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EXTREMAL CASES OF SN-MATRICES

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Abstract. We denote by $\mathcal{Q}(A)$ the set of all real matrices with the same sign pattern as a real matrix A. A matrix A is an SN-*matrix* provided there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$. Some properties of SN-matrices are investigated.

1. Introduction

The *sign* of a real number a is defined by

$$\operatorname{sign}(a) = \begin{cases} -1 & \text{if } a < 0, \\ 0 & \text{if } a = 0, \text{ and} \\ 1 & \text{if } a > 0. \end{cases}$$

The *sign pattern of a real matrix* A is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign. We denote by $\mathcal{Q}(A)$ the set of all real matrices with the same sign pattern as A. The *zero pattern* of a matrix A is the $(0, 1)$ matrix obtained from A by replacing each nonzero entry by 1.

A vector is *mixed* if it has a positive entry and a negative entry. A matrix is *row-mixed* if each of its rows is mixed. A vector is *balanced* if it is the zero vector or is mixed. The notion of a row-balanced matrix is defined analogously. A *signing* is a nonzero, diagonal $(0, 1, -1)$-matrix. A signing is *strict* if each of its diagonal entries is nonzero. A matrix B is *strictly row-mixable* provided there exists a strict signing D such that BD is row-mixed.

Let A be an m by n matrix and b an m by 1 vector. The linear system $Ax = b$ has *signed solutions* provided there exists a collection $\mathcal{S}$ of n by 1 sign patterns such that the set of sign patterns of the solutions to $\widetilde{A}x = \tilde{b}$

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is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. This notion generalizes that of a sign-solvable linear system (see [1] and references therein). The linear system, $Ax = b$, is *sign-solvable* provided each linear system $\widetilde{A}x = \tilde{b}$ ($\widetilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$) has a solution and all solutions have the same sign pattern. Thus, $Ax = b$ is sign-solvable if and only if $Ax = b$ has signed solutions and the set $\mathcal{S}$ has cardinality 1.

A matrix A is an *SN-matrix* provided $Ax = 0$ has signed solutions. Thus, A is an SN-matrix if and only if there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathcal{S}$, for each $\widetilde{A} \in \mathcal{Q}(A)$. An *$L$-matrix* is a matrix, A, with the property that each matrix in $\mathcal{Q}(A)$ has linearly independent rows. A square L-matrix is a *sign-nonsingular*, or SNS-matrix for short. A *totally L-matrix* is an $m \times n$ matrix such that each m by m submatrix is an SNS-matrix. It is known that totally L-matrices are SN-matrices[2].

Some properties of SN-matrices have been studied in [2, 3, 4, 5]. In [6] we proved that if a strictly row-mixable m by n SN-matrix is not conformally contractible, then it is permutation equivalent to

$$\begin{bmatrix} I_k & B \\ O & C \end{bmatrix} \tag{1}$$

where $2 \leq k \leq m$.

In this paper, considering a strictly row-mixable m by n SN-matrix of the form in (1) we find the range of n and characterize the matrices satisfying the extremal cases of n for $k = m$.

We use the following standard notations throughout the paper. If k is a positive integer, then $\langle k \rangle$ denotes the set $\{1, 2, \ldots, k\}$. Let A be an $m \times n$ matrix. If α is a subset of $\{1, 2, \ldots, m\}$ and β is a subset of $\{1, 2, \ldots, n\}$, then $A[\alpha|\beta]$ denotes the submatrix of A determined by the rows whose indices are in α and the columns whose indices are in β. The submatrix complementary to $A[\alpha|\beta]$ is denoted by $A(\alpha|\beta)$. In particular, $A(\alpha|-)$ denotes the submatrix obtained from A by deleting the rows whose indices are in α. Let $J_{m,n}$ denote the m by n matrix all of whose entries are 1 and let $\mathbf{e}_i$ denote the column vector all of whose entries are 0 except for the ith entry which is 1. O denotes a zero matrix.

2. Main Results

Let A be an m by n $(0, 1, -1)$-matrix. The matrix B is *conformally contractible* to A provided there exists an index k such that the rows and columns of B can be permuted so that B is $m+1$ by $n+1$ matrix of the form

$$\left[\begin{array}{c|c|c} A[\langle m\rangle|\langle n\rangle \setminus \{k\}] & x & y \\ \hline 0 \ \cdots \ 0 & 1 & -1 \end{array}\right],$$

where $x = [x_1, \dots, x_m]^T$ and $y = [y_1, \dots, y_m]^T$ are $(0, 1, -1)$ vectors such that $x_i y_i \geq 0$ for $i = 1, 2, \dots, m$, and the sign pattern of $x + y$ is the kth column of A. In this case we say that the zero pattern of A is obtained from the zero pattern of B by a contraction. More precisely, let $A = [a_{ij}]$ be an m by n $(0,1)$-matrix such that the row p of A contains exactly two 1's, say $a_{pr} = a_{ps} = 1$ whenever $r \neq s$. Let u be the m by 1 $(0,1)$ column vector whose ith entry is 1 if and only if $a_{ir} = 1$ or $a_{is} = 1$. Let B the $m-1$ by $n-1$ matrix obtained from A by replacing column s by u and then deleting row p and column r. We say that B is the matrix obtained from A by the contraction of columns r and s on row p. It is known that if B is conformally contractible to A, then A is an SN-matrix if and only if B is an SN-matrix[2].

It is easy to show that if an SN-matrix has two nonzero columns which are identical up to multiplication by -1, then the columns have exactly one nonzero entry. Whenever we consider a matrix A of the form in (1), we may assume that A is a strictly row-mixable SN-matrix which is not conformally contractible to a matrix, and each column of A is distinct. That is, each column of $\begin{bmatrix} B \\ C \end{bmatrix}$ has at least two nonzero entries.

Let $A = (\mathbf{a_1}, \dots, \mathbf{a_s}) = (\mathbf{c_1}, \dots, \mathbf{c_m})^{\mathbf{T}}$ be an m by s matrix and $B = (\mathbf{b_1}, \dots, \mathbf{b_t}) = (\mathbf{d_1}, \dots, \mathbf{d_n})^{\mathbf{T}}$ an n by t matrix. Write $A \Box B$ as

$$\begin{bmatrix} \mathbf{a_1} & \cdots & \mathbf{a_{s-1}} & \mathbf{a_s} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{b_1} & \mathbf{b_2} & \cdots & \mathbf{b_t} \end{bmatrix}$$

and $A \Diamond B$ as

$$\begin{bmatrix} \mathbf{c_1} & \\ \vdots & O \\ \mathbf{c_m} & \mathbf{d_1} \\ O & \vdots \\ & \mathbf{d_n} \end{bmatrix}.$$

Then $A \Box B$ is an $m+n$ by $s+t-1$ matrix and $A \Diamond B$ is an $m+n-1$ by $s+t$ matrix.

Now we want to investigate an m by n $(m < n)$ matrix A of the form in (1) with $k = 2$ or $k = m$. Let $\sigma(A)$ be the number of nonzero entries of A.

Let $k = 2$. Since A is an SN-matrix and every row of A has at least three nonzero entries, $n \geq m + 2$. The equality holds if and only if A is a totally L-matrix(see proposition 2 in [6]). In this case A can be obtained from

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

by a sequence of single extensions up to row and column permutations and multiplication of rows and columns by -1(for definition see p.88 in [1]). What are the maximum value ξ of n and the matrices corresponding to ξ? Let $J_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Let $H = \overbrace{J_2\diamondsuit \cdots \diamondsuit J_2}^{m-1}$ and $A = [\mathbf{e_1}\ \mathbf{H}\ \mathbf{e_m}]$. It is easy to show that A is a matrix of the form in (1) with $k = 2$. Hence $\xi \geq 2m$.

At present we cannot find the value ξ but we conjecture that $\xi = 2m$ for $m \geq 2$ and a matrix corresponding to $\xi = 2m$ is permutation equivalent to the matrix $A = [\mathbf{e_1}\ \mathbf{H}\ \mathbf{e_m}]$ up to multiplication of rows and columns by -1.

Let $\lfloor a \rfloor$ denote the smallest integer no less than a. Let $\mathcal{M}_m$ be the set of all m by $\lfloor \frac{m}{2} \rfloor + 1$ $(0,1)$-matrices defined inductively as follows:

For $m = 2$, let $\mathcal{M}_2 = \{J_{2,2}\}$. For any even number $m(\geq 4)$, $M_m \in \mathcal{M}_m$ if and only if M_m is permutation equivalent to

$$\left[\begin{array}{c|c} M_{m-2} & O \\ \hline \multirow{2}{*}{C} & 1 \\ & 1 \end{array}\right],$$

where $M_{m-2} \in \mathcal{M}_{m-2}$, and C has a column $(1,1)^T$ and other columns are all zero.

For odd number m, $M_m \in \mathcal{M}_m$ if and only if every row of M_m has exactly two nonzero entries and every columns of M_m has at least two nonzero entries, and there exists a row i of M_m such that the contraction of M_m on the row i is contained in $\mathcal{M}_{m-1}$. Thus $\mathcal{M}_3$ is the set of all matrices which are permutation equivalent to

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and $M_m \in \mathcal{M}_m (m \geq 5)$ if and only if M_m is permutation equivalent to one of the following matrices

$$\begin{bmatrix} M' & O \\ \mathbf{a} & \mathbf{b} \\ O & M'' \end{bmatrix} \tag{2}$$

where $M' \in \mathcal{M}_k$, $M'' \in \mathcal{M}_{m-k-1}$ for some even number k and the vectors $\mathbf{a}$ and $\mathbf{b}$ have exactly one nonzero entry respectively, or

$$\left[\begin{array}{c|c|c} M' & O & O \\ \hline & 1 & \\ S & 1 & T \\ & 0 & \\ \hline O & O & M'' \end{array}\right] \tag{3}$$

where $M' \in \mathcal{M}_k$, $M'' \in \mathcal{M}_{m-k-3}$ for some even number k, S and T have columns $(1, 0, 1)^T$ and $(0, 1, 1)^T$ respectively and other columns are zero.

Proposition 2.4 in [6] states that if A is strictly row-mixable m by n SN-matrix with no duplicate columns up to multiplication by -1 and every row has at least three non-zero elements, then A has at least two rows with exactly three nonzero entries. Using this property we can obtain the range of $\sigma(A)$ for a matrix A of the form in (1) with $k = m$ and we can characterize the matrices in the extremal cases of $\sigma(A)$. Let $\mathcal{N}_m$ be the set of all m by $2m - 2$ matrices B with $B^T \in \mathcal{M}_{2m-2}$.

Proposition 1. *Let A be a matrix of the form in (1) with $k = m (m \geq 2)$. Then $3m \leq \sigma(A) \leq 5m - 4$. Moreover, $\sigma(A) = 5m - 4$ if and only if the zero pattern of the matrix obtained from A by deleting the identity submatrix I_m is contained in $\mathcal{N}_m$.*

Proof. Since every row of A has at least three nonzero entries, $3m \leq \sigma(A)$. By the remark mentioned above, we may assume that A is of the form

$$(4)\qquad \left[\begin{array}{c|cccc} 1 & 1 & 1 & 0 & \cdots & 0 \\ \hline 0 & & & & & \\ \vdots & & & B & & \\ 0 & & & & & \end{array}\right].$$

Notice that all columns of $B(-|1,2)$ are distinct. If all columns of B are distinct, then $\sigma(B) \leq 5(m-1)-4$ by induction hypothesis. Hence $\sigma(A) = \sigma(B)+3 \leq 5(m-1)-4+3 < 5m-4$. Let B have duplicate columns up to multiplication by -1. Since such columns have only one nonzero entry, the zero pattern of A is permutation equivalent to one of the following matrices

$$(5)\qquad \left[\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & \cdots & 0 \\ \hline 0 & 1 & & & & \\ 0 & 0 & & & & \\ \vdots & \vdots & & & B' & \\ 0 & 0 & & & & \end{array}\right],$$

$$(6)\qquad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \cdots & 0 \\ \hline 0 & 1 & 1 & & & \\ 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & B' & \\ 0 & 0 & 0 & & & \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & \cdots & 0 \\ \hline 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & B' & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & & & \end{array}\right]$$

where the submatrix B' has all distinct columns. It is easy to show that $\sigma(B') \leq 5(m-1)-4$. Since $\sigma(B) \leq \sigma(B')+2$, $\sigma(A) = \sigma(B)+3 \leq \sigma(B')+2+3 \leq 5(m-1)-4+5 = 5m-4$.

Let A be a matrix of the form in (1) with $\sigma(A) = 5m-4$. If the zero pattern of A is of the form in (5), then $\sigma(B') = 5m-8 > 5(m-1)-4$, which is impossible. Hence the zero pattern of A should be one of the matrices in (6). Consider a matrix A of the second form in (6). Suppose that a matrix of the second form in (6) is the zero pattern of A. If the number of nonzero entries in the first row or the second row of B' is 2, it easy to show that $\sigma(B') < 5(m-1)-4$ and hence $\sigma(A) < 5m-4$. Thus each row of B' has at least three nonzero entries. Therefore the matrix obtained from B' by deleting the columns corresponding to the identity submatrix I_{m-1} is contained in $\mathcal{N}_{m-1}$ by induction hypothesis.

Then $J_{2,3}$ is the zero pattern of a submatrix of the conformal contraction on the first row of $A(-|1)$. By Theorem B in [6] A is not SN-matrix, which is a contradiction. Next consider a matrix of the first form in (6). If the number of nonzero entries in the first row of B' is 1 or 2, it is also easy to show that $\sigma(B') < 5(m-1)-4$. Thus each row of B' has at least three nonzero entries. Since $\sigma(B') = 5(m-1)-4$, the matrix obtained from B' by deleting the columns corresponding to the identity submatrix I_{m-1} is contained in $\mathcal{N}_{m-1}$ by induction hypothesis. Clearly the matrix obtained from the zero pattern of A by deleting the columns corresponding to I_m is contained in $\mathcal{N}_m$. □

Lemma 2. *Let A be an m by n (0,1)-matrix such that each row of A has exactly two nonzero entries ($m \geq 3$). If $n \leq \lfloor \frac{m}{2} \rfloor$, then there exists a matrix B obtained from A by a finite sequence of contractions on rows such that $J_{3,2}$ is a submatrix of B.*

Proof. We will prove it by induction on m. For $m = 3, 4$, there is nothing to prove it. Let $m \geq 5$. If A has $J_{2,2}$ as a submatrix, we may assume that A is of the form

$$A = [a_{ij}] = \begin{bmatrix} A_1 & A_2 \\ O & J_{2,2} \end{bmatrix}.$$

Suppose that $J_{3,2}$ is not a submatrix of A. Then $a_{i,n-1} \cdot a_{in} = 0$ for all $i \in \{1, 2, \cdots, m-2\}$. Let B be the contraction of A on the row m and let B' be the matrix obtained from B by deleting the last row. Then B' is an $m-2$ by $n-1$ matrix such that each row of B' has exactly two nonzero entries. Since $n \leq \lfloor \frac{m}{2} \rfloor$, $n-1 \leq \lfloor \frac{m}{2} \rfloor - 1 = \lfloor \frac{m-2}{2} \rfloor$. By induction there exists a matrix C obtained from B' by a finite sequence of contractions such that $J_{3,2}$ is a submatrix of C. Hence $J_{3,2}$ is a submatrix of a matrix obtained from A by a finite sequence of contractions. This is a contradiction.

If A does not have $J_{2,2}$ as a submatrix, we may assume that $A = [a_{ij}]$ is one of the forms

$$\left[\begin{array}{c|cc} A_1 & \multicolumn{2}{c}{O} \\ \hline O & 1 & 1 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|cc} A_1 & \multicolumn{2}{c}{A_2} \\ \hline \multirow{2}{*}{E} & 0 & 1 \\ & 1 & 1 \end{array}\right]$$

where E has only one nonzero entry in the first row and $a_{i,n-1} \cdot a_{in} = 0$ for all $i \in \{1, 2, \cdots, m-2\}$. In the former case, clearly A_1 satisfies the hypothesis. By induction A_1 and hence A satisfies the result. In the latter case, let B be the contraction of A on row m. Then B is an $m-1$

by $n-1$ matrix such that each row of B has exactly two nonzero entries and $n-1 \leq \lfloor \frac{m}{2} \rfloor - 1 = \lfloor \frac{m-2}{2} \rfloor \leq \lfloor \frac{m-1}{2} \rfloor$. By induction, B and hence A satisfies the result. □

Proposition 3. *Let A be an m by n matrix containing I_m as a submatrix such that each row of A has exactly three nonzero entries and $m \geq 2$. If A is an SN-matrix, then $n \geq \lfloor \frac{m}{2} \rfloor + m + 1$.*

Proof. The result is clear for $m = 2$. Let $m \geq 3$. Suppose that $n < \lfloor \frac{m}{2} \rfloor + m + 1$. Without loss of generality we may assume that $A = [I_m \ B]$. Then B is m by $n - m$ matrix such that each row of B has exactly two nonzero entries and $n - m \leq \lfloor \frac{m}{2} \rfloor$. By Lemma 2, there exists a matrix B obtained from A by a finite sequence of contractions on rows such that $J_{3,2}$ is a submatrix of B. Since A is an SN-matrix if and only if a conformal contraction of the matrix A is an SN-matrix, A is not an SN-matrix by Theorem B in [6]. Hence we have the result. □

In the following we can get matrices on which equality in Proposition 3 holds.

Proposition 4. *Let $A = [I_m \ B]$ be an m by n SN-matrix with no duplicate columns up to multiplication by -1 such that each row of A has exactly three nonzero entries, then $n = \lfloor \frac{m}{2} \rfloor + m + 1$ if and only if the zero pattern of B is in $\mathcal{M}_m$.*

Proof. (*Sufficiency.*) It is clear.
(*Necessity.*) We will prove it by induction on m. Since A is an SN-matrix with no duplicate columns up to multiplication by -1, each column of the matrix B has at least two nonzero entries. Let $\mathcal{Z}(X)$ denote the zero pattern of a matrix X. It is easy to show that $\mathcal{Z}(B)$ is $J_{2,2}$ if $m = 2$, and $\mathcal{Z}(B)$ is permutation equivalent to

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

if $m = 3$. Thus we are done for $m = 2,\ 3$. Let $m \geq 4$. First, let $\mathcal{Z}(B)$ have no submatrix of the form $J_{2,2}$. Without loss of generality, we may assume that $\mathcal{Z}(B)$ is of the form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & & & & \\ & & * & * & * & \end{bmatrix}.$$

Let C be the $m-1$ by $\lfloor \frac{m}{2} \rfloor$ matrix obtained from $\mathcal{Z}(B)$ by contraction on the first row. Then each row of C has exactly two nonzero entries. If m is even, the matrix A is not an SN-matrix by Lemma 2. Hence m is odd and by induction hypothesis $C \in \mathcal{M}_{m-1}$. Thus there exist permutation matrices P, Q such that $P\mathcal{Z}(B)Q$ has a submatrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

since $m \geq 5$. This is impossible since A is an SN-matrix.

Next, let $\mathcal{Z}(B)$ have $J_{2,2}$ as a submatrix. Without loss of generality, we may assume that $\mathcal{Z}(B)[1,2|1,2] = J_{2,2}$. Suppose that $B[3,\ldots,m|1,2] = O$. Then $[I_{m-2}\ B(1,2|1,2)]$ is an $m-2$ by $\lfloor \frac{m}{2} \rfloor + m - 3$ matrix each row of which has exactly three nonzero entries. Clearly $[I_{m-2}\ B(1,2|1,2)]$ is an SN-matrix. Hence the total number of columns of $[I_{m-2}\ B(1,2|1,2)]$ is no less than $\lfloor \frac{m}{2} \rfloor + m - 2$ by Proposition 3. This is a contradiction. Thus $B[3,\ldots,m|1,2] \neq O$.

Let $\sigma(B[3,\ldots,m|1,2]) = 1$. Then we may assume that $\mathcal{Z}(B)$ is of the form

$$\left[\begin{array}{cc|cccc} 1 & 1 & & & & \\ 1 & 1 & & \multicolumn{2}{c}{O} & \\ \hline 0 & 1 & 1 & 0 & \cdots & 0 \\ \hline \multicolumn{2}{c|}{O} & & \multicolumn{2}{c}{M} & \end{array}\right].$$

Let D be the matrix obtained from $\mathcal{Z}(B)$ by the contraction on the third row. Then each column of the matrix D has at least two nonzero entries and each row of D has exactly two nonzero entries. Clearly $[I_{m-1}\ D]$ is the zero pattern of an $m-1$ by $\lfloor \frac{m}{2} \rfloor + m - 1$ SN-matrix. By Proposition 3, we have $\lfloor \frac{m}{2} \rfloor \geq \lfloor \frac{m-1}{2} \rceil + 1$. This implies that m is odd and $\lceil \frac{m}{2} \rfloor = \lfloor \frac{m-1}{2} \rfloor + 1$. Hence $D \in \mathcal{M}_{m-1}$ by induction hypothesis and $M \in \mathcal{M}_{m-3}$. Thus $\mathcal{Z}(B) \in \mathcal{M}_m$.

Let $\sigma(B[3,\ldots,m|1,2]) \geq 2$. Let B' be the matrix obtained from B by the conformal contraction on the first row and then by deleting the first row. Then $[I_{m-2}\ B']$ is satisfies the hypothesis. If m is even, then we may assume that $\mathcal{Z}(B)$ is either

$$(7)\qquad \left[\begin{array}{cc|cccc} 1 & 1 & & O & & \\ 1 & 1 & & & & \\ \hline 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \multicolumn{2}{c|}{*} & & * & & \end{array}\right] \text{ or } \left[\begin{array}{cc|cccc} 1 & 1 & & O & & \\ 1 & 1 & & & & \\ \hline 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \multicolumn{2}{c|}{**} & & * & & \end{array}\right].$$

By contracting on the third row in the first matrix, we have $J_{3,2}$ as its submatrix. This implies that A is not an SN-matrix by Theorem B in [6]. It is impossible. Hence $\mathcal{Z}(B)$ is of the form in the second matrix. It is contained in $\mathcal{M}_m$.

Let m be odd. Then $\mathcal{Z}(B^{'})$ is one of matrices in (2) or (3) where $M^{'} \in \mathcal{M}_p$ and $M^{''} \in \mathcal{M}_q$ for some even numbers p, q and one of $M^{'}, M^{''}$ in (3) may be vacuous.

If $\mathcal{Z}(B^{'})$ is a matrix of the form in (2), then $\mathcal{Z}(B)$ has a submatrix which is permutation equivalent to one of the matrices in (7). The first matrix in (7) also does not occur and hence $\mathcal{Z}(B) \in \mathcal{M}_m$. Let $\mathcal{Z}(B^{'})$ be a matrix of the form in (3). If $M^{'}$ is not vacuous, it is easy to show that $\mathcal{Z}(B) \in \mathcal{M}_m$ by the similar method above. If $M^{'}$ is vacuous, then $\mathcal{Z}(B)$ has a submatrix which is permutation equivalent to

$$(8)\qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The first matrix in (8) does not occur and hence $\mathcal{Z}(B) \in \mathcal{M}_m$. Thus we have the result. □

Let

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & 1 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 1 & C_1 \\ C_2 & C_3 \end{bmatrix}.$$

Let $B \star C$ denote

$$\begin{bmatrix} B_1 & B_2 & O \\ B_3 & 1 & C_1 \\ O & C_2 & C_3 \end{bmatrix}.$$

Proposition 5. *Let $m > 1$ be an integer. For any s with $3m \leq s \leq 5m - 4$, there exists an m by n matrix A in (1) with $\sigma(A) = s$ and $k = m$.*

Proof. First of all, we show that there exists such a matrix A with $\sigma(A) = s$ for $4m-2 \leq s \leq 5m-4$. Let $A_t = (\overbrace{J_2 \star \cdots \star J_2}^{t})\diamondsuit(\overbrace{J_2\diamondsuit \cdots \diamondsuit J_2}^{m-t-1})$ where $1 \leq t \leq m-1$. Then $[I_m\ A_t]$ satisfies the conditions in (1) and $\sigma([I_m\ A_t]) = 5m - t - 3$.

Let m be any even integer. If $m = 2$, then $A = [I_2\ J_2]$ is an SN-matrix with $\sigma(A) = s(3m \leq s \leq 4m-2)$. Let $m \geq 4$. Let $B = \overbrace{J_2\Box \cdots \Box J_2}^{m/2}$. Then $A = [I_m\ B]$ is an SN-matrix with $\sigma(A) = 3m$. Consider a column $\mathbf{c}$ of B with at least three nonzero entries. We choose a nonzero entry a of $\mathbf{c}$. Then the matrix B has the unique submatrix J_2 which contains a. Let C be the unique submatrix J_2. Let $\mathbf{d}$ be the column vector obtained from $\mathbf{c}$ by replacing every nonzero entry which is not an entry of C with 0. Let B_1 be the matrix obtained from B by replacing a with 0 and by adding the column $\mathbf{d}$ as the last column. Then the matrix $A_1 = [I_m\ B_1]$ is an SN-matrix with $\sigma(A_1) = 3m+1$. Applying the process above to the matrix B_1, we choose a matrix B_2 which is an SN-matrix with $\sigma(A_2) = 3m+2$. We can continue the process until every column has exactly two nonzero entries. That is, we can find matrices $B_1, B_2, \ldots, B_{m-2}$. Let $A_i = [I_m\ B_i]$ for $i = 1, 2, \ldots, m-2$. Then A_i is is an SN-matrix with $\sigma(A_i) = 3m+i$ for $i = 1, 2, \ldots, m-2$. Thus we have the result.

Let m be any odd integer. First, if $m = 3$, then $A = [I_3\ B_3]$ is an SN-matrix with $\sigma(A) = s(3m \leq s < 4m-2)$ where

$$B_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

For $m \geq 5$, let $B = J_2\Box J_{1,2}\Box \overbrace{J_2\Box \cdots \Box J_2}^{(m-3)/2}$. Then the matrix $A = [I_m\ B]$ is an SN-matrix with $\sigma(A) = 3m$. By the similar method shown in the case of even m, we can find matrices $B_1, B_2, \ldots, B_{m-3}$ such that $A_i = [I_m\ B_i]$ is an SN-matrix with $\sigma(A_i) = 3m+i$ for $i = 1, 2, \ldots, m-3$. Thus we have the result. □

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