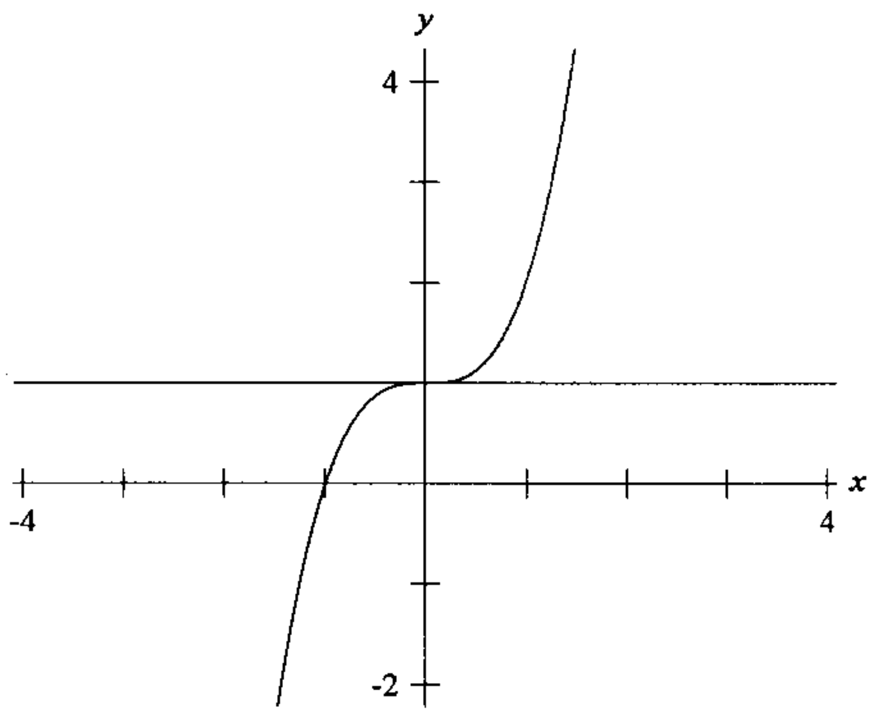
In the context of circle, the statement that 'a line that touches a circle meets at only one point' holds since a line that meets at only one point with a circle touches it. However, it becomes different in other curves such as parabolas, hyperbolas, etc. In the context of parabola, defining a tangent in terms of the number of meeting points with the curve confronts counter examples, thus calls for a modification. The counter example such as figure 4 should be excluded in modified definition. It is 'exclusive modification'.

The modified definition of a tangent 'a line which touches a curve but not cuts the curve (Definition 2)' can be taken in this context. Then, students are recommended to compare the new definition with the old definition. Some students may apply their new definition to the context of circle. But others may still look at the number of meeting points. The latter have to explore whether there is a line which touches a circle but does not meet with the circle at one point, or which meets with a circle at one point but does not touch a circle. From the exploration students can recognize the two conditions 'meets at one point' and 'touches a curve' are necessary and sufficient conditions in the context of circle. That is, Definition 1 was sufficient in the context of circle. But its ambiguity is eventually revealed in the context where 'meets at one point' is necessary but not sufficient condition of being a tangent. Here, students may feel that 'touches a curve' is a better for the definition of a tangent since it comprises both the context of circle and that of parabola.

At this point, teacher can challenge students by asking "would it be always true that a tangent of a curve meets with the curve at only one point?" It would make students critically reexamine the first definition of a tangent line. In the context of circle and parabola, it is true. But what about other general curves? Students may draw a general curve, check this, and find out that it is not always true. This is a simple activity indeed that does not take much time. Definition 1, served as a foundation once, has to be rejected in more general contexts. It is very important in that it highlights the problems of former definition students once thought as proper but recognize its limitations later, which finally leads to reject them. Through this procedure, students may be humble their thoughts and pleased with a feeling of an improvement.

Encountering a counter example that challenges present definition may perplex students. At the same time, however, it may inspire longings for proper modification. Introducing a new definition of a tangent without any challenge for present definition cannot impress students. If it is given to students without stimulating any reflection, it may be accepted as if it was determined from the beginning. There, a change of mind is hardly expectable.

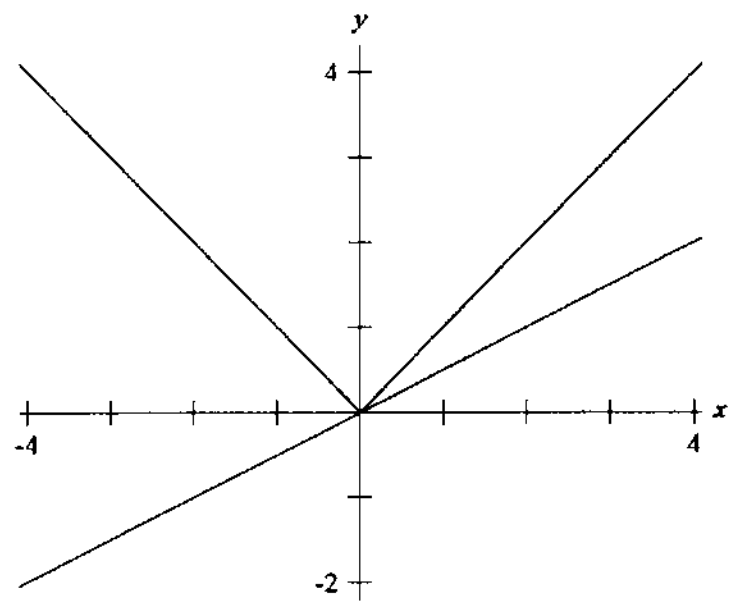
For some while, Definition 2 is satisfactory. Yet it is not the end. Some students come into to encountering some examples where it is not applied. The curve in Figure 5 is $y=x^3+1$ and the line is $y=1$. There, the line does not touch the curve but passes through it. Can it not be regarded as a tangent?



[Figure 5.] Tangent line that violates Definition 2

Students, perplexed, might try to apply the criterion of determinant. However, it does not hold either, since it is applied only to quadratic expressions. This problem is resolved with a reference to the concept of derivative. In the context of derivative the line $y=1$ is a tangent of the curve $y=x^3+1$ at $(0,1)$. The firmness of Definition 2 is weakened. Can the existence of a derivative be a new criterion for judging a tangent of a curve?

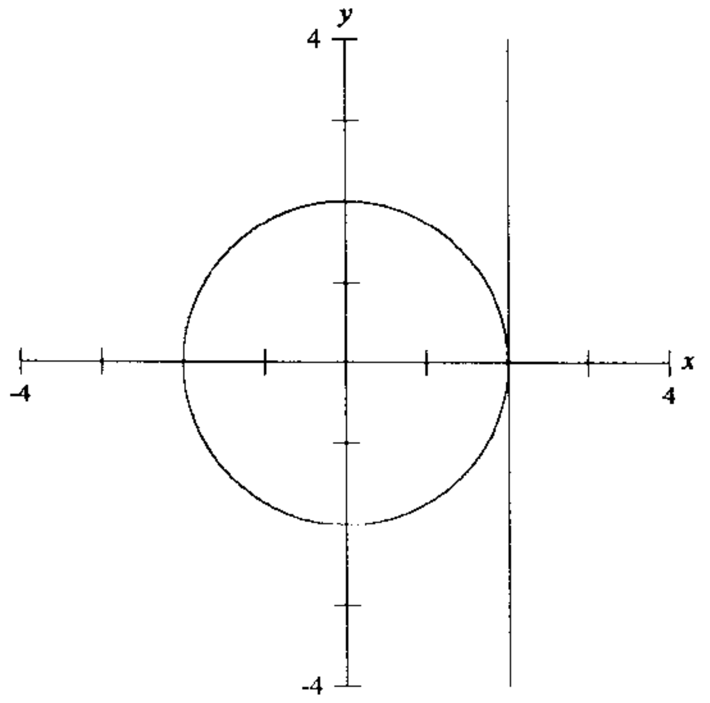
At the point $(0,0)$ on the curve $y=|x|$ in figure 6, although there seems to be plenty of lines that touch but not pass the curve, there exists no derivative at the point. In this example the old definition cannot decide whether there are plentiful of tangent lines or none. But, the new criterion is decisive. Thus Definition 2 must be modified or replaced with other definition.



[Figure 6.] $y=|x|$

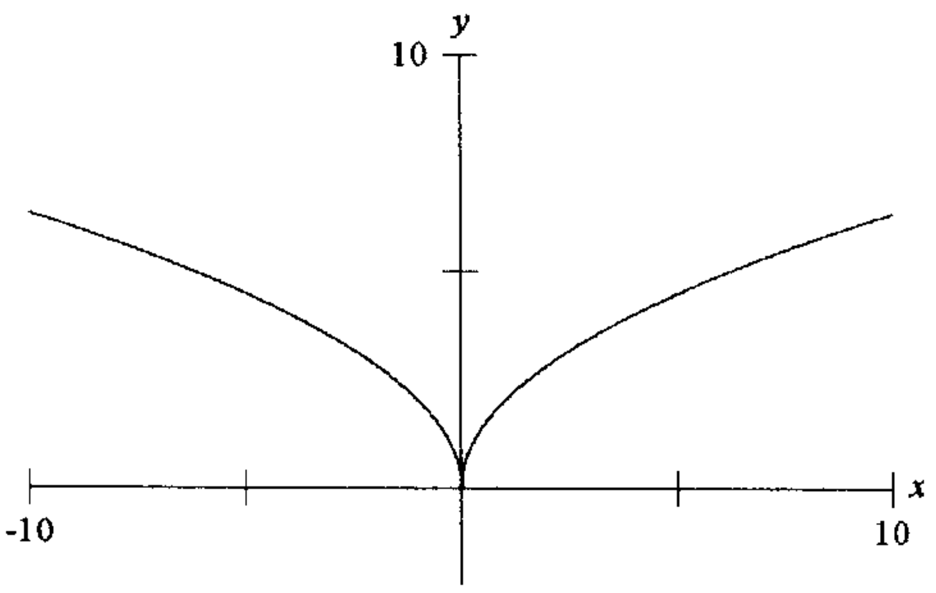
Even though the existence of a derivative is accepted as a criterion on which the existence of a tangent is judged, there still remains a problem. What does tangent mean particularly in geometric sense? A derivative interpreted as a slope of a tangent line at a point does not define 'tangent' but use the term in its interpretation. Point, line, plane and space are geometrical terms. Tangent is also a term that has to be understood geometrically. But the existence of a derivative cannot define it.

As an alternative, 'the limit of secants' can be proposed. It is defined as follows: when a point P on a curve is fixed, and another point Q on the curve approaches to P, a tangent of the curve at P is the limit of $\overline{PQ}$ (Definition 3). The limit of secants is quite inclusive and visual definition not requiring operations of algebraic expressions. With this definition, it is easily understand that the curves in figure 2, 3 and 5 have a tangent line at a point on the curves, and the curve in figure 6 does not have any tangent at the origin. It can also be applied where a tangent is parallel to y-axis (figure 7). The line $y=2$ in figure 7 can be accepted as a tangent line at point (2, 0) with Definition 2, since it touches the circle $x^2+y^2=4$ but not passes through it. Here, however, a derivative cannot be calculated because finding out a derivative at a point means getting a slope of a tangent line, and a slope cannot be defined in this case. It is resolved with Definition 3.



[Figure 7.] Tangent line parallel to y-axis

The curve in figure 8 is more complicated. Does the point (0, 0) on the curve have a tangent?



[Figure 8.] $y=2\sqrt{|x|}$

Since the origin is a cusp of the two curves $y=2\sqrt{x}$ and $y=2\sqrt{-x}$, a derivative at the point (0, 0) does not exist. That is, the differentiability cannot be a criterion of the existence of a tangent. However, according to Definition 3, it has a tangent line, the y-axis. The tangent does not touch the curve but passes it. Definition 3 is an 'inclusive modification' in that it includes counter examples such as figure 6 and figure 8.

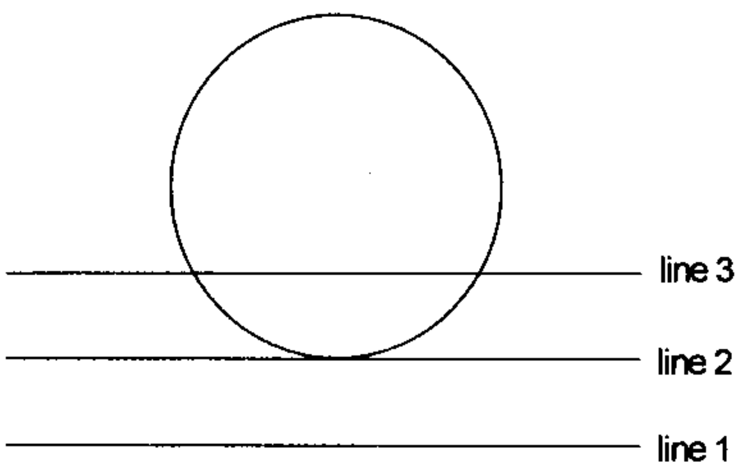
In summary, the context-dependent definition of a tangent has been modified step by step from 'a line which meets at only one point with a curve' to 'the limit of secants' through reflections and comparisons. In school mathematics where subjects matters are dealt with rather intuitively than rigorously, definitions are inevitably context-dependent. The definition of a tangent discussed above is an example that shows how initial intuitive and vague definitions are clarified and improved stepping on several stages with counter examples. Passing through these steps students possibly feel that the definition currently possessed may not a complete form. It may be modified by another counter example in future time. Even Definition 3 may not be accepted as a complete form or final essence, but an incomplete, context-dependent form that can be changed and improved further. The essence of a tangent may not be sufficiently revealed yet. This experience would be helpful for students to critically reexamine their concept or definition of a tangent, and to willingly modify them whenever a problem with the concept appears in the future.

Reflection and comparison of perspectives is basically social (Vygotsky, 1978; Balacheff, 1991). If students do not see any limitation of current perspectives, they would not feel any necessity of changing them. Therefore, challenging should be provided in advance to an alternative. It may be teachers' roles to challenge students' limited perspectives and show an alternative.

2. Revealing the ambiguities with context-dependent definitions

Following dialogues are excerpted from a math class in a middle school in Korea (Park and Yim, 2004).

Teacher: At how many points does line 3 meet with the circle?
Student: It meets at infinitely many points.
Teacher: Why do you think so?
Student: Because line 3 meets at infinitely many points inside the circle.



[Figure 9.] Lines meeting with a circle

In school mathematics, when a plane figure such as a polygon or a circle is said, does it include inner part or not? Mathematically, and in Korean middle school mathematics textbooks, a circle is defined as a figure or a set of points that are at the same distance from a point. In this definition, a circle does not include the inside. If a person claims that a secant meets at two points with a circle, which implies the student in the above dialogue is wrong, he/she would mean that a circle is only the boundary excluding the inside. However, in Korean elementary school mathematics textbooks, a circle is defined as a round figure that can be sketched copying a round thing. In this definition, it is not distinguishable whether a circle includes its inner part or not. That is, there is a conflict between the definition of a circle in elementary school textbooks and in middle school textbooks.

What about a polygon? Mathematically, a polygon is a subset of a single closed curve. Thus, it can be inferred that a polygon includes only the boundary without its inner part. In school mathematics textbooks, a polygon is sometimes defined as a figure surrounded with some segments. In this definition, a polygon includes the inside (or only the inside in extreme cases).

The problem whether a polygon or a circle is only the boundary or the boundary-and-inside is connected to the problem whether a polyhedron or a cylinder is filled in or empty. If a polygon or a circle is only the boundary, in consistency, a polyhedron or a cylinder should be empty. If a polygon or a circle includes inside, a polyhedron or a cylinder has to include inside, too.

Viewing that a polyhedron is an empty solid is regarding a rectangular parallelepiped as a solid abstracted as an empty box. This view matches the context that makes a development figure by cutting and unfolding the solid along the edges. However it has a problem. In the context of making a truncated pyramid cutting a pyramid with a plane parallel to the bottom, if the pyramid is empty solid, the resulting truncated pyramid is an open solid. Since its lid is open, the truncated pyramid cannot have two bottoms.

If a polyhedron includes its inside, a rectangular parallelepiped cannot be abstracted as an empty box but a solid like a brick. In this case a truncated pyramid has two bottoms. However, it is impossible to make a development figure by cutting and unfolding the solid along the edges. An empty box can be unfolded, but a brick cannot. In summary, either of the two views does not simultaneously satisfy both the two contexts 'cutting a solid by a plane' and 'unfolding a solid along its edges'. It is a dilemma.

This dilemma can be resolved with discrimination between a region of a figure and a figure. In strictly logical point of view, it would not be recommendable to change meaning of a term according to the context it is used. A good definition would hold a consistent and firm interpretation regardless of the context. Defining 'a triangle' and 'a region of a triangle' separately to avoid a logical conflict would be appropriate in this view. In Korean school mathematics in the late 1960s and early 1970s, when it was influenced by the 'New math movement', a figure and a region of a figure was distinguished. For example, a triangle was defined as "the union of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, when A, B, C are not on the same line" and the region of a triangle is "a set of points on a triangle and its inside (ME, 1979, p.227, 233)." But it is not defined as such in current school mathematics.

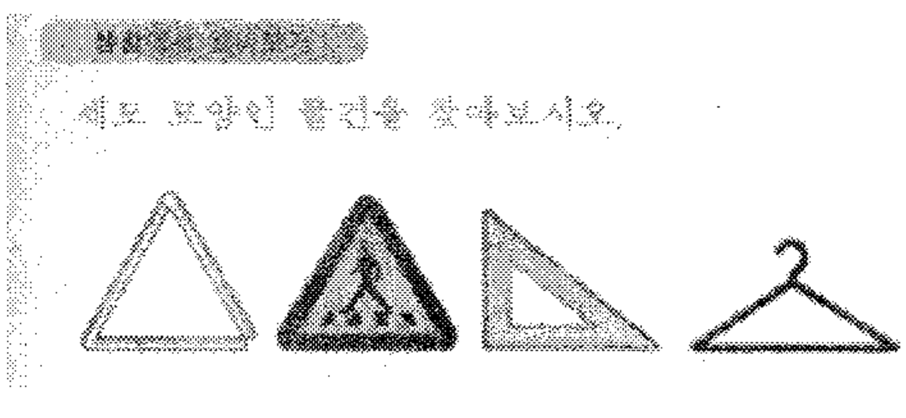
Schooling is accomplished over a long period. In the meanwhile, at certain stages, less strict definitions are introduced. In other words, 'context-dependent definitions' are allowed in school mathematics. For rectangular parallelepiped, as an example, a non-strict, rather obscure definition such as 'a solid surrounded by six rectangles' is allowed. With the definition, 'a filled solid' is adopted in the context of cutting whereas 'an empty solid' is chosen in the context of unfolding. It may be wrong in terms of logical consistency. However, it can be accepted in school mathematics, which is a peculiar logic of school mathematics.

The context-dependent definitions of a polygon and a polyhedron can be taught in a way that intuitive and vague definition is introduced in the beginning, and more modified or improved version in upper classes. As students ascend the grades, new context where the meaning of a term students already know has to be clarified or modified is provided. It is impossible to introduce all such contexts at the same time, particularly in the beginning. Various contexts may stimulate students' thought in different ways, thus should be introduced one after another over the school mathematics curriculum. The definitions in school mathematics are definitely not perfect in view of advanced mathematics. However, what is requested there is just being appropriate and relevant in the context.

A student in the 2nd grade of elementary school in Korea learns a circle as a round figure doing an activity of copying round materials such as coins, cans and cups. This activity could make the student perceive a circle as a certain figure. This context does not request to distinguish whether it means only 'boundary' or 'boundary-and-inside'. To the student, even if a round figure shown to him/her is not a perfect circle, he/she may not discriminate but acknowledge it as a circle. It would be acceptable at this stage. In other words, a second-grade-elementary-school-student does not have to possess a perfect concept of a circle. He/she will experience improvement from then on. Thus, it is enough for students to construct a concept to the extent of being relevant to the context given.

New context may provide an opportunity for students to reflect on the concept currently have. Someday, drawing a circle using a pair of compasses they may see the circle drawn with a new perspective such as a figure determined by a point (center) and a length (radius). In this context they may reflect their former concept of a circle. They may clarify the circularity, thinking that a circle is not roughly round but perfectly round. Or, they may notice the vagueness of the terms 'a round figure', and appreciate the clearness of the new definition, 'a set of points at a certain distance from a point or a figure composed of such points'. They may go further to pose a question whether a circle includes inside or not with recognition of the vagueness in the former definition.

The case of a polygon is also the same. A child in the 2nd grade of elementary school may have a concept of triangle as a figure surrounded by three segments, with activities of investigating and observing triangular shapes in their world such as triangles (musical instrument), traffic signs, triangular rulers, clothes hanger, and the like.



[Figure 10.] Exploring triangular figures (MEHRD, 2004, p.39)

Some of these shapes are empty whereas others are filled as figure 10 shows. However, emptiness or fullness is not important in this context, thus children cannot consciously inquire whether the definition includes inside or not. Unless given a new opportunity for reflection, children may regard a triangle meaning only its boundary, or its boundary and inside according to the contexts where it is placed, with no problems.

An introduction or change of other concepts could serve to look back on the concept of polygons. Students, who change their thoughts about a circle from a round figure to a set of points at a certain distance from a point, might ask questions such as "how about triangles or rectangles?" "Can they be defined in a similar manner?" "Is the concept sufficient that a triangle is a triangular figure or a figure surrounded by three segments?" or "If a circle means only its boundary, how about a triangle?" These conscious reexaminations of their concepts can be a fruitful base fostering an understanding of the relationship between closed curves and polygons.

Somewhat vague definitions like a round figure or a figure surrounded by three segments are allowed in school mathematics. We do not contend that the definition should be persistent throughout the whole mathematics curriculum from elementary school to high school. Education is a long-term process. Definitions should be reexamined and reformed step by step, not at once or at one level. A child who looks upon a circle with its boundary, later (for example at the first year of high school in Korea), may understand the difference between expressions of figures (equations) and expressions of regions (equalities) without much difficulty.

Here, the role of teachers needs to be emphasized. Teachers have to enter their classrooms with a full appreciation about why and how meanings of terms (or concepts) are varied at different contexts (or stages) in school mathematics with the merits and limitations of the definitions in each context, what kinds of conceptions are formed with context-dependent definitions, and how such conceptions are modified and improved over the whole school mathematics curriculum. Teachers, who are fully aware of the educational intentions inherent in the ways of treating a certain concept in school mathematics and who recognize the delicate problems or dilemmas derived from those treatments, may lead their students to critically reexamine and revise their conceptions. When they play these roles faithfully, educational values inherent in the subject can be revealed and fully appreciated.

V. Closing Remarks

Plato (1996) said that intellectual investigation starts from self-awareness of being unenlightened and of appreciating the wonderfulness of being enlightened. According to him, the mind which feels the wonderfulness of enlightenment has peculiar characteristics appeared in a person who loves wisdom. In reality, we have so many students who are not aware of being unenlightened, nor of the wonderfulness inherent in mathematical knowledge. We believe that they can have an opportunity of awareness reflecting their minds on mathematical knowledge. However, far from opening and taking the opportunity, many students rather give up mathematics with a thought that it has nothing to do with improvements of their minds.

Mathematics education should not be a procedure of accumulating mathematical knowledge but of changing students' minds. Students have to ascend their cognitive levels through the learning of mathematical knowledge recognizing the lowness of the levels where they are, e.g. the vagueness of their concepts and the flaws in the concepts.

Academic disciplines have some extent of truth (clearness) and some extent of flaws (vagueness) at the same time. No one can claim that he/she reaches at the final stage (the complete clearness) of academic exploration but continually strive to ascend to higher levels. The history of mathematics shows the progress of mathematical investigation which can be interpreted as a procedure of gradual revelation and clarification of the vagueness or the essence. Through the realization of the essence in mathematical content, students can be led to internal awakening, and to the cultivation of mind (Woo, 2007b).

From the intellectual experience of this kind where the essence reveals itself more clearly, students may have delight feeling that the world they view is different from the one before, and that they are improved. For example, when students notice the concept of a tangent such as 'a line meets at one point with a circle' was naive with a formation of the concept 'a line touches a curve', and when they realize the latter concept is also somewhat vague referring the concept of 'the limit of secants', they could feel improvements of their way of being and have desires for further improvement.

Woo inspires mathematics teachers "to open school mathematics through didactical analysis of it to enjoy the form and structure within its contents, revive the wonder and impression mathematicians felt when they first discovered it, and inspires it in students' mind (2007b, p.88)." We believe that learning can be acquired through the operation of intelligence, emotion and will altogether. The emotion in intellectual experience is composed of positive feelings such as delight, satisfaction, growth and improvement and also of negative feelings such as pity and despair for the limitations of human beings. The two feelings are like the two sides of a coin. Therefore, together with the delight of being improved, students may have the humble attitude that their current knowledge is not perfect. They may pursue the improvement of it, even though they cannot reach at the center of the essence no matter how hard they try to. They may respect the essence, the reality, or the world including other people with sympathy that we all are human beings who inevitably have limitations and deficiencies.

Lastly, we would like to emphasize that learning requests students' commitment as Polanyi (1962) insists. The precious cannot be acquired unless we do our best to have them. Education is the same. Students need to make great efforts to break out the confinements surrounding their current concepts, ideas, thoughts with a conviction of being improved.

Being enlightened through education means opening our eyes of mind, which would be more precious than any other things in the world. We believe that the humanity mathematics education can be realized when teachers and students participate in educational program do their best for teaching and learning mathematics with humble and open attitude.

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인간주의 수학교육:
수학적 개념의 모호성을 드러내고 명확히 하기

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본 연구는 심성함양으로서의 수학교육을 목표로 하는 인간주의 수학교육을 구현하기 위한 방안을 모색하고자 한 것이다. 수학적 개념의 본질은 현 개념의 모호함을 인식하고 이를 명확히 하면서 점진적으로 드러난다. 이것은 수학적 개념의 역사적 발달 과정에서도 나타난다. 본 연구에서는 이를 군과 연속함수로 예시하였다. 수학적 개념의 학습에서도 학생들은 다소 직관적이고 모호하며 맥락-의존적으로 정의되는 학교수학의 개념들을 이전 개념의 모호성을 인식하고 분명히 하는 과정을 통하여 학습하여야 한다. 본 연구에서는 이를 학교수학에서 접선과 다각형의 정의의 개선 과정으로 예시하였다. 이와 같은 수학적 개념의 학습을 통하여 학생은 자신의 사고의 한계를 깨닫고, 개선하게 된다. 이러한 지적 발전은 개선의 의지와 함께 겸손과 만족이라는 감정을 수반하면서, 수학교육을 통한 심성 함양이라는 인간주의 수학교육을 구현하는 방편이 될 수 있다.

* **Key words** : ambiguity(모호성), context-dependent definition(맥락-의존적 정의), continuity(연속), counterexample(반례), group(군), history of mathematics (수학사), humanity mathematics education(인간주의 수학교육), polygon(다각형), tangent(접선)

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