KYUNGPOOK Math. J. 48(2008), 685-703

Strong Convergence of Modified Iteration Processes for Relatively Nonexpansive Mappings

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Abstract. Motivated and inspired by ideas due to Matsushida and Takahashi [*J. Approx. Theory* **134**(2005), 257-266] and Martinez-Yanes and Xu [*Nonlinear Anal.* **64**(2006), 2400-2411], we prove some strong convergence theorems of modified iteration processes for a pair (or finite family) of relatively nonexpansive mappings in Banach spaces, which improve and extend the corresponding results of Matsushida and Takahashi and Martinez-Yanes and Xu in Banach and Hilbert spaces, repectively.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a mapping. We say that T is a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$. In particular, a Lipschitzian mapping T is called *nonexpansive* if $k_n = 1$ for all $n \geq 1$ and *asymptotically nonexpansive* [9] if $\lim_{n\to\infty} k_n = 1$, respectively. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an *asymptotic fixed point* of T [23] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

Let X be a smooth Banach space and let X^* be the dual of X. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in X$, where J denotes the normalized duality mapping from X to X^*. A mappings $T : C \to C$ is called *relatively nonexpansive* [18] if $F(T)$ is nonempty,

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Received May 30, 2007.
2000 Mathematics Subject Classification: 47H09, 65J15.
Key words and phrases: strong convergence, modified Ishikawa's iteration, relatively nonexpansive mapping, nonexpansive mappings.
Supported by a Pukyong National University Research Fund (PK2006-018).

$\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$, $p \in F(T)$; see also [3], [4], [5]. It is known in [18] that if X is strictly convex and T is relatively nonexpansive, then $F(T)$ is closed and convex.

Construction of approximating fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing. However, the sequence $\{T^n x\}$ of iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology. Thus three averaged iteration methods often prevail to approximate a fixed point of a nonexpansive mapping T. The first one is introduced by Halpern [10] and is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n x_0 + (1 - t_n) T x_n, \quad n \geq 0, \tag{1.1}$$

where $\{t_n\}$ is a sequence in the interval $[0, 1]$.

The second iteration process is now known as Mann's iteration process [16] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$.

The third iteration process is referred to as Ishikawa's iteration process [11] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \quad n \geq 0, \tag{1.3}$$

where the initial guess x_0 is taken in C arbitrarily and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$. By taking $\beta_n = 1$ for all $n \geq 0$ in (1.3), Ishikawa's iteration process reduces to the Mann's iteration process (1.2). It is known in [6] that the process (1.2) may fail to converge while the process (1.3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space.

In general, the iteration process (1.1) has been proved to be strongly convergent in both Hilbert spaces [10], [15], [26] and uniformly smooth Banach spaces [20], [24], [29], while Mann's iteration (1.2) has only weak convergence even in a Hilbert space [8].

Attempts to modify the Mann iteration method (1.2) or the Ishikawa iteration method (1.3) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [19] proposed the following modification of Mann's iteration process (1.2) for a single nonexpansive mapping T with $F(T) \neq \emptyset$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.4}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}x_0$. A recent extension of the process (1.4) to asymptotically nonexpansive mappings can be found in [14]. See also [13] for another modification of the Mann iteration process (1.2) which also has strong convergence. Very recently, Martinez-Yanez and Xu [17] generalized Nakajo and Takahashi's iteration process (1.4) to the following modification of Ishikawa's iteration process (1.3) for a nonexpansive mapping $T : C \to C$ with $F(T) \neq \emptyset$ in a Hilbert space H:

$$(1.5)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|z_n - v\|^2\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

and proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}x_0$ provided the sequence $\{\alpha_n\}$ is bounded above from one and $\lim_{n\to\infty}\beta_n = 1$.

On the other hand, Matsushita and Takahashi [18] extended Nakajo and Takahashi's iteration process (1.4) to the following modification of Mann's iteration process (1.2) using the hybrid method in mathematical programming for a relatively nonexpansive mapping $T : C \to C$ in a uniformly convex and uniformly smooth Banach space X:

$$(1.6)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases}$$

and they also proved that if the sequence $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\limsup_{n\to\infty}\alpha_n < 1$, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $\prod_{F(T)} x_0$, where $\prod_K$ denotes the generalized projection from X onto a closed convex subset K of X.

The purpose of this paper, motivated and inspired by ideas due to Martinez-Yanez and Xu [17] and Matsushita and Takahashi [18], is to prove some strong convergence theorems for a pair (or finite family) of relatively nonexpansive mappings in Banach spaces.

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle\cdot,\cdot\rangle$ the duality product. The normalized duality mapping from X to X^* is defined by

$$Jx = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in X$. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by

$$\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}.$$

A Banach space X is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all x, $y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be *uniformly convex* if $\|x_n - y_n\| \to 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \to 1$.

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Then the Banach space X is said to be *smooth* provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each x, $y \in U$. It is also said to be *uniformly smooth* if the limit in (2.1) is attained uniformly for $x, y \in U$. It is well known that if X is smooth, then the duality mapping J is single-valued. It is also known that if X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X. Some properties of the duality mapping have been given in [7], [22], [25]. A Banach space X is said to have the *Kadec-Klee* property if a sequence $\{x_n\}$ of X satisfying that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if X is uniformly convex, then X has the Kadec-Klee property; see [7], [25] for more details.

Let X be a smooth Banach space. Recall that the function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in X$. It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \tag{2.2}$$

for all $x, y \in X$. Further, we have that for any $x, y, z \in X$,

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle. \tag{2.3}$$

In particular, it is easy to see that if X is strictly convex, for $x, y \in X$, $\phi(y, x) = 0$ if and only if $y = x$ (see, for example, Remark 2.1 of [18]).

Let X be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed convex subset of X. Then, for any $x \in X$, there exists a unique element $\tilde{x} \in C$ such that

$$\phi(\tilde{x}, x) = \inf_{z \in C} \phi(z, x).$$

Then a mapping $\prod_C : X \to C$ defined by $\prod_C x = \tilde{x}$ is called the *generalized projection* (see [1], [2], [12]). In Hilbert spaces, notice that the generalized projection is clearly coincident with the metric projection.

The following result is well known (see, for example, [1], [2], [12]).

Proposition 2.1 ([1], [2], [12]). *Let K be a nonempty closed convex subset of a real Banach space X and let $x \in X$.*

(a) *If X is smooth, then, $\tilde{x} = \prod_K x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x} \rangle \geq 0$ for $y \in K$.*

(b) *If X is reflexive, strictly convex and smooth, then $\phi(y, \prod_K x) + \phi(\prod_K x, x) \leq \phi(y, x)$ for all $y \in K$.*

Lemma 2.2. *Let X be a smooth Banach space. Then, for any fixed $x \in X$, $\phi(\cdot, x)$ is weakly lower semicontinuous on X; moreover, it is continuous and convex on X.*

Proof. Fix $x \in X$ and let $x_n \rightharpoonup p \in X$. Clearly, $\langle x_n, Jx \rangle \to \langle p, Jx \rangle$, and using the weakly lower semicontinuity of the norm, we have

$$\begin{aligned} \phi(p, x) &= \|p\|^2 - 2\langle p, Jx \rangle + \|x\|^2 \\ &\leq \liminf_{n\to\infty} \left(\|x_n\|^2 - 2\langle x_n, Jx \rangle + \|x\|^2 \right) \\ &= \liminf_{n\to\infty} \phi(x_n, x). \end{aligned}$$

Hence $\phi(\cdot, x)$ is weakly lower semicontinuous on X. Obviously, the continuity and convexity of the function $\phi(\cdot, x)$ follow from the continuity and convexity of $\|\cdot\|^2$ and the linearity of Jx. □

Motivated by Lemmas 1.3 and 1.5 of Martinez-Yanes and Xu [17] in Hilbert spaces, we present the following two lemmas.

Lemma 2.3. *Let C be a nonempty closed convex subset of a smooth Banach space X, $x, y, z \in X$ and $\lambda \in [0, 1]$. Given also a real number $a \in \mathbb{R}$, the set*

$$D := \{v \in C : \phi(v, z) \leq \lambda\phi(v, x) + (1 - \lambda)\phi(v, y) + a\}$$

is closed and convex.

Proof. The closedness of D is obvious from the continuity of $\phi(\cdot, x)$ for $x \in X$. Now we show that D is convex. As a matter of fact, the defining inequality in D is equivalent to the inequality

$$\langle v, \lambda Jx + (1 - \lambda)Jy - Jz \rangle \leq \frac{1}{2}(\lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \|z\|^2 + a).$$

This inequality is affine in v and hence the set D is convex. □

Lemma 2.4. *Let X be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property, and let K be a nonempty closed convex subset of X. Let $x_0 \in X$ and $q := \prod_K x_0$, where $\prod_K$ denotes the generalized projection from X onto K. If $\{x_n\}$ is a sequence in X such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\phi(x_n, x_0) \leq \phi(q, x_0) \tag{2.4}$$

for all n. Then $x_n \to q = \prod_K x_0$.

Proof. By (2.4), $\{\phi(x_n, x_0)\}$ is bounded and, by (2.2), $\{x_n\}$ is bounded; so $\omega_w(x_n) \neq \emptyset$ by reflexivity of X. Since $\phi(\cdot, x_0)$ is weakly lower semicontinuous on X by Lemma 2.2, and, by using (2.4) again, we get $\phi(v, x_0) \le \phi(q, x_0)$ for all $v \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset K$ and $q = Q_K x_0$, we must have $v = q$ for all $v \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{q\}$ and $x_n \rightharpoonup q$. On the other hand, using the weakly lower semicontinuity of $\phi(\cdot, x_0)$ again, we have

$$\begin{aligned} \phi(q, x_0) &\le \liminf_{n\to\infty} \phi(x_n, x_0) \\ &\le \limsup_{n\to\infty} \phi(x_n, x_0) \\ &\le \phi(q, x_0) \qquad \text{by (2.4)} \end{aligned}$$

and so $\lim_{n\to\infty} \phi(x_n, x_0) = \phi(q, x_0)$. This implies $\lim_{n\to\infty} \|x_n\| = \|q\|$. By the Kadec-Klee property of X, we have $x_n \to q$.

Lemma 2.5 ([28]). *Let X be a uniformly convex Banach space and let $B_r = \{x \in X : \|x\| \le r\}$ be a closed ball with radius $r > 0$ in X. Then there is a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|) \tag{5}$$

for all x, $y \in B_r$ and $\alpha \in [0, 1]$.

Recently, Kamimura and Takahashi [12] proved the following result, which plays a crucial role in our discussion.

Proposition 2.6 ([12]). *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}, \{z_n\}$ be two sequences of X. If $\phi(x_n, z_n) \to 0$ and either $\{x_n\}$ or $\{z_n\}$ is bounded, then $x_n - z_n \to 0$.*

Here we give the following converse of Proposition 2.6.

Proposition 2.7. *Let X be a smooth Banach space and let $\{x_n\}, \{z_n\}$ be two sequences in X. If $x_n - z_n \to 0$ and either $\{x_n\}$ or $\{z_n\}$ is bounded, then $\phi(x_n, z_n) \to 0$.*

Proof. Since $x_n - z_n \to 0$, it is not hard to see that if either $\{x_n\}$ or $\{z_n\}$ is bounded, then the other is also bounded. Now let $x \in X$ be fixed. Then noticing that

$$\begin{aligned} |\phi(x_n, x) - \phi(z_n, x)| &= |\, \|x_n\|^2 - \|z_n\|^2 + 2\langle z_n - x_n, Jx\rangle \,| \\ &\le |\|x_n\| - \|z_n\||(\|x_n\| + \|z_n\|) + 2\|z_n - x_n\|\,\|x\| \\ &\le \|x_n - z_n\|(\|x_n\| + \|z_n\| + 2\|x\|) \to 0 \end{aligned}$$

and using the identity equation (2.3), we have

$$\begin{aligned} \phi(x_n, z_n) &= \phi(x_n, x) - \phi(z_n, x) + 2\langle x_n - z_n, Jx - Jz_n\rangle \\ &\le |\,\phi(x_n, x) - \phi(z_n, x)\,| + 2\|x_n - z_n\|(\|x\| + \|z_n\|) \to 0 \end{aligned}$$

and the proof is complete. □

Now combining Proposition 2.6 with Proposition 2.7 gives the following equivalent form in uniformly convex and smooth Banach spaces. This property will be frequently used for proving our main result.

Proposition 2.8. *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}, \{z_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{z_n\}$ is bounded, then $\phi(x_n, z_n) \to 0$ if and only if $x_n - z_n \to 0$.*

As an easy observation of Proposition 2.8, we obtain the following result.

Proposition 2.9. *Let C be a closed convex subset of a uniformly convex and smooth Banach space X and $T : C \to C$ be a relatively nonexpansive mapping. Then T is continuous on $F(T)$.*

Proof. Let $p \in F(T)$ and let $x_n \to p$. To claim that $Tx_n \to p$, by Proposition 2.8, it suffices to show that $\phi(p, Tx_n) \to 0$. Indeed, since J is norm-to-weak* continuous, $Jx_n \overset{*}{\rightharpoonup} Jp$; in particular, $\langle p, Jx_n\rangle \to \langle p, Jp\rangle$. Hence

$$\phi(p, x_n) = \|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2 \to \|p\|^2 - 2\langle p, Jp\rangle + \|p\|^2 = 0.$$

Now using the relative nonexpansivity of T, we get

$$\phi(p, Tx_n) \leq \phi(p, x_n) \to 0.$$ □

Next consider the relationship between the Kadec-Klee property and the following weak property which is motivated by Proposition 2.8:

(KT) *Given a sequence $\{x_n\}$ in a smooth Banach space X and $x(\neq 0) \in X$,* $\phi(x_n, x) \to 0$ if and only if $x_n \to x$.

Here, we prove that the property (KT) is equivalent to the Kadec-Klee property in a reflexive, strictly convex and smooth Banach space.

Proposition 2.10. *Let X be a smooth Banach space. Then,*

(a) $(KT) \;\Rightarrow\; (Kadec-Klee)$.

(b) *if X is reflexive and strictly convex,* $(Kadec-Klee) \;\Rightarrow\; (KT)$.

Proof. (a) Let $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Assume without loss of generality that $x \neq 0$. Then, we have

$$\phi(x_n, x) = \|x_n\|^2 - 2\langle x_n, Jx\rangle + \|x_n\|^2 \to \|x\|^2 - 2\langle x, Jx\rangle + \|x\|^2 = 0.$$

From (KT), it follows that $x_n \to x$. Hence X satisfies the Kadec-Klee property.

(b) Let $x\,(\neq 0) \in X$. Then, by virtue of Proposition 2.7, it suffices to show that if $\phi(x_n, x) \to 0$, then $x_n \to x$. Now let $\phi(x_n, x) \to 0$. Clearly, $\{\phi(x_n, x)\}$ is bounded; by (2.2), $\{x_n\}$ is bounded and so $\omega_w(x_n) \neq \emptyset$. Now if $x_{n_k} \rightharpoonup v \in \omega_w(x_n)$, then, since $\phi(\cdot, x)$ is weakly lower semicontinuous by Lemma 2.2,

$$\phi(v, x) \leq \liminf_{k\to\infty} \phi(x_{n_k}, x) = \lim_{k\to\infty} \phi(x_{n_k}, x) = 0,$$

which says that $\phi(v, x) = 0$. By strict convexity of X, we have $v = x$ for all $v \in \omega_w(x_n)$. Therefore, $\omega_w(x_n) = \{x\}$; so $x_n \rightharpoonup x$. On the other hand, since

$$(\|x_n\| - \|x\|)^2 \leq \phi(x_n, x) \to 0,$$

we have $\|x_n\| \to \|x\|$. By the Kadec-Klee property, we conclude that $x_n \to x$. □

3. Strong convergence theorems

In this section we first propose a modification of Ishikawa's iteration process (1.3), motivated by the idea due to [17], [18], to prove strong convergence for a pair of relatively nonexpansive mappings in a Banach space.

Theorem 3.1. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X. Let $\{T_1, T_2 : C \to C\}$ be a pair of relatively nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n JT_2 z_n + (1-\alpha_n) JT_1 x_n), \\ z_n = \beta_n x_n + (1-\beta_n) e_n, \\ H_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, z_n) + (1-\alpha_n)\phi(v, x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0\rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases}$$

where J is the normalized duality mapping on X and $\{e_n\}$ is a bounded sequence in C. If T_2 is uniformly continuous on C, then $x_n \to \prod_F x_0$.

Proof. We employ the methods of the proofs in [18] and [17]. First, observe that H_n is closed and convex by Lemma 2.3, and that W_n is obviously closed and convex for each $n \geq 0$. Next we show that $F \subset H_n$ for all n. Indeed, for all $p \in F$, we have, using convexity of $\|\cdot\|^2$ and relative nonexpansivity of T_i, $i = 1, 2$ (noticing that $z_n \in C$),

$$\begin{aligned} (3.1)\quad \phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n JT_2 z_n + (1-\alpha_n) JT_1 x_n)) \\ &= \|p\|^2 - 2\langle p, \alpha_n JT_2 z_n + (1-\alpha_n) JT_1 x_n\rangle + \|\alpha_n JT_2 z_n + (1-\alpha_n) JT_1 x_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n\langle p, JT_2 z_n\rangle - 2(1-\alpha_n)\langle p, JT_1 x_n\rangle + \alpha_n\|T_2 z_n\|^2 + (1-\alpha_n)\|T_1 x_n\|^2 \\ &= \alpha_n \phi(p, T_2 z_n) + (1-\alpha_n)\phi(p, T_1 x_n) \\ &\leq \alpha_n \phi(p, z_n) + (1-\alpha_n)\phi(p, x_n). \end{aligned}$$

So $p \in H_n$ for all n. Moreover, we show that

$$F \subset H_n \cap W_n \tag{3.2}$$

for all $n \geq 0$. It suffices to show that $F \subset W_n$ for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $F \subset C = W_0$. Assume that $F \subset W_k$ for some $k \geq 1$. Since x_{k+1} is the generalized projection of x_0 onto $H_k \cap W_k$, by Proposition 2.1 (a) we have

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1} \rangle \geq 0$$

for all $z \in H_k \cap W_k$. As $F \subset H_k \cap W_k$, the last inequality holds, in particular, for all $z \in F$. This together with the definition of W_{k+1} implies that $F \subset W_{k+1}$. Hence (3.2) holds for all $n \geq 0$. So, $\{x_n\}$ is well defined. Obviously, since $x_n = \prod_{W_n} x_0$ by the definition of W_n and Proposition 2.1 (a), and since $F \subset W_n$, it follows from the definition of $\prod_{W_n}$ that $\phi(x_n, x_0) \leq \phi(p, x_0)$ for all $p \in F$. In particular, we obtain that for all $n \geq 0$,

$$\phi(x_n, x_0) \leq \phi(q, x_0), \qquad \text{where} \quad q := \textstyle\prod_F x_0. \tag{3.3}$$

Therefore, $\{\phi(x_n, x_0)\}$ is bounded; so is $\{x_n\}$ by (2.2). Since $\{e_n\}$ is bounded, $\{z_n\}$ is also bounded. Noticing that $\phi(p, T_i x_n) \leq \phi(p, x_n)$ for all $p \in F(T_i)$, $\{T_i x_n\}$ is also bounded for $i = 1, 2$.

Now we show that

$$\|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

Indeed, by the definition of W_n and Proposition 2.1 (a), we have $x_n = \prod_{W_n} x_0$ which together with the fact that $x_{n+1} \in H_n \cap W_n \subset W_n$ implies that

$$\phi(x_n, x_0) = \min_{z \in W_n} \phi(z, x_0) \leq \phi(x_{n+1}, x_0),$$

which shows that the sequence $\{\phi(x_n, x_0)\}$ is nondecreasing and so the $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. Simultaneously, from Proposition 2.1 (b), we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi\left(x_{n+1}, \textstyle\prod_{W_n} x_0\right) \leq \phi(x_{n+1}, x_0) - \phi(\textstyle\prod_{W_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \to 0. \end{aligned} \tag{3.5}$$

Hence, (3.4) is satisfied from Proposition 2.8.

Since $\beta_n \to 1$, and $\{x_n\}, \{e_n\}$ are bounded, we have

$$\|x_n - z_n\| = (1 - \beta_n)\|x_n - e_n\| \to 0. \tag{3.6}$$

Combining with (3.4) gives $\|x_{n+1} - z_n\| \to 0$, which is equivalent to $\phi(x_{n+1}, z_n) \to 0$ by Proposition 2.8. Now since $x_{n+1} \in H_n$, we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, z_n) + (1 - \alpha_n)\phi(x_{n+1}, x_n) \to 0,$$

hence $\phi(x_{n+1}, y_n) \to 0$. Using Proposition 2.8 again, we obtain $\|x_{n+1} - y_n\| \to 0$. This, together with (3.4), implies that $\|x_n - y_n\| \to 0$ and also $\|z_n - y_n\| \to 0$.

Next, we show that $\|x_n - T_i x_n\| \to 0$ for all $i = 1, 2$. Since $\{T_1 x_n\}$ and $\{T_2 z_n\}$ are bounded, there exists $r > 0$ such that $\{T_1 x_n\} \cup \{T_2 z_n\} \subset B_r$. Applying for Lemma 2.5 yields

$$\begin{aligned} (3.7) \qquad & \|\alpha_n JT_2 z_n + (1-\alpha_n) JT_1 x_n\|^2 \\ \leq \quad & \alpha_n \|T_2 z_n\|^2 + (1-\alpha_n)\|T_1 x_n\|^2 - \alpha_n(1-\alpha_n) g(\|JT_2 z_n - JT_1 x_n\|), \end{aligned}$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$. Using (3.7) instead of convexity of $\|\cdot\|^2$ in (3.1), we have

$$\phi(p, y_n) \leq \alpha_n \phi(p, z_n) + (1-\alpha_n)\phi(p, x_n) - \alpha_n(1-\alpha_n) g(\|JT_2 z_n - JT_1 x_n\|)$$

and so

$$\begin{aligned} (3.8) \qquad & \alpha_n(1-\alpha_n) g(\|JT_2 z_n - JT_1 x_n\|) \\ \leq \quad & \alpha_n \phi(p, z_n) + (1-\alpha_n)\phi(p, x_n) - \phi(p, y_n). \end{aligned}$$

Notice that, for $p \in F$, using (2.3) repeatedly,

$$\begin{aligned} (3.9) \qquad \phi(p, y_n) &= \phi(p, z_n) + \phi(z_n, y_n) + 2\langle p - z_n, Jz_n - Jy_n \rangle, \\ &= \phi(p, z_n) + c_n \end{aligned}$$

and

$$\begin{aligned} (3.10) \qquad \phi(p, y_n) &= \phi(p, x_n) + \phi(x_n, y_n) + 2\langle p - x_n, Jx_n - Jy_n \rangle \\ &= \phi(p, x_n) + d_n, \end{aligned}$$

where $c_n := \phi(z_n, y_n) + 2\langle p - z_n, Jz_n - Jy_n \rangle \to 0$ and $d_n = \phi(x_n, y_n) + 2\langle p - x_n, Jx_n - Jy_n \rangle \to 0$ from Proposition 2.8. After multiplying α_n and $1 - \alpha_n$ in (3.9) and (3.10), respectively, summing both sides yields

$$\phi(p, y_n) = \alpha_n \phi(p, z_n) + (1-\alpha_n)\phi(p, x_n) + \alpha_n c_n + (1-\alpha_n) d_n.$$

Since $c_n,\ d_n \to 0$, we obtain

$$\alpha_n \phi(p, z_n) + (1-\alpha_n)\phi(p, x_n) - \phi(p, y_n) \to 0.$$

Then it follows from (3.8), together with $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, that

$$\lim_{n\to\infty} g(\|JT_2 z_n - JT_1 x_n\|) = 0.$$

Since g is continuous, strictly increasing and $g(0) = 0$, $\lim_{n\to\infty} \|JT_2 z_n - JT_1 x_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\|T_2 z_n - T_1 x_n\| \to 0.$$

Immediately, using convexity of $\|\cdot\|^2$ and Proposition 2.8 again, we have

$$\begin{aligned}\phi(T_1x_n, y_n) &= \|T_1x_n\|^2 - 2\langle T_1x_n, \alpha_n JT_2z_n + (1-\alpha_n)JT_1x_n\rangle \\ &\quad +\|\alpha_n JT_2z_n + (1-\alpha_n)JT_1x_n\|^2 \\ &\leq \alpha_n\phi(T_1x_n, T_2z_n) \to 0.\end{aligned}$$

Using Proposition 2.8 once more gives $\|T_1x_n - y_n\| \to 0$, this combined with $\|y_n - x_n\| \to 0$ implies

$$\|T_1x_n - x_n\| \to 0. \tag{3.11}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\|Jx_n - Jy_n\| \to 0, \quad \|JT_1x_n - Jx_n\| \to 0. \tag{3.12}$$

On the other hand, notice that

$$\begin{aligned}Jx_n - Jy_n &= Jx_n - (\alpha_n JT_2z_n + (1-\alpha_n)JT_1x_n) \\ &= \alpha_n(Jx_n - JT_2z_n) + (1-\alpha_n)(Jx_n - JT_1x_n)\end{aligned} \tag{3.13}$$

from the definition of y_n. Then using (3.12) and $\liminf_{n\to\infty}\alpha_n > 0$ yield

$$\begin{aligned}\|Jx_n - JT_2z_n\| &= \frac{1}{\alpha_n}\|(Jx_n - Jy_n) + (1-\alpha_n)(JT_1x_n - Jx_n)\| \\ &\leq \frac{1}{\alpha_n}(\|Jx_n - Jy_n\| + (1-\alpha_n)\|JT_1x_n - Jx_n\|) \to 0.\end{aligned}$$

Again, since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\|x_n - T_2z_n\| \to 0.$$

Since $\|z_n - x_n\| \to 0$ and T_2 is uniformly continuous, this yields

$$\|x_n - T_2x_n\| \leq \|x_n - T_2z_n\| + \|T_2z_n - T_2x_n\| \to 0. \tag{3.14}$$

With the help of (3.11) and (3.14), we have $\omega_w(x_n) \subset \hat{F}(T_1) \cap \hat{F}(T_2) = F(T_1) \cap F(T_2) = F$. Joining with (3.3) and Lemma 2.4 (with $K := F$), we conclude that $x_n \to q = \prod_F x_0$. □

Remark 3.2. Note that if $T_2 = I$, the processes of (3.7)-(3.11) are abundant. Also, the parameter assumption $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ in Theorem 3.1 can be weaken with $\limsup_{n\to\infty}\alpha_n < 1$ as readily seen in (3.13) to get $\|x_n - T_1x_n\| \to 0$.

Taking $\beta_n = 1$ for $n \geq 1$ in Theorem 3.1, we have the following modification of Mann's iteration process (1.2) to prove strong convergence for a pair of relatively nonexpansive mappings in a Banach space.

Theorem 3.3. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X. Let $\{T_1, T_2 : C \to C\}$ be a pair of relatively nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n JT_2x_n + (1-\alpha_n)JT_1x_n), \\ H_n = \{v \in C : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0\rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0. \end{cases}$$

If either T_1 or T_2 is uniformly continuous on C, then $x_n \to \prod_F x_0$.

Now taking $T_2 = I$, the identity operator of X and $T_1 = T$ in Theorem 3.3, since the control condition of $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ can be replaced with $\limsup_{n\to\infty} \alpha_n < 1$ by Remark 3.2, we have the following result due to Matsushita and Takahashi [18].

Corollary 3.4 ([18]). *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty bounded closed convex subset of X and let $T : C \to C$ be a relatively nonexpansive mapping with $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then the sequence $\{x_n\}$ generated by the algorithm (1.6) converges in norm to $\prod_{F(T)} x_0$.*

In Hilbert spaces, noticing that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$, we see that $\|Tx - Ty\| \leq \|x-y\|$ is equivalent to $\phi(Tx, Ty) \leq \phi(x, y)$. Also, the demiclosedness principle of a nonexpansive mapping T yields that $\hat{F}(T) = F(T)$. Therefore, every nonexpansive mapping is relatively nonexpansive (for more details, see the proof of Theorem 4.1 in [18]). Now we have the following two variants of Theorem 3.1 and 3.2 for a pair of nonexpansive mappings in Hilbert spaces.

Theorem 3.5. *Let C be a closed convex subset of a Hilbert space H and let $\{T_1, T_2 : C \to C\}$ be a pair of nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n\ (1-\alpha_n) > 0$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n T_2 z_n + (1-\alpha_n)T_1x_n, \\ z_n = \beta_n x_n + (1-\beta_n)e_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|z_n - v\|^2\} \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Then the sequence $\{x_n\}$ converges in norm to $P_F x_0$.

Theorem 3.6. *Let C be a closed convex subset of a Hilbert space H and let $\{T_1, T_2 : C \to C\}$ be a pair of nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n T_2 x_n + (1-\alpha_n) T_1 x_n, \\ C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\} \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

Then the sequence $\{x_n\}$ converges in norm to $P_F x_0$.

As recalling Remark 3.2 again, taking $T_2 = I$, $T_1 = T$ and the term $e_n = Tx_n$ for $n \geq 1$ in Theorem 3.5, and taking $T_2 = I$ and $T_1 = T$ in Theorem 3.6, respectively, we obtain the following subsequent results due to Martinez-Yanez and Xu [17] and Nakajo and Takahashi [19], respectively.

Corollary 3.7 ([17]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. Then the sequence $\{x_n\}$ defined by the algorithm (1.5) converges in norm to $P_{F(T)} x_0$.*

Corollary 3.8 ([19]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then the sequence $\{x_n\}$ defined by the algorithm (1.4) converges in norm to $P_{F(T)} x_0$.*

Now we propose another modification of Ishikawa's iteration process (1.3) to have strong convergence for a pair of relatively nonexpansive mappings defined on a Banach space.

Theorem 3.9. *Let X be a uniformly convex and uniformly smooth Banach space, and let $\{T_1, T_2 : X \to X\}$ be a pair of relatively nonexpansive mappings with $F := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that T_2 is uniformly continuous and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ by the algorithm:*

$$\begin{cases} x_0 \in X \text{ chosen arbitrarily,} \\ z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J e_n), \\ y_n = J^{-1}(\alpha_n J T_2 z_n + (1-\alpha_n) J T_1 x_n), \\ H_n = \{v \in X : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ W_n = \{v \in X : \langle x_n - v, J x_n - J x_0 \rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in X. Then $\{x_n\}$ converges in norm to $\prod_F x_0$.

Proof. Use the following (3.15)-(3.17) to prove $\|x_n - z_n\| \to 0$ of (3.6) in the proof of Theorem 3.1. Since $x_{n+1} \in H_n$, we have

$$\phi(x_{n+1}, y_n) \le \alpha_n \phi(x_{n+1}, x_n) + (1-\alpha_n)\phi(x_{n+1}, z_n). \tag{3.15}$$

However, using the convexity of $\|\cdot\|^2$ for the first inequality, and $\beta_n \to 1$, $\phi(x_{n+1}, x_n) \to 0$ and the boundedness of $\{x_n\}$ and $\{e_n\}$, we get

$$\begin{aligned} \phi(x_{n+1}, z_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n Jx_n + (1-\beta_n)Je_n\rangle \\ &\quad + \|\beta_n Jx_n + (1-\beta_n)Je_n\|^2 \\ &\le \|x_{n+1}\|^2 - 2\beta_n\langle x_{n+1}, Jx_n\rangle - 2(1-\beta_n)\langle x_{n+1}, Je_n\rangle \\ &\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|e_n\|^2 \\ &= \beta_n\phi(x_{n+1}, x_n) + (1-\beta_n)\phi(x_{n+1}, e_n) \to 0. \end{aligned} \tag{3.16}$$

Therefore, the right hand of (3.15) converges to 0; hence $\phi(x_{n+1}, y_n) \to 0$. Also, from Proposition 2.8, $\phi(x_{n+1}, z_n) \to 0$ implies that $\|x_{n+1} - z_n\| \to 0$, and this, together with (3.4), gives that

$$\|x_n - z_n\| \to 0. \tag{3.17}$$

Now repeating the remaining part of the proof of Theorem 3.1, we can prove that $x_n \to \prod_F x_0$. □

Using Lemma 2.5 and the induction method, we have the following easy observation.

Lemma 3.10. *Let X be a uniformly convex Banach space and let $B_r = \{x \in X : \|x\| \le r\}$ be a closed ball with radius $r > 0$ in X. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\sum_{i=0}^{n} \lambda_i x_i\|^2 \le \sum_{i=0}^{n} \lambda_i \|x_i\|^2 - \lambda_i\lambda_0 g(\|x_i - x_0\|) \tag{3.18}$$

for all n, $1 \le i \le n$, where all $x_i \in B_r$ and $\lambda_i \in [0,1]$ with $\sum_{i=0}^{n} \lambda_i = 1$.

We next prove strong convergence for a finite family of relatively nonexpansive mappings in a Banach space.

Theorem 3.11. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X. Given a positive integer $N \ge 1$, let $\{T_i\}_{i=1}^N$ be a finite family of relatively nonexpansive self-mappings of C with $F := \cap_{i=1}^N F(T_i) \ne \emptyset$. Assume that, for each n, $\{\alpha_n^{(i)}\}$ is a finite sequence in $[0,1]$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and also $\liminf_{n\to\infty} \hat{\alpha}_n > 0$, where $\hat{\alpha}_n = \alpha_n^{(0)} \min\{\alpha_n^{(i)}$:*

$1 \le i \le N\}$. *Define a sequence* $\{x_n\}$ *in* C *by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\sum_{i=0}^N \alpha_n^{(i)} JT_i x_n), \\ H_n = \{v \in C : \phi(v, y_n) \le \phi(v, x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0\rangle \le 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} x_0, \end{cases}$$

where $T_0 = I$ *is the identity operator of* X. *Then* $x_n \to \prod_F x_0$.

Proof. Obviously, H_n and W_n are closed and convex for each $n \ge 0$. Next we show that $F \subset H_n$ for all $n \ge 0$. Indeed, for all $p \in F$, we have, using convexity of $\|\cdot\|^2$ and relative nonexpansivity of T_i, $1 \le i \le N$,

$$\begin{aligned} \phi(p, y_n) &= \phi(p, J^{-1}(\textstyle\sum_{i=0}^N \alpha_n^{(i)} JT_i x_n)) \\ &= \|p\|^2 - 2\langle p, \textstyle\sum_{i=0}^N \alpha_n^{(i)} JT_i x_n\rangle + \|\textstyle\sum_{i=0}^N \alpha_n^{(i)} JT_i x_n\|^2 \\ &\le \textstyle\sum_{i=0}^N \alpha_n^{(i)} [\|p\|^2 - 2\langle p, JT_i x_n\rangle + \|T_i x_n\|^2] \\ &= \textstyle\sum_{i=0}^N \alpha_n^{(i)} \phi(p, T_i x_n) \\ &\le \textstyle\sum_{i=0}^N \alpha_n^{(i)} \phi(p, x_n) = \phi(p, x_n). \end{aligned} \tag{3.19}$$

So $p \in H_n$ for all $n \ge 0$. By mimicking the processes of the proof of Theorem 3.1, we can similarly prove the following properties:

(i) x_n is well defined for all $n \ge 0$.

(ii) $\phi(x_n, x_0) \le \phi(q, x_0)$ for all n, where $q := \prod_F x_0$.

(iii) $\|x_{n+1} - x_n\| \to 0$.

Noticing that $\phi(p, T_i x_n) \le \phi(p, x_n)$ for all $p \in F$, $\{T_i x_n\}$ is also bounded for $1 \le i \le N$. Since $x_{n+1} \in H_n$, we have

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n) \to 0,$$

hence $\phi(x_{n+1}, y_n) \to 0$. Using Proposition 2.8, we obtain $\|x_{n+1} - y_n\| \to 0$. This, together with (iii), implies that $\|x_n - y_n\| \to 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\|Jx_n - Jy_n\| \to 0 \tag{3.20}$$

and also

$$\phi(y_n, x_n) \to 0 \tag{3.21}$$

by virtue of Proposition 2.8.

Now we claim that

$$\|x_n - T_i x_n\| \to 0 \tag{3.22}$$

for $1 \le i \le N$. Since all $\{T_i x_n\}$ are bounded for $0 \le i \le N$, there exists $r > 0$ such that $\{x_n\} \cup \{T_1 x_n\} \cup \cdots \cup \{T_N x_n\} \subset B_r$. Applying for Lemma 3.10 yields

$$\begin{aligned} \textstyle \|\sum_{i=0}^{N} \alpha_n^{(i)} J T_i x_n\|^2 & \\ \le \textstyle \sum_{i=0}^{N} \alpha_n^{(i)} \|T_i x_n\|^2 - \alpha_n^{(i)} \alpha_n^{(0)} g(\|JT_i x_n - Jx_n\|), & \end{aligned} \tag{3.23}$$

for $1 \le i \le N$, where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$. Using (3.23) instead of convexity of $\|\cdot\|^2$ in (3.19), we similarly obtain

$$\phi(p, y_n) \le \phi(p, x_n) - \alpha_n^{(i)} \alpha_n^{(0)} g(\|JT_i x_n - Jx_n\|)$$

for $p \in F$ and $1 \le i \le N$. This with (2.3) yields

$$\begin{aligned} \alpha_n^{(i)} \alpha_n^{(0)} g(\|JT_i x_n - Jx_n\|) &\le \phi(p, x_n) - \phi(p, y_n) \\ &= \phi(y_n, x_n) + 2\langle p - y_n, Jy_n - Jx_n \rangle \end{aligned} \tag{3.24}$$

for $1 \le i \le N$. Using (3.20) and (3.21), we see the right hand of (3.24) converges to zero as $n \to \infty$. Since $\liminf_{n\to\infty} \hat{\alpha}_n > 0$ by assumption, we have

$$g(\|JT_i x_n - Jx_n\|) \to 0$$

for $1 \le i \le N$. Since g is continuous, strictly increasing and $g(0) = 0$, $\lim_{n\to\infty} \|JT_i x_n - Jx_n\| = 0$ for $1 \le i \le N$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\|T_i x_n - x_n\| \to 0$$

for $1 \le i \le N$, which proves (3.22).

It is not hard to derive from (3.22) that $\omega_w(x_n) \subset \cap_{i=1}^{N} \hat{F}(T_i) = F$. After joining this property with (ii), an application of Lemma 2.4 (with $K := F$) ensures that $x_n \to q = \prod_F x_0$. □

Remark 3.12. Note that taking $T_i = T$ for all $1 \le i \le N$ in Theorem 3.11 coincides with the case of taking $T_2 = I$ and $T_1 = T$ in Theorem 3.3.

Finally, we shall give examples of relatively nonexpansive self-mappings which are not nonexpansive. This is motivated by the example in the Hilbert space ℓ^2 of Goebel and Kirk [9].

Example 3.13. Let B denote the unit ball in the space $X = \ell^p$, where $1 < p < \infty$. Obviously, X is uniformly convex and uniformly smooth. Let $T : B \to B$ be defined by

$$Tx = (0, x_1^2, \lambda_2 x_2, \lambda_3 x_3, \cdots)$$

for all $x = (x_1, x_2, x_3, \cdots) \in B$, where $\lambda_n = 1 - \frac{1}{n^2}$ for $n \ge 2$ (hence $\prod_{n=2}^{\infty} \lambda_n = \frac{1}{2}$). Then T is Lipschitzian, i.e., $\|Tx - Ty\| \le 2\|x - y\|$ for all $x, y \in B$. Noticing that, for $x = (x_1, x_2, \cdots) \in B$,

$$T^n x = \Big(\overbrace{0, \cdots, 0}^{n}, \prod_{i=2}^{n} \lambda_i\, x_1^2, \prod_{i=2}^{n+1} \lambda_i\, x_2, \prod_{i=3}^{n+2} \lambda_i\, x_3, \cdots\Big)$$

and also for each $n \geq 2$, since $\prod_{i=2}^{n} \lambda_i = \frac{1}{2}\left(1 + \frac{1}{n}\right)$ and $\prod_{i=k}^{n+k-1} \lambda_i = \left(1 - \frac{1}{k}\right)\left(\frac{n+k}{n+k-1}\right) \uparrow 1$ as $k \to \infty$, we have

$$2\prod_{i=2}^{n} \lambda_i = 1 + \frac{1}{n} \geq \prod_{i=k}^{n+k-1} \lambda_i$$

for all $k \geq 2$. Thus we have $\|T^n x - T^n y\| \leq 2\prod_{i=2}^{n} \lambda_i \|x - y\|$ for all $n \geq 2$. Obviously, since $2\prod_{i=2}^{n} \lambda_i \downarrow 1$, T is asymptotically nonexpansive. On the other hand, since $\|Tx - Ty\| = \frac{3}{4} > \frac{1}{2} = \|x - y\|$ for $x = (1, 0, 0, \cdots)$ and $y = (1/2, 0, 0, \cdots)$, T is not nonexpansive. But T is relatively nonexpansive. Indeed, since $\|Tx\| \leq \|x\|$ for $x \in B$ and $F(T) = \{0\}$, where $0 = (0, 0, \cdots) \in B$, we can see that

$$\phi(0, Tx) = \|Tx\|^2 \leq \|x\|^2 = \phi(0, x)$$

for all $x \in B$. Also, from the demiclosedness principle of the asymptotically nonexpansive mapping T (see Theorem 2 of [27]) it follows immediately that $\hat{F}(T) \subset F(T)$. Since the converse inclusion always holds true, it must be $\hat{F}(T) = F(T)$. Therefore, T is relatively nonexpansive.

Next, consider an example in case $F(T)$ is not singleton set.

Example 3.14. Let $X = \ell^p$, where $2 < p < \infty$, and $C = \{x = (x_1, x_2, \cdots) \in X; 0 \leq x_n \leq 1\}$. Then C is a closed convex subset of X. Note that C is not bounded. Let $T : C \to C$ be defined by

$$Tx = (x_1, 0, x_2^2, \lambda_2 x_3, \lambda_3 x_4, \cdots)$$

for all $x = (x_1, x_2, x_3, \cdots) \in C$, where $\lambda_n = 1 - \frac{1}{n^2}$ for $n \geq 2$ as in Example 3.13. In a similar way to Example 3.13, we see that T is Lipschitzian, asymptotically nonexpansive, but not nonexpansive. Obviously, $F(T) = \{p = (p_1, 0, 0, \cdots) : 0 \leq p_1 \leq 1\}$ and $Jx = \frac{1}{\|x\|^{p-2}}(|x_1|^{p-1}\mathrm{sign}\, x_1, |x_2|^{p-1}\mathrm{sign}\, x_2, \cdots)$ for $x = (x_1, x_2, \cdots) \in X$. Now we claim that T is relatively nonexpansive. Indeed, since $\|Tx\| \leq \|x\|$ for $x \in C$, for $p = (p_1, 0, \cdots) \in F(T)$ and $x = (x_1, x_2, \cdots) \in C$, we have

$$\begin{aligned}\langle p, JTx\rangle &= p_1 x_1^{p-1}/\|Tx\|^{p-2} \\ &\geq p_1 x_1^{p-1}/\|x\|^{p-2} = \langle p, Jx\rangle,\end{aligned}$$

and so

$$\phi(p, Tx) = \|p\|^2 - 2\langle p, JTx\rangle + \|Tx\|^2 \leq \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2 = \phi(p, x).$$

Similarly to the argument of Example 3.13, we have $\hat{F}(T) = F(T)$. Thus, T is relatively nonexpansive.

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