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Biideals in BCK/BCI-Bialgebras

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Abstract. The biideal structure in BCK/BCI-bialgebras is discussed. Relationships between sub-bialgebras, biideals and IC-ideals (and/or CI-ideals) are considered. Conditions for a biideal to be a sub-bialgebra are provided, and conditions for a subset to be a biideal (resp. IC-ideal, CI-ideal) are given.

1. Introduction

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers. Bialgebraic structures, for example, bisemigroups, bigroups, bigroupoids, biloops, birings, bisemirings, binear-rings, etc., are discussed in [4]. In [2], Jun et al. established the structure of BCK/BCI-bialgebras, and investigated some properties. In this paper, we introduce the notion of biideals, IC-ideals and/or CI-ideals in BCK/BCI-bialgebras. We discuss relationships between biideals, IC-ideals (and/or CI-ideals) and sub-bialgebras, and give conditions for a biideal to be a sub-bialgebra. We also provide conditions for a subset to be a biideal (resp. IC-ideal, CI-ideal).

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

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then X is called a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset H of a BCK/BCI-algebra X is called an *ideal* of X, written by $H \lhd X$, if it satisfies the following axioms:

- $0 \in H$,
- $(\forall x \in X)\ (\forall y \in H)\ (x * y \in H \Rightarrow x \in H)$.

Any ideal H of a BCK/BCI-algebra X satisfies the following implication:

$$(\forall x \in X)\,(\forall y \in H)\,(x \leq y \Rightarrow x \in H).$$

A subset A of a BCI-algebra X is called a *closed ideal* of X, denoted by $A \lhd_c X$, if it is an ideal of X such that $0 * x \in A$ for all $x \in A$. We refer the reader to the book [3] for further information regarding BCK/BCI-algebras.

3. Biideals of BCK/BCI-bialgebras

Definition 3.1 ([2]). Let $X = (X, *, \oplus, 0)$ be an algebra of type $(2, 2, 0)$. Then $X = (X, *, \oplus, 0)$ is called a *BCK-bialgebra* (resp. *BCI-bialgebra*) if there exists two distinct proper subsets X_1 and X_2 of X such that

(i) $X = X_1 \cup X_2$.

(ii) $(X_1, *, 0)$ is a BCK-algebra (resp. BCI-algebra).

(iii) $(X_2, \oplus, 0)$ is a BCK-algebra (resp. BCI-algebra).

Denote by $X = K(X_1) \uplus K(X_2)$ (resp. $X = I(X_1) \uplus I(X_2)$) the BCK-bialgebra (resp. BCI-bialgebra). If $(X_1, *, 0)$ is a BCK-algebra (resp. BCI-algebra) and $(X_2, \oplus, 0)$ is a BCI-algebra (resp. BCK-algebra), then we say that $X = (X, *, \oplus, 0)$ is a BCKI-bialgebra (resp. BCIK-bialgebra), and denoted by $X = K(X_1) \uplus I(X_2)$ (resp. $X = I(X_1) \uplus K(X_2)$).

Definition 3.2. Let $X = K(X_1) \uplus K(X_2)$ (or $X = K(X_1) \uplus I(X_2)$, $X = I(X_1) \uplus K(X_2)$, $X = I(X_1) \uplus I(X_2)$) be a BCK-bialgebra (or a BCKI-bialgebra, a BCIK-bialgerba, a BCI-bialgebra). A subset $H(\neq \emptyset)$ of X is called a *biideal* of X if there exist distinct proper subsets H_1 and H_2 of X_1 and X_2, respectively, such that $H = H_1 \cup H_2$ and $H_i \lhd X_i$ for $i = 1, 2$.

We illustrate this definition by the following examples.

Example 3.3. Let $X = \{0, a, b, c, d, x, y\}$ and consider two proper subsets $X_1 = \{0, a, b, c, d\}$ and $X_2 = \{0, a, b, x, y\}$ of X together with Cayley tables respectively as follows:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	c	c	0	c
d	d	d	d	d	0

$\oplus$	0	a	b	x	y
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	a
x	x	x	x	0	x
y	y	y	y	y	0

Then $X = K(X_1) \uplus K(X_2)$. Note that $H_1 = \{0, a, b\} \lhd X_1$ and $H_2 = \{0, a, b, y\} \lhd X_2$. Hence $H = \{0, a, b, y\}$ is a biideal of X.

Example 3.4. Let $X = \{0, a, b, c, x\}$ and consider two proper subsets $X_1 = \{0, a, b, c\}$ and $X_2 = \{0, a, x\}$ of X together with Cayley tables respectively as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

$\oplus$	0	a	x
0	0	0	x
a	a	0	x
x	x	x	0

Then $X = K(X_1) \uplus I(X_2)$, and $I_1 = \{0, c\} \lhd X_1$ and $I_2 = \{0, a\} \lhd X_2$. Therefore $I = \{0, a, c\}$ is a biideal of X.

Example 3.5. Let $X = \{0, a, b, c, d, e, f, g, x, y\}$ and consider two proper subsets $X_1 = \{0, a, b, c, d, e, f, g\}$ and $X_2 = \{0, a, x, y\}$ of X together with Cayley tables respectively as follows:

$*$	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

$\oplus$	0	a	x	y
0	0	0	x	x
a	a	0	x	x
x	x	x	0	0
y	y	x	a	0

Then $X = I(X_1) \uplus I(X_2)$, and $I_1 = \{0, d\} \lhd X_1$ and $I_2 = \{0, a\} \lhd X_2$. Therefore $I = \{0, a, d\}$ is a biideal of X. Note that $I = \{0, a, d\}$ is not an ideal of $(X_1, *, 0)$ since $e * d = a \in I$ and $e \notin I$.

Example 3.6. Let $X = \mathbb{Q}^* \cup X_2$, where $\mathbb{Q}^*$ is the set of all nonzero rational numbers and X_2 is a BCK-algebra under the operation $\oplus$ that satisfies the following implication:

$$(\forall x, y, z \in X_2)\,(x \oplus y \leq z,\ y \leq z \ \Rightarrow\ x \leq z).$$

Note that $(\mathbb{Q}^*, \div, 1)$ is a BCI-algebra. Thus $X = I(\mathbb{Q}^*) \uplus K(X_2)$. Let $J = A(a) \cup \mathbb{Z}^*$, where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ and $A(a) = \{x \in X_2 \mid x \leq a\}$ for a fixed element a of X_2. Then $A(a)$ and $\mathbb{Z}^*$ are ideals of X_2 and $\mathbb{Q}^*$, respectively. Hence J is a biideal of X.

We provide conditions for a subset to be a biideal.

Theorem 3.7. *Let $X = K(X_1) \uplus K(X_2)$ (resp. $X = K(X_1) \uplus I(X_2)$, $X = I(X_1) \uplus K(X_2)$, $X = I(X_1) \uplus I(X_2)$). If A is a nonempty subset of X such that $A \cap X_1 \lhd (X_1, *, 0)$ and $A \cap X_2 \lhd (X_2, \oplus, 0)$, then A is a biideal of X.*

Proof. It is sufficient to show that $(A \cap X_1) \cup (A \cap X_2) = A$. Now,

$$\begin{aligned}(A \cap X_1) \cup (A \cap X_2) &= \big((A \cap X_1) \cup A\big) \cap \big((A \cap X_1) \cup X_2\big)\\ &= \big(A \cap (X_1 \cup A)\big) \cap \big((A \cup X_2) \cap X\big)\\ &= A \cap (A \cup X_2)\\ &= A.\end{aligned}$$

Hence A is a biideal of X. □

Definition 3.8 ([2]). Let $X = K(X_1) \uplus K(X_2)$ (resp. $X = K(X_1) \uplus I(X_2)$, $X = I(X_1) \uplus K(X_2)$, $X = I(X_1) \uplus I(X_2)$). A subset $H(\neq \emptyset)$ of X is called a *sub-bialgebra* of X if there exist subsets H_1 and H_2 of X_1 and X_2, respectively, such that

(i) $H_1 \neq H_2$ and $H = H_1 \cup H_2$,

(ii) $(H_1, *, 0)$ is a subalgebra of $(X_1, *, 0)$,

(iii) $(H_2, \oplus, 0)$ is a subalgebra of $(X_2, \oplus, 0)$.

Theorem 3.9. *Let $X = K(X_1) \uplus K(X_2)$ be a BCK-bialgebra. Then any biideal of X is a sub-bialgebra of X.*

Proof. Straightforward. □

The following example shows that the converse of Theorem 3.9 is not true in general.

Example 3.10. Let $X = \{0, a, b, 1, 2, 3, 4\}$ and consider two proper subsets $X_1 = \{0, a, b\}$ and $X_2 = \{0, 1, 2, 3, 4\}$ of X together with Cayley tables respectively as follows:

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

$\oplus$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	1	1	0	0
4	4	1	1	1	0

Then $X = K(X_1) \uplus K(X_2)$. Note that $S_1 = \{0, a\}$ and $S_2 = \{0, 1, 2, 3\}$ are subalgebras of X_1 and X_2, respectively. Hence $S = \{0, a, 1, 2, 3\}$ is a sub-bialgebra of X. But S_1 is not an ideal of X_1 since $b * a = a \in S_1$ and $b \notin S_1$. Also, S_2 is not an ideal of X_2 because $4 \oplus 2 = 1 \in S_2$ and $4 \notin S_2$. Therefore S is not a biideal of X.

Example 3.11. Let $X = \{0, a, x, y, 1, 2, 3, 4\}$ and consider two proper subsets $X_1 = \{0, 1, 2, 3, 4\}$ and $X_2 = \{0, a, x, y\}$ of X together with Cayley tables respectively as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

$\oplus$	0	a	x	y
0	0	0	x	x
a	a	0	x	x
x	x	x	0	0
y	y	x	a	0

Then $X = K(X_1) \uplus I(X_2)$. Note that $H_1 = \{0, 1, 2\}$ is a subalgebra of X_1 which is an ideal of X_1, and $H_2 = \{0, a, x\}$ is a subalgebra of X_2 but not an ideal of X_2 since $y \oplus a = x \in H_2$ and $y \notin H_2$. Hence $H = \{0, 1, 2, a, x\}$ is a sub-bialgebra of X which is not a biideal of X.

Note that any biideal in a BCK-bialgebra $X = K(X_1) \uplus K(X_2)$ is a sub-bialgebra (see Theorem 3.9). But, in a BCKI-bialgebra $X = K(X_1) \uplus I(X_2)$, any biideal is not a sub-bialgebra in general as seen in the following example.

Example 3.12. In Example 3.6, we know that $\mathbb{Z}^*$ is an ideal of $\mathbb{Q}^*$, but not a subalgebra. So, we know that any biideal is not a sub-bialgebra in $X = K(X_1) \uplus I(X_2)$, $X = I(X_1) \uplus K(X_2)$, or $X = I(X_1) \uplus I(X_2)$.

Example 3.13. Let $X = Y \cup \mathbb{Z}$, where $Y = \{0, a, b, c, d\}$ is a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Note that $(\mathbb{Z}, -, 0)$ is a BCI-algebra. Hence $X = K(Y) \uplus I(\mathbb{Z})$. It is easy to show that $G_1 = \{0, a, c\}$ is an ideal of Y which is also a subalgebra of Y, and the set $G_2 = \{x \in \mathbb{Z} \mid 0 \leq x\}$ is an ideal of $\mathbb{Z}$ which is not a subalgebra. Hence $G := G_1 \cup G_2$ is a biideal of X which is not a sub-bialgebra of X.

Definition 3.14. Let $X = K(X_1) \uplus I(X_2)$ (resp. $X = I(X_1) \uplus K(X_2)$). A subset

$A(\neq \emptyset)$ of X is called an *IC-ideal* (resp. *CI-ideal*) of X if there exist distinct proper subsets A_1 and A_2 of X_1 and X_2, respectively, such that

(i) $A = A_1 \cup A_2$,

(ii) $A_1 \lhd X_1$ and $A_2 \lhd_c X_2$ (resp. $A_1 \lhd_c X_1$ and $A_2 \lhd X_2$).

Note that any IC-ideal (resp. CI-ideal) in $X = K(X_1) \uplus I(X_2)$ (resp. $X = I(X_1) \uplus K(X_2)$) is a biideal, but the converse is not true in general.

Example 3.15. (1) In Example 3.13, $G := G_1 \cup G_2$ is a biideal which is not an IC-ideal since G_2 is not closed.

(2) In Example 3.5, $I = \{0, a, d\}$ is an IC-ideal of X.

Theorem 3.16. *Let $X = K(X_1) \uplus I(X_2)$* (resp. $X = I(X_1) \uplus K(X_2)$)*, where $|X_2| < \infty$* (resp. $|X_1| < \infty$)*. Then every biideal of X is an IC-ideal* (resp. *CI-ideal*) *of X.*

Proof. Assume that $X = K(X_1) \uplus I(X_2)$ and $|X_2| = n < \infty$. Let A be a biideal of X. Then there are distinct subsets A_1 and A_2 of X_1 and X_2, respectively, so that $A = A_1 \cup A_2$ and $A_i \lhd X_i$ for $i = 1, 2$. For every $a, b \in A_2$ and $k \in \mathbb{N}$, denote

$$a \oplus b^k = (\cdots((a \oplus \underbrace{b) \oplus b) \oplus \cdots) \oplus b}_{k\text{-times}}.$$

Now, for each $a \in A_2$, consider $n + 1$ elements as follows:

$$0, 0 \oplus a, 0 \oplus a^2, \cdots, 0 \oplus a^n.$$

Since $|X_2| = n$, it follows that two of them must be equal so that there exist $r, s \in \mathbb{N}$ such that $s < r \leq n$ and $0 \oplus a^r = 0 \oplus a^s$. Then

$$0 = (0 \oplus a^r) \oplus (0 \oplus a^s) = ((0 \oplus a^s) \oplus a^{r-s}) \oplus (0 \oplus a^s) = 0 \oplus a^{r-s} \in A_2,$$

and so $0 \oplus a \in A_2$ since $A_2 \lhd X_2$. Thus $A_2 \lhd_c X_2$. Therefore A is an IC-ideal of X. Similarly we get desired result for the case $X = I(X_1) \uplus K(X_2)$ with $|X_1| < \infty$. □

Corollary 3.17. *Let $X = I(X_1) \uplus I(X_2)$, where $|X_1| < \infty$ and $|X_2| < \infty$. Then every biideal of X is a CC-ideal of X.*

Theorem 3.18. *Let $X = K(X_1) \uplus I(X_2)$* (resp. $X = I(X_1) \uplus K(X_2)$)*. Then any IC-ideal* (resp. *CI-ideal*) *of X is a sub-bialgebra of X.*

Proof. It is straightforward because any closed ideal of a BCI-algebra is a subalgerba, and any ideal of a BCK-algebra is a subalgebra. □

The following example shows that the converse of Theorem 3.18 is not true in general.

Example 3.19. Let $X = \{0, a, b, 1, 2, 3, 4\}$ and consider two proper subsets $X_1 =$

$\{0, 1, 2, 3, 4\}$ and $X_2 = \{0, a, b\}$ of X together with Cayley tables respectively as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$\oplus$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Then $X = K(X_1) \uplus I(X_2)$, and $H_1 = \{0, 1, 2, 4\}$ is a subalgebra of X which is also an ideal. But $H_2 = \{0, b\}$ is a subalgebra of X_2 which is not an ideal of X_2. Hence $H := \{0, 1, 2, 4, b\}$ is a sub-bialgebra of X which is not an IC-ideal.

Corollary 3.20. *Let $X = K(X_1) \uplus I(X_2)$* (resp. $X = I(X_1) \uplus K(X_2)$)*, where $|X_2| < \infty$* (resp. $|X_1| < \infty$)*. Then every biideal of X is a sub-bialgebra of X.*

Theorem 3.21. *Let $X = K(X_1) \uplus I(X_2)$ in which X_2 satisfies the following inequality:*

$$(\forall x \in X_2)\,(0 \oplus x \leq x).$$

Then any biideal of X is an IC-ideal of X and hence is a sub-bialgebra of X.

Proof. Let A be a biideal of X. Then $A = A_1 \uplus A_2$ and $A_i \lhd X_i$, $i = 1, 2$, for some $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$ with $A_1 \neq A_2$. Let $y \in A_2$. Since $0 \oplus y \leq y$ by assumption, it follows that $0 \oplus y \in A_2$ so that $A_2 \lhd_c X_2$. Hence A is an IC-ideal of X. □

Theorem 3.22. *Let $X = K(X_1) \uplus I(X_2)$* (resp. $X = I(X_1) \uplus K(X_2)$) *and let A be a subset of X such that $A \cap X_1 \lhd (X_1, *, 0)$ and $A \cap X_2 \lhd_c (X_2, \oplus, 0)$* (resp. *$A \cap X_1 \lhd_c (X_1, *, 0)$ and $A \cap X_2 \lhd (X_2, \oplus, 0)$*)*. Then A is an IC-ideal* (resp. *CI-ideal*) *of X.*

Proof. Similar to the proof of Theorem 3.7. □

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