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On the Ideal Extensions in Γ-Semigroups

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Abstract. In 1981, Sen [4] have introduced the concept of Γ-semigroups. We have known that Γ-semigroups are a generalization of semigroups. In this paper, we introduce the concepts of the extensions of s-prime ideals, prime ideals, s-semiprime ideals and semiprime ideals in Γ-semigroups and characterize the relationship between the extensions of ideals and some congruences in Γ-semigroups.

1. Preliminaries

Let M and Γ be any two nonempty sets. M is called a Γ-*semigroup* [5], [7] if for all $a, b, c \in M$ and $\gamma, \mu \in \Gamma$, we have (i) $a\gamma b \in M$ and (ii) $(a\gamma b)\mu c = a\gamma(b\mu c)$. A Γ-semigroup M is called a *commutative* Γ-*semigroup* if $a\gamma b = b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$. A nonempty subset K of a Γ-semigroup M is called a *sub*-Γ-*semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

For examples of Γ-semigroups, see [1], [3], [5], [6], [7].

Let S be a semigroup and $\Gamma = \{1\}$. We define a mapping $S \times \Gamma \times S \longrightarrow S$ by $a1b = ab$ for all $a, b \in S$. Then S is a Γ-semigroup. Hence we have known that Γ-semigroups are a generalization of semigroups.

For nonempty subsets A and B of a Γ-semigroup M and a nonempty subset Γ' of Γ, let $A\Gamma' B := \{a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $\{a\}\Gamma' B$ as $a\Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$. A nonempty subset I of a Γ-semigroup M is called an *ideal* of M if $M\Gamma I \subseteq I$ and $I\Gamma M \subseteq I$. The intersection of all ideals of a Γ-semigroup M containing a nonempty subset A of M is the *ideal of M generated by* A, and will be denoted by $I(A)$. If $A = \{x\}$, then we also write $I(\{x\})$ as $I(x)$. An ideal I of a Γ-semigroup M is called an *s-prime ideal* [3] of M if for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in I$ implies $a \in I$ or $b \in I$. Equivalently, for any $A, B \subseteq M$ and $\gamma \in \Gamma, A\gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. An ideal I of a Γ-semigroup M is called a *prime ideal* of M if for any $a, b \in M, a\Gamma b \subseteq I$ implies $a \in I$ or $b \in I$. Equivalently, for any $A, B \subseteq M, A\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

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An ideal I of a Γ-semigroup M is called an *s-semiprime ideal* of M if for any $a \in M$ and $\gamma \in \Gamma, a\gamma a \in I$ implies $a \in I$. Equivalently, for any $A \subseteq M$ and $\gamma \in \Gamma, A\gamma A \subseteq I$ implies $A \subseteq I$. An ideal I of a Γ-semigroup M is called a *semiprime ideal* of M if for any $a \in M, a\Gamma a \subseteq I$ implies $a \in I$. Equivalently, for any $A \subseteq M, A\Gamma A \subseteq I$ implies $A \subseteq I$. Hence we have the following statements for Γ-semigroups.

(1) Every s-prime ideal is a prime ideal.
(2) Every prime ideal is a semiprime ideal.
(3) Every s-prime ideal is an s-semiprime ideal.
(4) Every s-semiprime ideal is a semiprime ideal.

For a Γ-semigroup M, let

$$\begin{aligned} P(M) &:= \{A : A \text{ is a prime ideal of } M\}, \\ SP(M) &:= \{A : A \text{ is an } s\text{-prime ideal of } M\}. \end{aligned}$$

Then $\emptyset \neq SP(M) \subseteq P(M)$. A sub-$\Gamma$-semigroup F of a Γ-semigroup M is called a *filter* [3] of M if for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in F$ implies $a, b \in F$. The intersection of all filters of a Γ-semigroup M containing a nonempty subset A of M is the filter of M generated by A. For $A = \{x\}$, let $n(x)$ denote the filter of M generated by $\{x\}$. An equivalence relation σ on a Γ-semigroup M is called a *congruence* [2], [6] if for any $a, b, c \in M$ and $\gamma \in \Gamma, (a, b) \in \sigma$ implies $(a\gamma c, b\gamma c) \in \sigma$ and $(c\gamma a, c\gamma b) \in \sigma$. Let σ be a congruence on a Γ-semigroup M and $M/\sigma := \{(x)_\sigma : x \in M\}$. We define $(x)_\sigma\gamma(y)_\sigma = (x\gamma y)_\sigma$ for all $(x)_\sigma, (y)_\sigma \in M/\sigma$ and $\gamma \in \Gamma$. It is easy to verify that the definition is well-defined and M/σ is a Γ-semigroup. A congruence σ on a Γ-semigroup M is called a *semilattice congruence* [8] if for all $a, b \in M$ and $\gamma \in \Gamma, (a\gamma b, b\gamma a) \in \sigma$ and $(a\gamma a, a) \in \sigma$. For an ideal I of a Γ-semigroup M and $A \subseteq M$, the set $< A, I > := \{x \in M : A\Gamma x \subseteq I\}$ is called the *extension* of I by A. If $A = \{a\}$, then we also write $< \{a\}, I >$ as $< a, I >$. For an ideal I of a Γ-semigroup M, we define equivalence relations on M as follows:

$$\begin{aligned} \sigma_I &:= \{(x, y) \in M \times M : x, y \in I \text{ or } x, y \notin I\}, \\ \phi_I &:= \{(x, y) \in M \times M : < x, I > = < y, I >\}, \\ n &:= \{(x, y) \in M \times M : n(x) = n(y)\}. \end{aligned}$$

Example 1.([3]) Let $M = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ with the multiplication defined by

$$x\gamma y = \begin{cases} b & \text{if } x, y \in \{a, b\}, \\ c & \text{otherwise.} \end{cases}$$

Then M is a Γ-semigroup. We can easily get all ideals of M as follows:

$$P_1 = M, P_2 = \{c, d\}, P_3 = \{b, c\}, P_4 = \{c\}, P_5 = \{a, b, c\}, P_6 = \{b, c, d\}.$$

It is easy to see that P_1 and P_2 are s-prime ideals of M, so P_1 and P_2 are semiprime ideals of M. Let

$$\begin{aligned} \sigma_1 &= M \times M, \\ \sigma_2 &= \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}. \end{aligned}$$

It is easy to see that σ_1 and σ_2 are semilattice congruences on M.

Example 2. For $n \in \{1, 2\}$, let $M = \{n, n+1, n+2, \cdots\}$ and $\Gamma = \{-n\}$. Then M is a Γ-semigroup under usual addition. Let $I = \{2n, 2n+1, 2n+2, \cdots\}$. It is easy to verify that I is a semiprime ideal of M and $\sigma = \{(n, n)\}$ is a semilattice congruence on M.

The following theorem is obtained similarly in [3] and the following lemmas will be used frequently in this paper.

Theorem 1.1. *If M is a Γ-semigroup, then*

$$n = \bigcap_{I \in SP(M)} \sigma_I.$$

In this paper, we consider the ideal extensions in a commutative Γ-semigroup. From now on, M stands for a commutative Γ-semigroup. The next two lemmas are easy to verify.

Lemma 1.2. *If A is a subset of M, then $I(A) = A \cup M\Gamma A$.*

Lemma 1.3. *Let I be an ideal of M and $A \subseteq B \subseteq M$. Then $< B, I > \subseteq < A, I >$.*

Lemma 1.4. *Let I be an ideal of M, $A \subseteq M$ and $\gamma \in \Gamma$. Then we have the following statements:*

(a) *$< A, I >$ is an ideal of M.*

(b) *$I \subseteq < A, I > \subseteq < A\Gamma A, I > \subseteq < A\gamma A, I >$.*

(c) *If $A \subseteq I$, then $< A, I > = M$.*

Proof. (a) Let $x \in < A, I >, y \in M$ and $\gamma \in \Gamma$. Then $A\Gamma(x\gamma y) = (A\Gamma x)\gamma y \subseteq I\Gamma M \subseteq I$, so $x\gamma y \in < A, I >$. Hence $< A, I >$ is an ideal of M.

(b) If $x \in I$, then $A\Gamma x \subseteq M\Gamma I \subseteq I$. Thus $x \in < A, I >$. If $x \in < A, I >$, then $(A\Gamma A)\Gamma x = A\Gamma(A\Gamma x) \subseteq M\Gamma I \subseteq I$. Thus $x \in < A\Gamma A, I >$. If $x \in < A\Gamma A, I >$, then $(A\gamma A)\Gamma x \subseteq (A\Gamma A)\Gamma x \subseteq I$. Thus $x \in < A\gamma A, I >$. Hence $I \subseteq < A, I > \subseteq < A\Gamma A, I > \subseteq < A\gamma A, I >$.

(c) Let $A \subseteq I$ and $x \in M$. Then $A\Gamma x \subseteq I\Gamma M \subseteq I$, so $x \in < A, I >$. Hence $< A, I > = M$. □

Lemma 1.5. *Let I be an ideal of M and $A \subseteq M$. Then*

$$< A, I > = \bigcap_{a \in A} < a, I > = < A \setminus I, I > .$$

Proof. By Lemma 1.3, we have $< A, I > \subseteq \bigcap_{a \in A} < a, I >$. Let $x \in \bigcap_{a \in A} < a, I >$. Then $a\Gamma x \subseteq I$ for all $a \in A$, so $A\Gamma x \subseteq I$. Thus $x \in < A, I >$, so $\bigcap_{a \in A} < a, I > \subseteq <$

$A, I >$. Hence $< A, I >= \bigcap_{a\in A} < a, I >$. By Lemma 1.4 ($c$), we have $< A, I >= \bigcap_{a\in A} < a, I >=< A \setminus I, I >$. □

Lemma 1.6. *Let I be an ideal of M. Then I is a prime ideal of M if and only if $< A, I >= I$ for all $A \not\subseteq I$.*

Proof. Assume that I is a prime ideal of M and $A \not\subseteq I$. Let $x \in< A, I >$. Then $A\Gamma x \subseteq I$. By hypothesis and $A \not\subseteq I$, $x \in I$. Thus $< A, I >\subseteq I$. By Lemma 1.4 (b), $< A, I >= I$.

Conversely, assume that $< A, I >= I$ for all $A \not\subseteq I$. Let $A, B \subseteq M$ be such that $A\Gamma B \subseteq I$ and $A \not\subseteq I$. Then $B \subseteq< A, I >= I$. Hence I is a prime ideal of M. □

We can easily prove the last lemma.

Lemma 1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subfamilies of $P(M)$ and $SP(M)$, respectively. Then we have the following statements:*

(a) $\bigcap_{P\in\mathcal{A}} P$ *is a semiprime ideal of M if* $\bigcap_{P\in\mathcal{A}} P \neq \emptyset$.

(b) $\bigcup_{P\in\mathcal{B}} P$ *is a prime ideal of M.*

(c) $\bigcap_{P\in\mathcal{B}} P$ *is an s-semiprime ideal of M if* $\bigcap_{P\in\mathcal{B}} P \neq \emptyset$.

(d) $\bigcup_{P\in\mathcal{B}} P$ *is an s-prime ideal of M.*

2. Main theorems

In this section, we give some characterizations of the relationship between the extensions of ideals and some congruences in Γ-semigroups.

Theorem 2.1. *Let P be a prime ideal of M and $A \subseteq M$. Then $< A, P >$ is a prime ideal of M. Furthermore, $< A, \bigcap_{P\in P(M)} P >$ is a semiprime ideal of M if $\bigcap_{P\in P(M)} P \neq \emptyset$.*

Proof. If $A \subseteq P$, then it follows from Lemma 1.4 (c) that $< A, P >= M$. If $A \not\subseteq P$, then it follows from Lemma 1.6 that $< A, P >= P$. Hence $< A, P >$ is a prime ideal of M. Now,

$$\begin{aligned} x \in < A, \bigcap_{P \in P(M)} P > &\Leftrightarrow A\Gamma x \subseteq \bigcap_{P \in P(M)} P \\ &\Leftrightarrow A\Gamma x \subseteq P \text{ for all } P \in P(M) \\ &\Leftrightarrow x \in < A, P > \text{ for all } P \in P(M) \\ &\Leftrightarrow x \in \bigcap_{P \in P(M)} < A, P > . \end{aligned}$$

Hence

$$< A, \bigcap_{P \in P(M)} P > = \bigcap_{P \in P(M)} < A, P > .$$

It follows from Lemma 1.7 (a) that $< A, \bigcap_{P \in P(M)} P >$ is a semiprime ideal of M. □

Theorem 2.2. *Let $A, B \subseteq M$ and $A \subseteq M\Gamma A$. Then $I(A) \subseteq I(B)$ if and only if for every ideal J of M, $< B, J > \subseteq < A, J >$.*

Proof. Assume that $I(A) \subseteq I(B)$. Let J be an ideal of M and $x \in < B, J >$. By Lemma 1.2, we have $A \subseteq I(B) = B \cup M\Gamma B$. For any $a \in A$, if $a = y\alpha b$ for some $y \in M, b \in B$ and $\alpha \in \Gamma$, then $a\gamma x = (y\alpha b)\gamma x = y\alpha(b\gamma x) \in M\Gamma J \subseteq J$ for all $\gamma \in \Gamma$. Hence $a\gamma x \in J$ for all $\gamma \in \Gamma$, so $x \in < a, J >$. If $a = b$ for some $b \in B$, then $a\gamma x = b\gamma x \in J$ for all $\gamma \in \Gamma$. Hence $a\gamma x \in J$ for all $\gamma \in \Gamma$, so $x \in < a, J >$. Therefore $< B, J > \subseteq \bigcap_{a \in A} < a, J >$. It follows from Lemma 1.5 that $< B, J > \subseteq < A, J >$.

Conversely, assume that $< B, J > \subseteq < A, J >$ for all ideal J of M. Then $< B, I(B) > \subseteq < A, I(B) >$. Since $B \subseteq I(B)$, it follows from Lemma 1.4 (c) that $< B, I(B) > = M$. Thus $< A, I(B) > = M$, so $M\Gamma A \subseteq I(B)$. Hence $A \subseteq M\Gamma A \subseteq I(B)$. This implies that $I(A) \subseteq I(B)$. □

Theorem 2.3. *If I is an s-semiprime ideal of M, then ϕ_I is a semilattice congruence on M.*

Proof. Let $(x, y) \in \phi_I, c \in M$ and $\gamma \in \Gamma$. Then $< x, I > = < y, I >$. Thus

$$\begin{aligned} a \in < x\gamma c, I > &\Leftrightarrow (x\gamma c)\Gamma a \subseteq I \\ &\Leftrightarrow x\Gamma(c\gamma a) \subseteq I \\ &\Leftrightarrow c\gamma a \in < x, I > \\ &\Leftrightarrow c\gamma a \in < y, I > \\ &\Leftrightarrow y\Gamma(c\gamma a) \subseteq I \\ &\Leftrightarrow (y\gamma c)\Gamma a \subseteq I \\ &\Leftrightarrow a \in < y\gamma c, I > . \end{aligned}$$

Hence $(x\gamma c, y\gamma c) \in \phi_I$. Similarly, we can show that $(c\gamma x, c\gamma y) \in \phi_I$. Hence ϕ_I is a congruence on M. Let $x \in M$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} a \in < x\gamma x, I > &\Rightarrow (x\gamma x)\Gamma a \subseteq I \\ &\Rightarrow (x\gamma x\Gamma a)\Gamma a \subseteq I\Gamma M \subseteq I \\ &\Rightarrow (x\Gamma a)\gamma(x\Gamma a) \subseteq I \\ &\Rightarrow x\Gamma a \subseteq I \\ &\Rightarrow a \in < x, I > . \end{aligned}$$

Thus $< x\gamma x, I > \subseteq < x, I >$. By Lemma 1.4 (b), $< x, I > \subseteq < x\gamma x, I >$. Hence $< x\gamma x, I > = < x, I >$, so $(x\gamma x, x) \in \phi_I$. Therefore ϕ_I is a semilattice congruence on M. □

Theorem 2.4. *If I is an s-prime ideal of M, then $\phi_I = \sigma_I$ and $n \subseteq \phi_I$.*

Proof. Let $(x, y) \in \phi_I$. Then $< x, I > = < y, I >$. Suppose that $(x, y) \notin \sigma_I$. Without loss of generality, we may assume that $x \in I$ but $y \notin I$. By Lemma 1.4 (c) and Lemma 1.6, we have $< x, I > = M$ and $< y, I > = I$. Thus $I = M$, so $y \notin M$. This is a contradiction. Hence $(x, y) \in \sigma_I$, so $\phi_I \subseteq \sigma_I$. Let $(x, y) \in \sigma_I$. If $x \in I$, then $y \in I$. By Lemma 1.4 (c), $< x, I > = M = < y, I >$. If $x \notin I$, then $y \notin I$. By Lemma 1.6, $< x, I > = I = < y, I >$. Hence $(x, y) \in \phi_I$, so $\sigma_I \subseteq \phi_I$. Therefore $\phi_I = \sigma_I$. It follows from Theorem 1.1 that

$$n = \bigcap_{J \in SP(M)} \sigma_J = \bigcap_{J \in SP(M)} \phi_J \subseteq \phi_I.$$

Hence the proof is completed. □

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