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Poisson Banach Modules over a Poisson C^*-Algebra

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Abstract. It is shown that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a unital Poisson C^*-algebra $\mathcal{A}$ to a unital Poisson C^*-algebra $\mathcal{B}$ is a Poisson C^*-algebra homomorphism when $h(2^n uy) = h(2^n u)h(y)$ or $h(3^n uy) = h(3^n u)h(y)$ for all $y \in \mathcal{A}$, all unitary elements $u \in \mathcal{A}$ and $n = 0, 1, 2, \cdots$, and that every almost linear almost multiplicative mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism when $h(2x) = 2h(x)$ or $h(3x) = 3h(x)$ for all $x \in \mathcal{A}$. Here the numbers $2, 3$ depend on the functional equations given in the almost linear mappings or in the almost linear almost multiplicative mappings. We prove the Cauchy–Rassias stability of Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras, and of homomorphisms in Poisson Banach modules over a unital Poisson C^*-algebra.

1. Introduction

A *Poisson C^*-algebra* $\mathcal{A}$ is a C^*-algebra with a C-bilinear map $\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called a *Poisson bracket*, such that $(\mathcal{A}, \{\cdot, \cdot\})$ is a complex Lie algebra and

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

for all $a, b, c \in \mathcal{A}$. Poisson algebras have played an important role in many mathematical areas and have been studied to find sympletic leaves of the corresponding Poisson varieties. It is also important to find or construct a Poisson bracket in the theory of Poisson algebra.

A *Poisson Banach module* X over a Poisson C^*-algebra $\mathcal{A}$ is a left Banach $\mathcal{A}$-module endowed with a C-bilinear map $\{\cdot, \cdot\} : \mathcal{A} \times X \to X$ such that

$$\begin{aligned} \{\{a, b\}, x\} &= \{a, \{b, x\}\} - \{b, \{a, x\}\}, \\ \{a, b\} \cdot x &= a \cdot \{b, x\} - \{b, a \cdot x\} \end{aligned}$$

for all $a, b \in \mathcal{A}$ and all $x \in X$. Here $\cdot$ denotes the associative module action (see [3], [7], [8], [19]).

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in R$ for each fixed

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$x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Rassias [12] showed that there exists a unique R-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. Găvruta [2] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$. C. Park [9] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Jun and Lee [4] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 3^{-j}\varphi(3^j x, 3^j y) < \infty$$

for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in X \setminus \{0\}$. C. Park and W. Park [11] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra. Several authors have investigated functional equations (see [1], [10], [13]-[18]).

Using the stability methods of linear functional equations, we prove that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism when $h(2^n uy) = h(2^n u)h(y)$ or $h(3^n uy) = h(3^n u)h(y)$ for all $y \in \mathcal{A}$, all unitary elements

$u \in \mathcal{A}$ and $n = 0, 1, 2, \cdots$, and that every almost linear almost multiplicative mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism when $h(2x) = 2h(x)$ or $h(3x) = 3h(x)$ for all $x \in \mathcal{A}$. We moreover prove the Cauchy–Rassias stability of Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras, and of homomorphisms in Poisson Banach modules over a unital Poisson C^*-algebra.

2. Homomorphisms between Poisson C^*-algebras

Throughout this section, let $\mathcal{A}$ be a unital Poisson C^*-algebra with norm $||\cdot||$, unit e and unitary group $\mathcal{U}(\mathcal{A})$, and $\mathcal{B}$ a unital Poisson C^*-algebra with norm $\|\cdot\|$.

Definition 2.1. A C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ is called a *Poisson C^*-algebra homomorphism* if $H : \mathcal{A} \to \mathcal{B}$ satisfies

$$H(\{a, b\}) = \{H(a), H(b)\}$$

for all $a, b \in \mathcal{A}$.

We are going to investigate Poisson C^*-algebra homomorphisms between Poisson C^*-algebras associated with the Cauchy functional equation.

Theorem 2.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ such that*

$$\text{(2.i)} \qquad \widetilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty,$$

$$\text{(2.ii)} \qquad \|h(\mu x + \mu y + \{z, w\}) - \mu h(x) - \mu h(y) - \{h(z), h(w)\}\| \le \varphi(x, y, z, w),$$

$$\text{(2.iii)} \qquad \|h(2^n u^*) - h(2^n u)^*\| \le \varphi(u, u, 0, 0)$$

for all $\mu \in T^1 := \{\lambda \in C \mid |\lambda| = 1\}$, all $x, y, z, w \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. Assume that (2.iv) $\lim_{n\to\infty} \dfrac{h(2^n e)}{2^n}$ *is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism.*

Proof. Put $z = w = 0$ and $\mu = 1 \in T^1$ in (2.ii). It follows from Găvruta Theorem [2] that there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\text{(2.†)} \qquad \|h(x) - H(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x, 0, 0)$$

for all $x \in \mathcal{A}$. The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$\text{(2.1)} \qquad H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in \mathcal{A}$.

By the assumption, for each $\mu \in T^1$,

$$\|h(2^n \mu x) - 2\mu h(2^{n-1}x)\| \le \varphi(2^{n-1}x, 2^{n-1}x, 0, 0)$$

for all $x \in \mathcal{A}$. And one can show that

$$\|\mu h(2^n x) - 2\mu h(2^{n-1}x)\| \le |\mu| \cdot \|h(2^n x) - 2h(2^{n-1}x)\| \le \varphi(2^{n-1}x, 2^{n-1}x, 0, 0)$$

for all $\mu \in T^1$ and all $x \in \mathcal{A}$. So

$$\begin{aligned}\|h(2^n \mu x) - \mu h(2^n x)\| \le& \|h(2^n \mu x) - 2\mu h(2^{n-1}x)\| + \|2\mu h(2^{n-1}x) - \mu h(2^n x)\| \\ \le& \varphi(2^{n-1}x, 2^{n-1}x, 0, 0) + \varphi(2^{n-1}x, 2^{n-1}x, 0, 0)\end{aligned}$$

for all $\mu \in T^1$ and all $x \in \mathcal{A}$. Thus $2^{-n}\|h(2^n \mu x) - \mu h(2^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in T^1$ and all $x \in \mathcal{A}$. Hence

$$H(\mu x) = \lim_{n\to\infty} \frac{h(2^n \mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu h(2^n x)}{2^n} = \mu H(x) \tag{2.2}$$

for all $\mu \in T^1$ and all $x \in \mathcal{A}$.

Now let $\lambda \in C$ $(\lambda \neq 0)$ and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [5, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in T^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. And $H(x) = H(3 \cdot \frac{1}{3}x) = 3H(\frac{1}{3}x)$ for all $x \in \mathcal{A}$. So $H(\frac{1}{3}x) = \frac{1}{3}H(x)$ for all $x \in \mathcal{A}$. Thus by (2.2)

$$\begin{aligned}H(\lambda x) &= H(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot H(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}H(3\frac{\lambda}{M}x) \\ &= \frac{M}{3}H(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)H(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}H(x) = \lambda H(x)\end{aligned}$$

for all $x \in \mathcal{A}$. Hence

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in C(\zeta, \eta \neq 0)$ and all $x, y \in \mathcal{A}$. And $H(0x) = 0 = 0H(x)$ for all $x \in \mathcal{A}$. So the unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ is a C-linear mapping.

Since $h(2^n uy) = h(2^n u)h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$,

$$H(uy) = \lim_{n\to\infty} \frac{1}{2^n}h(2^n uy) = \lim_{n\to\infty} \frac{1}{2^n}h(2^n u)h(y) = H(u)h(y) \tag{2.3}$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. By the additivity of H and (2.3),

$$2^n H(uy) = H(2^n uy) = H(u(2^n y)) = H(u)h(2^n y)$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. Hence

$$H(uy) = \frac{1}{2^n} H(u) h(2^n y) = H(u) \frac{1}{2^n} h(2^n y) \tag{2.4}$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. Taking the limit in (2.4) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{2.5}$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. Since H is C-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [6, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in C, u_j \in \mathcal{U}(\mathcal{A})$),

$$\begin{aligned} H(xy) &= H(\sum_{j=1}^m \lambda_j u_j y) = \sum_{j=1}^m \lambda_j H(u_j y) = \sum_{j=1}^m \lambda_j H(u_j) H(y) \\ &= H(\sum_{j=1}^m \lambda_j u_j) H(y) = H(x)H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism.

By (2.3) and (2.5),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \dfrac{h(2^n e)}{2^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

It follows from (2.1) that

$$H(x) = \lim_{n\to\infty} \frac{h(2^{2n} x)}{2^{2n}} \tag{2.6}$$

for all $x \in \mathcal{A}$. Let $x = y = 0$ in (2.ii). Then we get

$$\|h(\{z, w\}) - \{h(z), h(w)\}\| \le \varphi(0, 0, z, w)$$

for all $z, w \in \mathcal{A}$. So

$$\begin{aligned} &\frac{1}{2^{2n}} \|h(\{2^n z, 2^n w\}) - \{h(2^n z), h(2^n w)\}\| \\ \le\ &\frac{1}{2^{2n}} \varphi(0, 0, 2^n z, 2^n w) \le \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w) \end{aligned} \tag{2.7}$$

for all $z, w \in \mathcal{A}$. By (2.i), (2.6), and (2.7),

$$\begin{aligned} H(\{z, w\}) &= \lim_{n\to\infty} \frac{h(2^{2n}\{z, w\})}{2^{2n}} = \lim_{n\to\infty} \frac{h(\{2^n z, 2^n w\})}{2^{2n}} \\ &= \lim_{n\to\infty} \frac{1}{2^{2n}} \{h(2^n z), h(2^n w)\} = \lim_{n\to\infty} \{\frac{h(2^n z)}{2^n}, \frac{h(2^n w)}{2^n}\} \\ &= \{H(z), H(w)\} \end{aligned}$$

for all $z, w \in \mathcal{A}$.

By (2.i) and (2.iii), we get

$$\begin{aligned} H(u^*) &= \lim_{n\to\infty} \frac{h(2^n u^*)}{2^n} = \lim_{n\to\infty} \frac{h(2^n u)^*}{2^n} = (\lim_{n\to\infty} \frac{h(2^n u)}{2^n})^* \\ &= H(u)^* \end{aligned}$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since H is C-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in C, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* = (\sum_{j=1}^m \lambda_j H(u_j))^* \\ &= H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$. So the mapping $h = H : \mathcal{A} \to \mathcal{B}$ is an involutive mapping.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism, as desired. □

Corollary 2.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} &\|h(\mu x + \mu y + \{z, w\}) - \mu h(x) - \mu h(y) - \{h(z), h(w)\}\| \\ \leq\ &\theta(||x||^p + ||y||^p + ||z||^p + ||w||^p), \\ &\|h(2^n u^*) - h(2^n u)^*\| \leq 2\theta \end{aligned}$$

for all $\mu \in T^1$, all $x, y, z, w \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. Assume that $\lim_{n\to\infty} \dfrac{h(2^n e)}{2^n}$ is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism.

Proof. Define $\varphi(x, y, z, w) = \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p)$, and apply Theorem 2.1. □

Theorem 2.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying* (2.i)*,* (2.iii) *and* (2.iv) *such that*

$$\|h(\mu x+\mu y+\{z, w\})-\mu h(x)-\mu h(y)-\{h(z), h(w)\}\| \leq \varphi(x, y, z, w) \tag{2.v}$$

for $\mu = 1, i$, and all $x, y, z, w \in \mathcal{A}$. If $h(tx)$ is continuous in $t \in R$ for each fixed $x \in \mathcal{A}$, then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism.

Proof. Put $z = w = 0$ and $\mu = 1$ in (2.v). By the same reasoning as in the proof of

Theorem 2.1, there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.†). The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in \mathcal{A}$. By the same reasoning as in the proof of [12, Theorem], the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is R-linear.

Put $y = z = w = 0$ and $\mu = i$ in (2.v). By the same method as in the proof of Theorem 2.1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{h(2^n ix)}{2^n} = \lim_{n\to\infty} \frac{ih(2^n x)}{2^n} = iH(x)$$

for all $x \in \mathcal{A}$. For each element $\lambda \in C$, $\lambda = s + it$, where $s, t \in R$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) \\ &= (s + it)H(x) = \lambda H(x) \end{aligned}$$

for all $\lambda \in C$ and all $x \in \mathcal{A}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in C$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is C-linear.

The rest of the proof is the same as in the proof of Theorem 2.1. □

Theorem 2.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(2x) = 2h(x)$ for all $x \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying* (2.i)*,* (2.ii)*,* (2.iii) *and* (2.iv) *such that*

$$\|h(2^n uy) - h(2^n u)h(y)\| \le \varphi(u, y, 0, 0) \tag{2.‡}$$

for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique C-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.†).

By (2.‡) and the assumption that $h(2x) = 2h(x)$ for all $x \in \mathcal{A}$,

$$\begin{aligned} &\|h(2^n uy) - h(2^n u)h(y)\| \\ &= \frac{1}{4^m}\|h(2^m 2^n u \cdot 2^m y) - h(2^m 2^n u)h(2^m y)\| \\ &\le \frac{1}{4^m}\varphi(2^m u, 2^m y, 0, 0) \le \frac{1}{2^m}\varphi(2^m u, 2^m y, 0, 0), \end{aligned}$$

which tends to zero as $m \to \infty$ by (2.i). So

$$h(2^n uy) = h(2^n u)h(y)$$

for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. But by (2.1),

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) = h(x)$$

for all $x \in \mathcal{A}$.

The rest of the proof is the same as in the proof of Theorem 2.1. $\square$

Now we are going to investigate Poisson C^*-algebra homomorphisms between Poisson C^*-algebras associated with the Jensen functional equation.

Theorem 2.5. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(3^n uy) = h(3^n u)h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : (\mathcal{A} \setminus \{0\})^4 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 3^{-j} \varphi(3^j x, 3^j y, 3^j z, 3^j w) < \infty, \tag{2.vi}$$

$$\|2h(\frac{\mu x + \mu y + \{z, w\}}{2}) - \mu h(x) - \mu h(y) - \{h(z), h(w)\}\| \le \varphi(x, y, z, w), \tag{2.vii}$$

$$\|h(3^n u^*) - h(3^n u)^*\| \le \varphi(u, u, 0, 0) \tag{2.viii}$$

for all $\mu \in T^1$, all $x, y, z, w \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. Assume that $lim_{n\to\infty} \frac{h(3^n e)}{3^n}$ is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism.

Proof. Put $z = w = 0$ and $\mu = 1 \in T^1$ in (2.vii). It follows from Jun and Lee Theorem [4, Theorem 1] that there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \le \frac{1}{3}(\widetilde{\varphi}(x, -x, 0, 0) + \widetilde{\varphi}(-x, 3x, 0, 0))$$

for all $x \in \mathcal{A} \setminus \{0\}$. The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n x)$$

for all $x \in \mathcal{A}$.

By the assumption, for each $\mu \in T^1$,

$$\|2h(3^n \mu x) - \mu h(2 \cdot 3^{n-1} x) - \mu h(4 \cdot 3^{n-1} x)\| \le \varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0)$$

for all $x \in \mathcal{A} \setminus \{0\}$. And one can show that

$$\begin{aligned} &\|\mu h(2 \cdot 3^{n-1} x) + \mu h(4 \cdot 3^{n-1} x) - 2\mu h(3^n x)\| \\ &\le |\mu| \cdot \|h(2 \cdot 3^{n-1} x) + h(4 \cdot 3^{n-1} x) - 2h(3^n x)\| \\ &\le \varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0) \end{aligned}$$

for all $\mu \in T^1$ and all $x \in \mathcal{A} \setminus \{0\}$. So

$$\begin{aligned}
&\|h(3^n\mu x) - \mu h(3^n x)\| \\
&= \|h(3^n\mu x) - \frac{1}{2}\mu h(2 \cdot 3^{n-1}x) - \frac{1}{2}\mu h(4 \cdot 3^{n-1}x) \\
&+ \frac{1}{2}\mu h(2 \cdot 3^{n-1}x) + \frac{1}{2}\mu h(4 \cdot 3^{n-1}x) - \mu h(3^n x)\| \\
&\le \frac{1}{2}\|2h(3^n\mu x) - \mu h(2 \cdot 3^{n-1}x) - \mu h(4 \cdot 3^{n-1}x)\| \\
&+ \frac{1}{2}\|\mu h(2 \cdot 3^{n-1}x) + \mu h(4 \cdot 3^{n-1}x) - 2\mu h(3^n x)\| \\
&\le \frac{2}{2}\varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x, 0, 0)
\end{aligned}$$

for all $\mu \in T^1$ and all $x \in \mathcal{A} \setminus \{0\}$. Thus $3^{-n}\|h(3^n\mu x) - \mu h(3^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in T^1$ and all $x \in \mathcal{A} \setminus \{0\}$. Hence

$$H(\mu x) = \lim_{n\to\infty} \frac{h(3^n\mu x)}{3^n} = \lim_{n\to\infty} \frac{\mu h(3^n x)}{3^n} = \mu H(x)$$

for all $\mu \in T^1$ and all $x \in \mathcal{A} \setminus \{0\}$.

By the same reasoning as in the proof of Theorem 2.1, the unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ is a C-linear mapping.

By a similar method to the proof of Theorem 2.1, one can show that the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism. □

Corollary 2.6. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(3^n uy) = h(3^n u)h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}
&\|2h(\frac{\mu x + \mu y + \{z, w\}}{2}) - \mu h(x) - \mu h(y) - \{h(z), h(w)\}\| \\
&\le \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p), \\
&\|h(3^n u^*) - h(3^n u)^*\| \le 2\theta
\end{aligned}$$

for all $\mu \in T^1$, all $x, y, z, w \in \mathcal{A}\setminus\{0\}$, all $u \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. Assume that $lim_{n\to\infty} \frac{h(3^n e)}{3^n}$ is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Poisson C^*-algebra homomorphism.

Proof. Define $\varphi(x, y, z, w) = \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p)$, and apply Theorem 2.5. □

One can obtain similar results to Theorems 2.3 and 2.4 for the Jensen functional equation.

Now we are going to prove the Cauchy–Rassias stability of Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras associated with the Cauchy functional equation.

Theorem 2.7. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying* (2.i), (2.ii) *and* (2.iii) *such that*

$$\|h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)\| \leq \varphi(2^n u, 2^n v, 0, 0) \tag{2.ix}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. Then there exists a unique Poisson C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.†).

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique C-linear involutive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.†).

By (2.ix),

$$\begin{aligned} &\frac{1}{4^n}\|h(2^n u \cdot 2^n v) - h(2^n u)h(2^n v)\| \\ &\leq \frac{1}{4^n}\varphi(2^n u, 2^n v, 0, 0) \leq \frac{1}{2^n}\varphi(2^n u, 2^n v, 0, 0), \end{aligned}$$

which tends to zero by (2.i) as $n \to \infty$. By (2.1),

$$\begin{aligned} H(uv) &= \lim_{n\to\infty} \frac{h(2^n u \cdot 2^n v)}{4^n} = \lim_{n\to\infty} \frac{h(2^n u)h(2^n v)}{4^n} \\ &= \lim_{n\to\infty} \frac{h(2^n u)}{2^n}\frac{h(2^n v)}{2^n} = H(u)H(v) \end{aligned}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is C-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in C, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned} H(xv) &= H(\sum_{j=1}^m \lambda_j u_j v) = \sum_{j=1}^m \lambda_j H(u_j v) = \sum_{j=1}^m \lambda_j H(u_j)H(v) \\ &= H(\sum_{j=1}^m \lambda_j u_j)H(v) = H(x)H(v) \end{aligned}$$

for all $x \in \mathcal{A}$ and all $v \in \mathcal{U}(\mathcal{A})$. By the same method as given above, one can obtain that

$$H(xy) = H(x)H(y)$$

for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is an algebra $*$-homomorphism.

The rest of the proof is the same as in the proof of Theorem 2.1. □

One can obtain similar results to Theorem 2.7 for the Jensen functional equation and the Trif functional equation.

3. Homomorphisms between Poisson Banach modules over a unital Poisson C^*-algebra

Throughout this section, assume that $\mathcal{A}$ is a unital Poisson C^*-algebra with unitary group $\mathcal{U}(\mathcal{A})$, and that X and Y are left Poisson Banach $\mathcal{A}$-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Definition 3.1. A C-linear mapping $H : X \to Y$ is called a *Poisson module homomorphism* if $H : X \to Y$ satisfies

$$H(\{\{a,b\},x\}) = \{\{a,b\},H(x)\},$$
$$H(\{a,b\}\cdot x) = \{a,b\}\cdot H(x)$$

for all $a, b \in \mathcal{A}$ and all $x \in X$.

We are going to prove the Cauchy–Rassias stability of homomorphisms in Poisson Banach modules over a unital Poisson C^*-algebra associated with the Cauchy functional equation.

Theorem 3.1. *Let $h : X \to Y$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty, \tag{3.i}$$

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \le \varphi(x,y), \tag{3.ii}$$

$$\|h(\{\{u,v\},x\}) - \{\{u,v\},h(x)\}\| \le \varphi(x,x), \tag{3.iii}$$

$$\|h(\{u,v\}\cdot x) - \{u,v\}\cdot h(x)\| \le \varphi(x,x) \tag{3.iv}$$

for all $\mu \in T^1$, all $x, y \in X$ and all $u, v \in \mathcal{U}(\mathcal{A})$. Then there exists a unique Poisson module homomorphism $H : X \to Y$ such that

$$\|h(x) - H(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x) \tag{3.v}$$

for all $x \in X$.

Proof. By the same reasoning as in the proof of Theorem 2.1, one can show that there exists a unique C-linear mapping $H : X \to Y$ satisfying (3.v). The C-linear mapping $H : X \to Y$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) \tag{3.1}$$

for all $x \in X$.

By (3.iii),

$$\begin{aligned}
&\|\frac{1}{2^n} h(2^n\{\{u,v\},x\}) - \{\{u,v\},\frac{1}{2^n} h(2^n x)\}\| \\
&= \frac{1}{2^n}\|h(\{\{u,v\},2^n x\}) - \{\{u,v\},h(2^n x)\}\| \\
&\le \frac{1}{2^n}\varphi(2^n x, 2^n x),
\end{aligned}$$

which tends to zero for all $x \in X$ by (3.i). So

$$\begin{aligned}H(\{\{u,v\},x\}) &= \lim_{n\to\infty} \frac{1}{2^n} h(2^n\{\{u,v\},x\}) \\ &= \lim_{n\to\infty} \{\{u,v\}, \frac{1}{2^n} h(2^n x)\} = \{\{u,v\}, H(x)\}\end{aligned}$$

for all $x \in X$ and all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is C-linear and $\{\cdot,\cdot\}$ is C-bilinear and since each $a \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $a = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in C, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned}H(\{\{a,v\},x\}) &= H(\{\{\sum_{j=1}^m \lambda_j u_j, v\}, x\}) = \sum_{j=1}^m \lambda_j H(\{\{u_j,v\},x\}) \\ &= \sum_{j=1}^m \lambda_j \{\{u_j,v\}, H(x)\} = \{\{\sum_{j=1}^m \lambda_j u_j, v\}, H(x)\} = \{\{a,v\}, H(x)\}\end{aligned}$$

for all $x \in X$, all $a \in \mathcal{A}$ and all $v \in \mathcal{U}(\mathcal{A})$. Similarly, one can show that

$$H(\{\{a,b\},x\}) = \{\{a,b\}, H(x)\}$$

for all $x \in X$ and all $a, b \in \mathcal{A}$.

By (3.iv),

$$\begin{aligned}&\|\frac{1}{2^n} h(2^n\{u,v\}\cdot x) - \{u,v\}\cdot \frac{1}{2^n} h(2^n x)\| \\ &= \frac{1}{2^n}\|h(\{u,v\}\cdot 2^n x) - \{u,v\}\cdot h(2^n x)\| \\ &\le \frac{1}{2^n}\varphi(2^n x, 2^n x),\end{aligned}$$

which tends to zero for all $x \in X$ by (3.i). So

$$\begin{aligned}H(\{u,v\}\cdot x) &= \lim_{n\to\infty} \frac{1}{2^n} h(2^n\{u,v\}\cdot x) \\ &= \lim_{n\to\infty} (\{u,v\}\cdot \frac{1}{2^n} h(2^n x)) = \{u,v\}\cdot H(x)\end{aligned}$$

for all $x \in X$ and all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is C-linear and $\{\cdot,\cdot\}$ is C-bilinear and since each $a \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $a = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in C, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned}H(\{a,v\}\cdot x) &= H(\{\sum_{j=1}^m \lambda_j u_j, v\}\cdot x) = \sum_{j=1}^m \lambda_j H(\{u_j,v\}\cdot x) \\ &= \sum_{j=1}^m \lambda_j \{u_j,v\}\cdot H(x) = \{\sum_{j=1}^m \lambda_j u_j, v\}\cdot H(x) = \{a,v\}\cdot H(x)\end{aligned}$$

for all $x \in X$, all $a \in \mathcal{A}$ and all $v \in \mathcal{U}(A)$. Similarly, one can show that

$$H(\{a,b\} \cdot x) = \{a,b\} \cdot H(x)$$

for all $x \in X$ and all $a, b \in \mathcal{A}$. Thus $H : X \to Y$ is a Poisson module homomorphism.

Therefore, there exists a unique Poisson module homomorphism $H : X \to Y$ satisfying (3.v). □

Corollary 3.2. *Let $h : X \to Y$ be a mapping satisfying $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| &\leq \theta(||x||^p + ||y||^p), \\ \|h(\{\{u,v\}, x\}) - \{\{u,v\}, h(x)\}\| &\leq 2\theta||x||^p, \\ \|h(\{u,v\} \cdot x) - \{u,v\} \cdot h(x)\| &\leq 2\theta||x||^p \end{aligned}$$

for all $\mu \in T^1$, all $x, y \in X$ and all $u, v \in \mathcal{U}(\mathcal{A})$. Then there exists a unique Poisson module homomorphism $H : X \to Y$ such that

$$\|h(x) - H(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x,y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 3.1. □

Theorem 3.3. *Let $h : X \to Y$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (3.i), (3.iii) *and* (3.iv) *such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \varphi(x,y)$$

for $\mu = 1, i$, and all $x, y \in X$. If $h(tx)$ is continuous in $t \in R$ for each fixed $x \in X$, then there exists a unique Poisson module homomorphism $H : X \to Y$ satisfying (3.v).

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.1. □

Now we are going to prove the Cauchy–Rassias stability of homomorphisms in Poisson Banach modules over a unital Poisson C^*-algebra associated with the Jensen functional equation.

Theorem 3.4. *Let $h : X \to Y$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : (X \setminus \{0\})^2 \to [0, \infty)$ satisfying* (3.iii) *and* (3.iv) *such that*

$$\begin{aligned} \widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} 3^{-j} \varphi(3^j x, 3^j y) < \infty, \\ \|2h(\frac{\mu x + \mu y}{2}) - \mu h(x) - \mu h(y)\| \leq \varphi(x,y), \end{aligned}$$

for all $\mu \in T^1$ *and all* $x, y \in X$. *Then there exists a unique Poisson module homomorphism* $H : X \to Y$ *such that*

$$\|h(x) - H(x)\| \le \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in X \setminus \{0\}$.

Proof. The proof is similar to the proofs of Theorems 2.5 and 3.1. □

Corollary 3.5. *Let* $h : X \to Y$ *be a mapping satisfying* $h(0) = 0$ *for which there exist constants* $\theta \ge 0$ *and* $p \in [0, 1)$ *such that*

$$\begin{aligned} \|2h(\frac{\mu x + \mu y}{2}) - \mu h(x) - \mu h(y)\| &\le \theta(||x||^p + ||y||^p), \\ \|h(\{\{u, v\}, x\}) - \{\{u, v\}, h(x)\}\| &\le 2\theta||x||^p, \\ \|h(\{u, v\} \cdot x) - \{u, v\} \cdot h(x)\| &\le 2\theta||x||^p \end{aligned}$$

for all $\mu \in T^1$, *all* $x, y \in X \setminus \{0\}$, *and all* $u, v \in \mathcal{U}(\mathcal{A})$. *Then there exists a unique Poisson module homomorphism* $H : X \to Y$ *such that*

$$\|h(x) - H(x)\| \le \frac{(3 + 3^p)\theta}{3 - 3^p}||x||^p$$

for all $x \in X \setminus \{0\}$.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 3.4. □

One can obtain a similar result to Theorem 3.3 for the Jensen functional equation.

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