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TOTAL SCALAR CURVATURE AND EXISTENCE OF STABLE MINIMAL SURFACES

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Abstract. On a compact n-dimensional manifold M, it has been conjectured that a critical point metric of the total scalar curvature, restricted to the space of metrics with constant scalar curvature of volume 1, should be Einstein. The purpose of the present paper is to prove that a 3−dimensional manifold (M, g) is isometric to a standard sphere if $\ker s_g^* \neq 0$ and there is a lower Ricci curvature bound. We also study the structure of a compact oriented stable minimal surface in M.

1. Introduction

Let M be an n-dimensional compact orientable manifold and $\mathcal{M}_1$ the set of smooth Riemannian structures on M of volume 1. The total scalar curvature $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ is defined by

$$\mathcal{S}(g) = \int_M s_g dv_g$$

for $g \in \mathcal{M}_1$, where s_g is the scalar curvature and dv_g the volume form determined by the metric and orientation. Let $\mathcal{C} = \{g \in \mathcal{M}_1 | \ s_g \text{ constant } \}$. It has been conjectured that the critical points of the total scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{C}$ are Einstein ([1]).

The Euler-Lagrange equation of $\mathcal{S}$ restricted to $\mathcal{C}$ may be written as the following critical point equation(CPE,hereafter):

$$z_g = s_g'^*(f), \tag{1}$$

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where z_g is the traceless Ricci tensor, f is a function on M with vanishing mean value, and

$$s_g'^*(f) = D_g df - g\Delta_g f - f r_g. \tag{2}$$

Here, $D_g df$ is the Hessian of f and r_g the Ricci curvature of g. We define the Laplacian Δ_g of f to be the trace of its Hessian with respect to g. Taking the trace of (1) gives $\Delta f = -\frac{s}{n-1} f$. In particular, if s is non-positive, f is trivial. Therefore from now on, we will assume that s is a positive constant.

There has been partial results concerning about this conjecture. Lafontaine showed that if a solution metric g of CPE is conformally flat, such a metric is Einstein ([8]). The author showed that the Ricci curvature satisfies that

$$r_g > \frac{1}{\mu}\frac{s}{n} \tag{3}$$

for $\mu = \inf_{M^n}(1+f)$, then there is no embedded compact oriented stable minimal hypersurface in M^n ([3]). Here, $\mu < 0$, otherwise g is Einstein. The geometric structure of an Einstein solution was known to be simple due to Obata, who showed that such a solution is isometric to a standard n-sphere ([9]). In particular $\dim \ker s_g'^* = n+1$.

Throughout the present paper, we will assume that $\ker s_g'^* \neq 0$ and $n = 3$, except in Lemma 2.1. With this assumption, the following holds ([7]):

Theorem 1.1. *Let (g, f) be a non-trivial solution of (1) on a 3-dimensional compact oriented manifold M. If $\ker s_g'^* \neq 0$ and $H_2(M, \mathbb{Z}) = 0$, then (M, g) is isometric to a standard sphere S^3.*

It is shown in [6] that $H_2(M, \mathbb{Z}) = 0$ if and only if there exists no embedded compact oriented stable minimal surface in M^3. Thus, in virtue of Theorem 1.1, it may be restated that if there is no compact oriented stable minimal surface in M^3, then M is isometric to a standard sphere S^3.

This paper is motivated by the following questions: can the condition (3) be relaxed to include the equality? The second question is what the structure of a compact oriented stable minimal surface Σ is if Σ exists in M. The following theorem gives affirmative answer to the first question.

Theorem 1.2. *Let (g, f) be a non-trivial solution of (1) on a 3-dimensional compact oriented manifold M. If $\ker s_g'^* \neq 0$ and $r_g \geq \frac{1}{\mu}\frac{s}{3}$, then (M, g) is isometric to a standard sphere S^3.*

It is clearly a good improvement of Theorem 2 of [3] and Theorem 3.1 of [7].

Now we consider the case when such a surface Σ exists. It is known that the value of $(1+f)^3 z(\nu,\nu)$ for a unit normal vector field ν is constant on Σ ([7]). Let $c = (1+f)^3 z(\nu,\nu)_{|\Sigma}$. It is easy to see that there exist $t_0 > 0$ such that

$$t_0 = \inf\{\, t > 0 \,|\, r_g \geq \frac{1}{\mu}\frac{s}{3} t \,\}.$$

If $t_0 < 1$, this condition is just (3). The condition $t_0 = 1$ is also ruled out by Theorem 1.2. Thus the remaining condition to consider is the case $t_0 > 1$. The next result gives a bound of c in terms of t_0 and μ, which gives some perspective in the second question mentioned above.

THEOREM 1.3. *Let (g, f) be a non-trivial solution of (1) on a 3-dimensional compact oriented manifold M with* $\ker s_g'^* \neq 0$. *Let Σ be a compact oriented stable minimal surface in M. Then, $t_0 > 1$ and*

$$0 < c < \frac{s}{3}(t_0 - \mu - 1)^2 (t_0 - \mu). \tag{4}$$

In Section 2, Theorem 1.2 will be proved. We will study the structure of a compact oriented stable minimal surface and prove Theorem 1.3 in Section 3.

2. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. In virtue of Theorem 1.1 and the following statements in Section 1, it suffices to prove that there is no embedded compact oriented stable minimal surface Σ in M if $r_g \geq \frac{1}{\mu}\frac{s}{3}t_0$ with $t_0 = 1$. We will assume that such a surface exists in M and derive a contradiction.

Let Σ be a connected and embedded compact oriented stable minimal surface in M^3. According to [3] and [5], Σ is properly contained in M_0, where $M_0 = \{x \in M^3 \,|\, f(x) < -1\}$. In other words, $h < 0$ on Σ for $h = 1 + f$. It is also known that, for a non-trivial function $\varphi \in \ker s_g'^*$, Σ is a connected component of $\Gamma = \varphi^{-1}(0)$. Here, the set Γ is a union of totally geodesic hypersurfaces of M ([2]), and at least one connected component of Γ is homeomorphic to S^2 ([10]).

For the proof of Theorem 1.2, we need the following two lemmas:

LEMMA 2.1. *Let (g, f) be a non-trivial solution of (1) on a n-dimensional compact oriented manifold M. If $r_g \geq \frac{1}{\mu}\frac{s}{n}$ and there exists a* *compact oriented stable minimal hypersurface* Σ *in* M, *then* f *is constant on* Σ.

PROOF. The equation (1) can rewritten as

$$hz = Ddf + \frac{s}{n(n-1)} fg. \tag{5}$$

First we claim that on Σ, the following relation holds:

$$hr(\nu, \nu) = -\Delta' f + \frac{s}{n}, \tag{6}$$

where Δ' denote the intrinsic Laplacian on the surface Σ. It follows from (5) and the fact that the Laplacian Δ_g and the intrinsic Laplacian Δ' are related by

$$-\frac{s}{n-1} f = \Delta_g f = \Delta' f + Ddf(\nu, \nu) + m\langle \nu, df \rangle,$$

and the mean curvature m of Σ is zero.

In virtue of the assumption on r_g and (6),

$$\frac{1}{\mu}\frac{s}{n} \le r(\nu, \nu) \le -\Delta' f + \frac{1}{h}\frac{s}{n} \le -\Delta' f + \frac{1}{\mu}\frac{s}{n}.$$

Here, the last inequality comes from the definition of μ. It follows that $\Delta' f \ge 0$ on Σ since $h < 0$. Hence we may conclude that f is constant on Σ. □

It should be noted that Lemma 2.1 holds for an arbitrary dimension n and without the assumption $\ker s_g'^* \ne 0$. In the following we shall prove that f cannot be constant on all of Σ. The case $df \ne 0$ is excluded due to the following technical result from Lemma 3.1 of [6]:

LEMMA 2.2. *Let* Σ^A *be the subset of* Σ *such that* f *is constant and* $df \ne 0$. *Then there is no open set in* Σ^A.

Thus in order for f to be constant by Lemma 2.1, $df(p) = 0$ for some point p in Σ. Then, from the fact that $\langle df, \nu \rangle$ is constant on Σ for a unit normal vector field ν to Σ ([4]), $\langle df, \nu \rangle \equiv 0$ and so $df \equiv 0$ on Σ. It is easy to construct a function $\tilde{f}$ such that $\tilde{f}$ is a solution of (1), and $d\tilde{f} \ne 0$ with $\tilde{f} = f$ on Σ (Lemma 5 in [4]). In particular, $\tilde{f}$ is constant on Σ. Then the same statement of Lemma 2.2 can be used to argue that $\tilde{f}$ cannot be constant on all of Σ. In other words, f cannot be constant on Σ, which is a contradiction to Lemma 2.1. This contradiction leads to the conclusion that Σ does not exist, completing the proof of Theorem 1.2.

3. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3.

First we claim that the constant $c = h^3 z(\nu,\nu)_{|\Sigma}$ for a unit normal vector field ν to Σ, which is mentioned in Section 1, is positive on Σ. Note that by (6)

$$hz(\nu,\nu) = -\Delta' f - \frac{s}{3}f \tag{7}$$

on Σ. Thus we have

$$c\int_\Sigma \frac{1}{h^2} = -\frac{s}{3}\int_\Sigma f > 0,$$

which implies that $c > 0$. In particular, $z(\nu,\nu) \neq 0$ on Σ. In other words, g is not Einstein.

As a consequence of this claim, we may deduce from $r_g \geq \frac{1}{\mu}\frac{s}{3}t_0$ and $r(\nu,\nu) = \frac{s}{3} + \frac{c}{h^3}$ with $\frac{h}{\mu} \leq 1$ that

$$h^2 \geq \frac{3c}{s}\frac{1}{t_0-\mu} > 0, \tag{8}$$

since

$$\frac{c}{h^2} \leq \frac{s}{3}\frac{h}{\mu}(t_0-\mu) \leq \frac{s}{3}(t_0-\mu).$$

In order to determine the sign of $r(\nu,\nu)$, we need to know the upper and lower bound of c. The following lemma gives such bounds.

Lemma 3.1. *On Σ, we have*

$$\beta \leq c \leq \frac{s}{3}(t_0-\mu-1)^2(t_0-\mu), \tag{9}$$

where

$$\beta = \frac{s\,Area(\Sigma)}{3\int_\Sigma(-h^{-3})}.$$

Proof. From $r(\nu,\nu) \geq \frac{s}{3}\frac{t_0}{\mu}$,

$$hz(\nu,\nu) \leq \frac{s}{3}\frac{h}{\mu}(t_0-\mu) \leq \frac{s}{3}(t_0-\mu). \tag{10}$$

By (8), $1+f = h \leq -\gamma^{\frac{1}{2}}$, where $\gamma = \frac{3c}{s}\frac{1}{t_0-\mu}$. Using (7) and (10), we have

$$\frac{s}{3}(1+\gamma^{\frac{1}{2}})\,Area\,\Sigma \leq -\frac{s}{3}\int_\Sigma f = \int_\Sigma hz(\nu,\nu) \leq \frac{s}{3}(t_0-\mu)\,Area\,\Sigma.$$

Thus

$$1+\gamma^{\frac{1}{2}} \leq t_0 - \mu.$$

This explains the second inequality. The first inequality directly follows from the stability condition on Σ:

$$\int_{\Sigma} r(\nu,\nu) \leq 0. \tag{11}$$

□

Now, we are ready to prove Theorem 1.3.

PROOF OF THEOREM 1.3. In virtue of Lemma 3.1, it suffices to prove that $c \neq \frac{s}{3}(t_0-\mu-1)^2(t_0-\mu)$.

Suppose that $c = \frac{s}{3}(t_0-\mu-1)^2(t_0-\mu)$. Then, from

$$\frac{1}{\mu}\frac{s}{3}t_0 \leq r(\nu,\nu) = \frac{s}{3} + \frac{c}{h^3} \leq \frac{s}{3} + \frac{c}{\mu^3},$$

we have

$$\frac{s}{3}\left(\frac{t_0}{\mu} - 1\right) \leq \frac{c}{\mu^3} = \frac{s}{3}\frac{(t_0-\mu-1)^2(t_0-\mu)}{\mu^3}.$$

Therefore

$$\mu^2(t_0-\mu) \geq (t_0-\mu-1)^2(t_0-\mu). \tag{12}$$

However, the equation (12) does not holds for $t_0 > 1$ and $\mu < 0$; let $F(t) = (t-\mu-1)^2(t-\mu) - \mu^2(t-\mu)$, then it is easy to see that $F(1) = 0$ and $F'(t) > 0$ for $t > 1$. Therefore $F(t) > 0$ for $t > 1$, contradicting (12). This contradiction implies that $c \neq \frac{s}{3}(t_0-\mu-1)^2(t_0-\mu)$, completing the proof of Theorem 1.3. □

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