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THE EXACT SOLUTION OF KLEIN-GORDON'S EQUATION BY FORMAL LINEARIZATION METHOD

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Abstract. In this paper we discuss on the formal linearization and exact solution of Klein-Gordon's equation

(1) $$u_{tt} - au_{xx} + bu - cu^3 = 0 \ a, b, c \in R^+$$

So that we know an efficient method for constructing of particular solutions of some nonlinear partial differential equations is introduced.

1. Introduction

Many years ago there was interest in constructing solutions of nonlinear partial differential equations in the form of infinite series.The direct linearization of certain famous integrable nonlinear equations was carried out in[1].The possibility to use such series for some other equations was discussed in [4].Exponential series were used also for investigating nonlinear elliptic equations[10]. In this paper we consider the class of equations and systems containing arbitrary linear differential operators with constant coefficients and arbitrary nonlinear analytic functions of dependent variables and their derivatives up to some finite order in assumption that these equations possess a constant solution.Our method is based on formal linearization of a nonlinear partial differential equation to the system of linear ordinary differential equations,describing some finite-dimensional subspace of the space of solutions of the linearized equation.It allows us to develop a very simple technique of finding the linearizing transformation and to apply the method to nonintegrable equations as well as to integrable ones.Solutions have the form of exponential or Fourier series.

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2. The method of formal linearization

Let us consider equations of the following form

$$\hat{L}(D_t, D_x)u(t,x) = N[u], \tag{2}$$

where

$$\hat{L}(D_t, D_x) = \sum_{k=0}^{K}\sum_{m=0}^{M} l_{km} D_t^k D_x^m \tag{3}$$

is a linear differential operator with constant coefficients and

$$N[u] = N(u, u_1, u_2, \ldots, u_p),\ u_p = \frac{\partial^{p_1+p_2}u}{\partial t^{p_1}\partial x^{p_2}},\ p = (p_1, p_2)$$

is an arbitrary analytic function of u and of its derivatives up to some finite order p. We suppose that Eq.(2)possesses the constant solution.Without loss of generality we assume that

$$N[0] = 0,\ \frac{\partial N[0]}{\partial u} = 0,\ \frac{\partial N[0]}{\partial u_1} = 0, \ldots,\ \frac{\partial N[0]}{\partial u_p} = 0.$$

We consider Eq.(2) in connection with the equation linearized near a zero solution:

$$\hat{L}(D_t, D_x)w(t,x) = 0 \tag{4}$$

Let L be the vector space of solutions of Eq.(4) and $P^N \subset L$ be the N-dimensional subspace with the basis

$$w_i = W_i exp(\alpha_i \xi_i),\ \xi_i = x - s_i t,\ i = 1, ..., N.$$

Here s_i and W_i are some constants.The constants $\alpha_i = \alpha_i(s_i)$ are assumed to satisfy the dispersion relation

$$\hat{L}(-\alpha_i s_i, \alpha_i) = 0.$$

The subspace $P^N = \{\sum_{i=1}^N C_i w_i | C_i = const\}$ is specified by the system of N linear ordinary differential equations

$$\frac{dw_i}{d\xi_i} = \alpha_i w_i, i = 1, ..., N.$$

We use the following notation:

$$w_{(N)}^{\delta} = w_1^{\delta_1} w_2^{\delta_2} \ldots w_N^{\delta_N}$$

$$\delta = (\delta_1, \delta_2, \ldots, \delta_N)$$

$$|\delta| = \Sigma_{i=1}^N \delta_i.$$

It is obvious that the monomials $w^{\delta}_{(N)}$ are the eigenfunctions of the operator(3):

$$\hat{L}(D_t, D_x)w^{\delta}_{(N)} = \lambda_\delta w^{\delta}_{(N)}$$

with the eigenvalues

$$\lambda_\delta = \sum_{k=0}^{K}\sum_{m=0}^{M} l_{km}(-\sum_{i=1}^{N}\alpha_i s_i \delta_i)^k(\sum_{i=1}^{N}\alpha_i\delta_i)^m.$$

Theorem 1. If $\lambda_\delta \neq 0$ for every multiindex δ with positive integer components $\delta_i \in Z_+, i = 1, ..., N$, satisfying the condition $|\delta| \neq 0, 1$, then $Eq.(2)$ possesses solutions connected with solutions form P^N by the formal transformation

$$u = \sum_{n=1}^{\infty} \epsilon^n \phi_n(w_1, w_2, \ldots, w_N), \tag{5}$$

where

$$\phi_n = \sum_{|\delta|=n} (A_n)_\delta w^{\delta}_{(N)} \tag{6}$$

are homogeneous polynomials of degree n in the variables w_i .This transformation is unique(for the first term $\phi_1 \in P^N$ fixed).

Remark 1. Here ε is the grading parameter ,finally we can put $\varepsilon = 1$. The proof of the theorem is constructive. Substituting (5) into (2) ,expanding $N[u]$ into the power series in ε ,and then collecting equal powers of ε , we obtain the determining equations for the functions ϕ_n and show that if $\lambda_\delta \neq 0$,then these equations possess the solution(6) with the coefficients $(A_n)_\delta$ uniquely determined through the coefficients $(A_1)_\delta$ by the recursion relation.Thus, the theorem gives us the method for constructing particular solutions of $Eq.(2)$.

3. The solution of Klein-Gordon's equation

Let us consider the Klein-Gordon's equation

$$\hat{L}(D_t, D_x)u(t, x) = cu^3, \tag{7}$$

$$\hat{L}(D_t, D_x) = D_t^2 - aD_x^2 + b.$$

For simplicity we look for a solution of (7) in the form

$$u = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(w_1, w_2), \tag{8}$$

where

$$w_i = W_i exp[\sqrt{\frac{b}{a - s_i^2}}(x - s_i t)], \ i = 1, 2$$

is the basis of the subspace $P^2 \subset L$ (let s_i and W_i be some real constants). Substituting (8) into (7) and collecting equal powers of ε we obtain the determining equations for the functions ϕ_n as follows

$$\hat{L}\phi_1 = 0, \ \hat{L}\phi_2 = 0, \ \hat{L}\phi_n = c\sum_{k=2}^{n-1}\phi_{n-k}\sum_{l=1}^{k-1}\phi_l\phi_{k-l}, \ n \geq 3.$$

These equations possess the solution

$$\phi_{2p+1} = \sum_{n=0}^{2p+1} A_n^p w_1^n w_2^{2p+1-n}, \phi_{2p+2} = 0, p \geq 0$$

Where, if $p \geq 1, 0 \leq n \leq 2p+1$, then

$$A_n^p = \frac{c}{\lambda_{(n,2p+1-n)}}\sum_{m=0}^{p-1}\sum_{r=0}^{p-m-1}\sum_{k=0}^{2m+1}\sum_{l=0}^{2r+1} A_k^m A_l^r A_{n-k-l}^{p-m-r-1}, \tag{9}$$

If $n < 0$ *or* $n > 2p+1$ then $A_n^p = 0$

$$\lambda_{(n,2p+1-n)} = b[1 - n^2 - (2p+1-n)^2 - 2\frac{a - s_1 s_2}{\sqrt{(a - s_1^2)(a - s_2^2)}}n(2p+1-n)].$$

Here either $A_0^0 = A_1^0 = 1$ (in this case $A_n^p = A_{2p+1-n}^p$) or $A_0^0 = 0, A_1^0 = 1$. If $|s_1| < \sqrt{a}$ and $|s_2| < \sqrt{a}$,then $\lambda_{(n,2p+1-n)} \neq 0$ for every pair $(n, 2p+1)$ with $n, p \in Z^+, p \geq 1, 0 \leq n \leq 2p+1$.
If $A_0^0 = 0$, then by (8) we obtain

$$u = \sum_{p=0}^{\infty}(-\frac{c}{8b})^p(\varepsilon w_1)^{2p+1} = \frac{\varepsilon w_1}{1 + \frac{c}{8b}(\varepsilon w_1)^2} = \frac{2\sqrt{2b}w}{\sqrt{c}(1 + w^2)}$$

where

$$w = \frac{\sqrt{c}\varepsilon w_1}{2\sqrt{2b}}.$$

Thus,in(t,x)-variables we have

$$u = \pm\sqrt{\frac{2b}{c}}sech(\frac{\sqrt{b}(x - st + x_0)}{\sqrt{a - s^2}}).$$

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