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SOME PROPERTIES OF PARALLEL SURFACES IN EUCLIDEAN 3-SPACES

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Abstract. In this paper, we study some properties about the parallel surfaces of ruled surfaces in a Euclidean 3-space. Furthermore, we classify the parallel surfaces of ruled surfaces in a Euclidean 3-space satisfying a linear type and a quadric type with respect to the Gaussian curvature and the mean curvature.

1. Introduction

Let f and g be smooth functions on a surface M in a Euclidean 3-space. The Jacobi function $\Phi(f,g)$ formed with f,g is defined by $\Phi(f,g) = \det\begin{pmatrix} f_s & f_t \\ g_s & g_t \end{pmatrix}$ where $f_s = \frac{\partial f}{\partial s}$ and $f_t = \frac{\partial f}{\partial t}$. In particular, a surface satisfying the Jacobi condition $\Phi(K,H)=0$ with respect to the Gaussian curvature K and the mean curvature H on a surface M is called a *Weingarten surface* or a *W-surface*. The classification of the Weingarten surfaces in a Euclidean space is almost completely open today. These surfaces were introduced by the very Weingarten in the context of the problem of finding all surfaces isometric to a given surface of revolution ([cf. 8]). Also, if a surface satisfies a linear type with respect to K and H, that is, $aK+bH=c$ $(a,b,c\in\mathbb{R})$, then it is said to be a *linear Weingarten surface* and we abbreviate it by LW-surface. The first examples of LW-surfaces are those with constant mean curvature($a=0$) and those with constant Gaussian curvature($b=0$). Although these two kinds of surfaces have been extensively studied in the literature, the classification of LW-surfaces in the general case is almost completely open today. Several geometers ([3,4,7,8,12,13,14,16])

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have studied W-surfaces and LW-surfaces and obtained many interesting results. Recently, N.G. Kim and D.W.Yoon ([6]) studied the ruled surfaces in a Euclidean 3-space satisfying the quadric type with respect to the Gaussian curvature, the mean curvature and the second mean curvature. The second mean curvature is the mean curvature of non-degenernate second fundamental form of a surface. Also, Y.H. Kim and D.W. Yoon ([5]) investigated the ruled surfaces in a Minkowski 3-space satisfying the quadric type with respect to the Gaussian curvature, the mean curvature and the second Gaussian curvature. The second Gaussian curvature is that of the non-degenernate second fundamental form of a surface.

In this paper, we will study the parallel surface of a ruled surface in a Euclidean 3-space $\mathbb{R}^3$ satisfying the conditions

$$a\overline{K} + b\overline{H} = c, \quad b \neq 0, \tag{1.1}$$

$$a\overline{K}^2 + b\overline{K}\,\overline{H} + c\overline{H}^2 = d, \quad b^2 - 4ac > 0, \tag{1.2}$$

where a, b, c, d are constants, and $\overline{K}, \overline{H}$ the Gaussian curvature and the mean curvature of the parallel surface of a ruled surface. If a surface satisfies the equation (1.2), then a surface is said to be $\overline{K}\,\overline{H}$*-quadric.*

On the other hand, the parallel surfaces of a cylindrical ruled surface are ruled surfaces, but the parallel surfaces of a non-cylindrical ruled surface cannot be ruled surfaces ([10]).
Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless stated otherwise.

2. Preliminaries

A surface $\overline{M}$ whose points are at a constant distance along the normal from another surface M is said to be *parallel* to M. So, there are infinite numbers of parallel surfaces because we choose the constant distance along the normal arbitrarily. From the definition it follows that a parallel surface can be regarded as the locus of points which are on the normals to M at a non-zero constant distance λ from M.

First, we obtain the representation of points on $\overline{M}$ using the representations of points on M.
Let $\mathbf{x}$ be the position vector of a point P on M and $\overline{\mathbf{x}}$ be the position vector of a point $\overline{P}$ on the parallel surface $\overline{M}$. Then $\overline{P}$ is at a constant distance λ from P along the normal to the surface M. Therefore the

parametrization for $\overline{M}$ is given by

$$\overline{\mathbf{x}}(s,t) = \mathbf{x}(s,t) + \lambda\mathbf{n}(s,t), \tag{2.1}$$

where λ is a constant scalar and $\mathbf{n}$ is the unit normal vector field on M.

Let I, II, K, H be the first fundamental, the second fundamental form, the Gaussian curvature and the mean curvature of M, respectively, and let $\overline{\mathrm{I}}, \overline{\mathrm{II}}$, $\overline{K}, \overline{H}$ be the corresponding ones for $\overline{M}$. With the parametrization for a parallel surface, the following proposition holds.

Proposition 2.1([cf. 11]). *Let $\overline{M}$ be a parallel surface of a surface M in a Euclidean 3-space. Then we have*

(1) $\overline{\mathrm{I}} = (1-\lambda^2 K)\mathrm{I} - 2\lambda(1-\lambda H)\mathrm{II}$,

(2) $\overline{\mathrm{II}} = \lambda K\mathrm{I} + (1-2\lambda H)\mathrm{II}$,

(3) $\overline{K} = \frac{K}{1-2\lambda H+\lambda^2 K}$,

(4) $\overline{H} = \frac{H-\lambda K}{1-2\lambda H+\lambda^2 K}$.

From Proposition 2.1, differentiating $\overline{K}$ and $\overline{H}$ with respect to s and t respectively, we get

$$\begin{aligned}
(\overline{K})_s &= \frac{1}{(1-2\lambda H+\lambda^2 K)^2}(K_s - 2\lambda K_s H + 2\lambda K H_s),\\
(\overline{K})_t &= \frac{1}{(1-2\lambda H+\lambda^2 K)^2}(K_t - 2\lambda K_t H + 2\lambda K H_t),\\
(\overline{H})_s &= \frac{1}{(1-2\lambda H+\lambda^2 K)^2}(H_s - \lambda^2 K H_s - \lambda K_s + \lambda^2 H K_s),\\
(\overline{H})_t &= \frac{1}{(1-2\lambda H+\lambda^2 K)^2}(H_t - \lambda^2 K H_t - \lambda K_t + \lambda^2 H K_t),
\end{aligned} \tag{2.2}$$

which imply the Jacobian function of the Gaussian curvature $\overline{K}$ and the mean curvature $\overline{H}$ is given by

$$\begin{aligned}
\Phi(\overline{K},\overline{H}) &= \frac{1}{1-2\lambda H+\lambda^2 K}(K_s H_t - K_t H_s)\\
&= \frac{1}{1-2\lambda H+\lambda^2 K}\Phi(K,H).
\end{aligned}$$

From the relationship of the above Jacobian function, we have thus the following theorem.

Theorem 2.2. *Let $\overline{M}$ be a parallel surface of a surface M in a Euclidean 3-space. If $\overline{M}$ is a Weingarten surface if and only if M is a Weingarten surface.*

3. Main Results

In this section, we study the parallel surface $\overline{M}$ of a ruled surface in a Euclidean 3-space $\mathbb{R}^3$ which satisfies a linear Weingarten equation (1.1) and a quadric equation (1.2) with respect to the Gaussian curvature $\overline{K}$ and the mean curvature $\overline{H}$ of the parallel surface $\overline{M}$. It is well known that a cylindrical ruled surface is developable, i.e., the Gaussian curvature K is identically zero. Therefore, from (3) of Proposition 2.1 $\overline{K}$ is identically zero. Thus, non-cylindrical ruled surfaces are meaningful for our study.

Let M be a non-cylindrical ruled surface in $\mathbb{R}^3$. Then the parametrization for M is given by

$$\mathbf{x} = \mathbf{x}(s,t) = \alpha(s) + t\beta(s),$$

where $\langle\beta,\beta\rangle = 1, \langle\beta',\beta'\rangle = 1$ and $\langle\alpha',\beta'\rangle = 0$. In this case α is the striction curve of $\mathbf{x}$, and the parameter s is the arc-length on the spherical curve β. And we have the natural frame $\{\mathbf{x}_s, \mathbf{x}_t\}$ given by $\mathbf{x}_s = \alpha' + t\beta'$ and $\mathbf{x}_t = \beta$. Then, the components of the first fundamental form of M are given by $E = \langle\alpha',\alpha'\rangle + t^2$, $F = \langle\alpha',\beta\rangle$, $G = 1$. We put $D = \sqrt{EG - F^2}$. In terms of the orthonormal frame $\{\beta, \beta', \beta\times\beta'\}$ we obtain

$$\alpha' = F\beta + Q\beta\times\beta', \quad \beta'' = -\beta - J\beta\times\beta', \quad \alpha'\times\beta = Q\beta', \tag{3.1}$$

where $Q = \langle\alpha',\beta\times\beta'\rangle$, $\quad J = \langle\beta'',\beta'\times\beta\rangle$. Thus, we get $D = \sqrt{Q^2+t^2}$, from which the unit normal vector $\mathbf{n}$ of M is written as

$$\mathbf{n} = \frac{1}{D}(\alpha'\times\beta + t\beta'\times\beta) = \frac{1}{D}(Q\beta' - t\beta\times\beta').$$

This leads to the components e, f and g of the second fundamental form of M

$$e = \frac{1}{D}(Q(F+QJ) - Q't + Jt^2), \ f = \frac{Q}{D}, \ g = 0.$$

Therefore, using the data described above, the Gaussian curvature K and the mean curvature H of M are given respectively by

$$K = -\frac{Q^2}{D^4}, \quad H = \frac{1}{2D^3}A, \tag{3.2}$$

where we put $A = Jt^2 - Q't + Q(QJ - F)$. Differentiating K and H with respect to t respectively, we get

$$K_t = \frac{4Q^2}{D^6}t, \quad H_t = \frac{1}{2D^5}B, \tag{3.3}$$

where we put $B = -Jt^3 + 2Q't^2 + Q(-QJ + 3F)t - Q^2Q'$.

Let $\overline{M}$ be a parallel surface of a non-cylindrical ruled surface M in $\mathbb{R}^3$. Then, by the definition of a parallel surface, the parametrization for $\overline{M}$ is given by

$$\overline{\mathbf{x}}(s,t) = \mathbf{x}(s,t) + \lambda \mathbf{n}(s,t).$$

Suppose that a parallel surface $\overline{M}$ in $\mathbb{R}^3$ is a linear Weingarten surface. Then by (1.1), (2.2), (3.2) and (3.3) we have

(3.4)
$$(-8a\lambda Q^2At - 2a\lambda Q^2B + bBD^4 + b\lambda^2Q^2B + 4b\lambda^2Q^2At)^2 = 64(a-b\lambda)^2A^4t^2D^6.$$

From the functions D, A and B the equation (3.4) becomes the polynomial in t whose coefficients are functions of variable s. Then, by the coefficient of the highest order t^{14}, we have $b^2J^2 = 0$, from which $J = 0$ because of $b \neq 0$. Therefore, the functions A and B can be rewritten in the form

$$A = -Q't - QF, \qquad B = 2Q't^2 + 3QFt - Q^2Q'. \tag{3.5}$$

By (3.5) and the coefficient of t^{12} of (3.4), we have $4b^2Q'^2 = 0$, from which $Q' = 0$. In this case the coefficient of t^{10} of (3.4) is given by $9b^2Q^2F^2 = 0$, which implies $F = 0$ because the ruled surface M is non-developable, that is, $Q \neq 0$. Thus, from (3.2) M is minimal, that is, it is a helicoid. On the other hand, the coefficients of t^8, t^6, t^4 and t^2 are given as follows:

$$t^8 : -64Q^{10}(a-b\lambda)^2 = 0, \quad t^6 : -192Q^6(a-b\lambda)^2 = 0,$$
$$t^4 : -192Q^8(a-b\lambda)^2 = 0, \quad t^2 : -64Q^{10}(a-b\lambda)^2 = 0.$$

From which we have $\lambda = \frac{a}{b}$.

Thus, we have

Theorem 3.1. *Let $\overline{M}$ be a parallel surface of non-cylindrical ruled surface M in a Euclidean 3-space. If $\overline{M}$ is a linear Weingarten surface satisfying $a\overline{K} + b\overline{H} = c$ $(a, b \neq 0, c \in \mathbb{R})$. Then $\overline{M}$ is parametrized by*

$$\overline{\mathbf{x}}(s,t) = (s\cos t, s\sin t, ht) + \frac{a}{b\sqrt{t^2+h^2}}(h\sin t, -h\cos t, s), \quad h \neq 0.$$

Next, we consider parallel surfaces of non-cylindrical ruled surfaces satisfying the condition (1.2).

Theorem 3.2. *Let $\overline{M}$ be a parallel surface of non-cylindrical ruled surface M in a Euclidean 3-space and let a, b, c, d be constants such that $b^2 - 4ac > 0$ and $c \neq 0$. If $\overline{M}$ is a $\overline{K}\ \overline{H}$-quadric surface satisfying*

$a\overline{K}^2 + b\overline{K}\,\overline{H} + c\overline{H}^2 = d$. *Then* $\overline{M}$ *is parametrized by*

$$\overline{\mathbf{x}}(s,t) = (s\cos t, s\sin t, ht) + \frac{b \pm \sqrt{b^2 - 4ac}}{2c\sqrt{t^2+h^2}}(h\sin t, -h\cos t, s), \quad h \neq 0.$$

Proof. Let $\overline{M}$ be a parallel surface of non-cylindrical ruled surface $\mathbf{x}(s,t) = \alpha(s) + t\beta(s)$ in $\mathbb{R}^3$. Then the parametrization for $\overline{M}$ is given by

$$\overline{\mathbf{x}}(s,t) = \alpha(s) + t\beta(s) + \lambda\mathbf{n}(s,t),$$

where $\langle\beta,\beta\rangle = 1, \langle\beta',\beta'\rangle = 1$ and $\langle\alpha',\beta'\rangle = 0$.

Suppose that a parallel surface $\overline{M}$ is $\overline{K}\,\overline{H}$-quadric. Then, by using (2.2) the equation (1.2) implies

$$\begin{aligned}(3.6)\qquad &(-16aQ^4tD^2 - 4b\lambda Q^2tA^2D^4 - b\lambda Q^2ABD^4 + 16b\lambda Q^4tD^2 \\ &\quad + cAB + 4c\lambda^2Q^2tA^2D^4 - 16c\lambda^2Q^4tD^2)^2 \\ &= (16a\lambda Q^4tAD^4 + 4a\lambda Q^4BD^4 + 4bQ^2tA - 8b\lambda^2Q^4tAD^4 \\ &\quad - 3b\lambda^2Q^2BD^4 - bQ^2B - 4b\lambda^2Q^4tAD^4 + c\lambda^2Q^2ABD^4 \\ &\quad - 8c\lambda Q^2tA + 2c\lambda Q^2B + 2c\lambda^3Q^4BD^4 + 8c\lambda^3Q^4tAD^4)^2D^2.\end{aligned}$$

From the functions D, A and B the equation (3.6) becomes the polynomial in t whose coefficients are functions of variable s. Then, by the coefficient of the highest order t^{20}, we have $c^2\lambda^2Q^4J^4 = 0$, from which $J = 0$ because $c \neq 0$. In this case, by the coefficient of t^{16} of the equation (3.6), we have $4c^2\lambda^4Q^4Q'^4 = 0$, which implies $Q' = 0$. From $J = Q' = 0$, the coefficient of t^{12} of the equation (3.6) is given by $12c^2\lambda^5Q^9F^3 = 0$, so we have $F = 0$. Thus, the surface M is minimal by (3.2). In this case, we can show that the another coefficients of the equation (3.6) are given as follows:

$$\begin{aligned}&256Q^8(a - b\lambda + c\lambda^2)^2 = 0, \quad 512Q^{10}(a - b\lambda + c\lambda^2)^2 = 0, \\ &256Q^{12}(a - b\lambda + c\lambda^2)^2 = 0,\end{aligned}$$

which imply $a - b\lambda + c\lambda^2 = 0$, that is, $\lambda = \frac{b\pm\sqrt{b^2-4ac}}{2c}$. This completes the proof. □

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Figure 1. Surfaces parallel to a helicoid with $\lambda = 0, 0.5, 1, 2, 4$.

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