Honam Mathematical J. **30** (2008), No. 4, pp. 685–691

SELF-ADJOINT INTERPOLATION ON $AX = Y$ IN $\mathcal{B}(\mathcal{H})$

Sung Kon Kwak and Ki Sook Kim

Abstract. Given operators X_i and Y_i $(i = 1, 2, \cdots, n)$ acting on a Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A acting on $\mathcal{H}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$. In this article, if the range of X_k is dense in $\mathcal{H}$ for a certain k in $\{1, 2, \cdots, n\}$, then the following are equivalent:

(1) There exists a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

(2) $\sup\left\{\frac{\|\sum_{i=1}^{n} Y_i f_i\|}{\|\sum_{i=1}^{n} X_i f_i\|} : f_i \in \mathcal{H}\right\} < \infty$ and $< X_k f, Y_k g > = < Y_k f, X_k g >$ for all f, g in $\mathcal{H}$.

1. Introduction

Given operators X and Y on a Hilbert space $\mathcal{H}$, when is there a bounded linear operator A (usually satisfying some other conditions) such that $AX = Y$? The "other conditions" that have been of interest to us involve restricting A to lie in the algebra associated with a subspace lattice. We consider this problem for vectors. An interpolation question for a given subalgebra $\mathcal{C}$ of $\mathcal{B}(\mathcal{H})$ asks for which x and y is there a bounded operator $A \in \mathcal{C}$ that map x to y. Lance [6] initiated the discussion by considering a nest $\mathcal{N}$ and asking what conditions on x and y will guarantee the existence of an operator A in Alg$\mathcal{N}$ such that $Ax = y$. Hopenwasser [3] extended Lance's result to the case where the nest $\mathcal{N}$ is replaced by an arbitrary commutative subspace lattice $\mathcal{L}$; the conditions in both cases read the same. Munch [7] considered the problem of finding a Hilbert-Schmit operator A in Alg$\mathcal{N}$ that maps x to y, whereupon Hopenwasser [4] again extended to Alg$\mathcal{L}$. In [1], authors

Received September 25, 2008. Accepted October 8, 2008.

2000 Mathematics Subject Classification: 47L35.

Key words and phrases: Self-Adjoint Operator, Self-Adjoint Interpolation Problem, Subspace Lattice.

This paper is partially supported by Daegu University for sabatical year 2007.

studied the problem of finding A so that $Ax = y$ and A is required to lie in certain ideals contained in Alg$\mathcal{L}$ (for a nest $\mathcal{L}$).

Roughly speaking, when an operator maps one thing to another, we think of the operator as the interpolating operator and the equation representing the mapping as the interpolation equation. The equations $Ax = y$ and $AX = Y$ are indistinguishable if spoken aloud, but we mean the change to capital letters to indicate that we intend to look at fixed operators X and Y, and ask under what conditions there will exist an operator A satisfying the equation $AX = Y$.

Self-adjoint interpolation problem is the following : Given operators X and Y acting on $\mathcal{H}$, when does there exist a self-adjoint operator A in a subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on $\mathcal{H}$ such that $AX = Y$?

First, we establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The algebra Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Results

We give necessary and sufficient conditions that there exists a self-adjoint operator A such that $AX = Y$ for given a pair of bounded operators X and Y acting on $\mathcal{H}$. The result will extend to the case when finitely many pairs of operators are given. We also prove similar results for countable bounded operators $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$.

Let $\mathcal{H}$ be a Hilbert space. A bounded operator A on $\mathcal{H}$ is called *self-adjoint* if $A = A^*$.

Theorem 1. **[R. G. Douglas[2]]** Let X and Y be bounded operators acting on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:

(1) range$Y^* \subseteq$ range X^*

(2) $Y^*Y \leq \lambda^2 X^*X$ for some $\lambda \geq 0$

(3) there exists a bounded operator A on $\mathcal{H}$ so that $AX = Y$.

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator A so that

(a) $\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$

(b) $\ker Y^* = \ker A^*$ and

(c) $\text{range} A^* \subseteq \text{range} X^-$.

Theorem 2. Let X and Y be bounded operators acting on a Hilbert space $\mathcal{H}$. If there is a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX = Y$, then $\text{range} Y^* \subseteq \text{range} X^*$ and $< Xf, Yg >=< Yf, Xg >$ for all f, g in $\mathcal{H}$.

Proof. If there is an operator A in $\mathcal{B}(\mathcal{H})$ such that $AX = Y$, then $\text{range} Y^* \subseteq \text{range} X^*$ by Theorem 1. We shall show that $< Xf, Yg >= < Yf, Xg >$ for all f, g in $\mathcal{H}$. Let f, g be vectors in $\mathcal{H}$. Then $< Xf, Yg >=< Xf, AXg >=< A^*Xf, Xg >=< AXf, Xg >=< Yf, Xg >$. □

Theorem 3. Let X and Y be bounded operators acting on a Hilbert space $\mathcal{H}$. Assume that the range of X is dense in $\mathcal{H}$. If $\text{range} Y^* \subseteq \text{range} X^*$ and if $< Xf, Yg >=< Yf, Xg >$ for all f, g in $\mathcal{H}$, then there is a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX = Y$.

Proof. If $\text{range} Y^* \subseteq \text{range} X^*$, then there is an operator A in $\mathcal{B}(\mathcal{H})$ such that $AX = Y$ by Theorem 1. Let f, g be vectors in $\mathcal{H}$. Since $\overline{\text{range} X} = \mathcal{H}$, there are sequences $\{f_n\}, \{g_n\}$ in $\mathcal{H}$ such that $Xf_n \longrightarrow f$ and $Xg_n \longrightarrow g$. Then

$$
\begin{aligned}
< Af, g > &= < A \lim_{n\to\infty} Xf_n, \lim_{n\to\infty} Xg_n > \\
&= < \lim_{n\to\infty} AXf_n, \lim_{n\to\infty} Xg_n > \\
&= < \lim_{n\to\infty} Yf_n, \lim_{n\to\infty} Xg_n > \\
&= \lim_{n\to\infty} < Yf_n, Xg_n > \\
&= \lim_{n\to\infty} < Xf_n, Yg_n > \\
&= < \lim_{n\to\infty} Xf_n, \lim_{n\to\infty} AXg_n > \\
&= < \lim_{n\to\infty} Xf_n, A \lim_{n\to\infty} Xg_n > \\
&= < f, Ag > .
\end{aligned}
$$

Hence A is self-adjoint. □

From Theorems 2 and 3, we have the following theorem.

Theorem 4. Let X and Y be bounded operators acting on a Hilbert space $\mathcal{H}$. If the range of X is dense in $\mathcal{H}$, then the following are equivalent:

(1) There exists a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX = Y$.
(2) range$Y^* \subseteq$ rangeX^* and $< Xf, Yg >=< Yf, Xg >$ for all f, g in $\mathcal{H}$.

Inductively, we can state the following theorem.

Theorem 5. Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be bounded operators on a Hilbert space $\mathcal{H}$. If the range of X_k is dense in $\mathcal{H}$ for some k in $\{1, 2, \cdots, n\}$, then the following are equivalent:

(1) $\sup\left\{\frac{\|\sum_{i=1}^{n} Y_i f_i\|}{\|\sum_{i=1}^{n} X_i f_i\|} : f_i \in \mathcal{H}\right\} < \infty$ and $< X_k f, Y_k g >=< Y_k f, X_k g >$ for all f, g in $\mathcal{H}$.
(2) There exists a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

Proof. If $\sup\left\{\frac{\| \sum_{i=1}^{n} Y_i f_i\|}{\| \sum_{i=1}^{n} X_i f_i\|} : f_i \in \mathcal{H}\right\} < \infty$, then there exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$ by Theorem 2.3 [5]. In addition, for f and g in $\mathcal{H}$, there are sequences $\{f_n\}, \{g_n\}$ in $\mathcal{H}$ such that $X_k f_n \longrightarrow f$ and $X_k g_n \longrightarrow g$ since $\overline{\text{range} X_k} = \mathcal{H}$. Since $< X_k f, Y_k g >=< Y_k f, X_k g >$, for all f, g in $\mathcal{H}$,

$$\begin{aligned}
< Af, g > &= < A \lim_{n\to\infty} X_k f_n, \lim_{n\to\infty} X_k g_n > \\
&= < \lim_{n\to\infty} Y_k f_n, \lim_{n\to\infty} X_k g_n > \\
&= \lim_{n\to\infty} < Y_k f_n, X_k g_n > \\
&= \lim_{n\to\infty} < X_k f_n, Y_k g_n > \\
&= < \lim_{n\to\infty} X_k f_n, \lim_{n\to\infty} AX_k g_n > \\
&= < f, Ag > .
\end{aligned}$$

Hence $A = A^*$

Conversely, if there exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$, then $\sup\left\{\frac{\| \sum_{i=1}^{n} Y_i f_i\|}{\| \sum_{i=1}^{n} X_i f_i\|} : f_i \in \mathcal{H}\right\} < \infty$ by Theorem 2.3

[5]. Also since A is self-adjoint,

$$\begin{aligned} < X_i f, Y_i g > &= < X_i f, AX_i g > \\ &= < A^* X_i f, X_i g > \\ &= < AX_i f, X_i g > \\ &= < Y_i f, X_i g > \text{ for } i = 1, 2, \cdots, n . \quad \square \end{aligned}$$

Theorem 6. **[5]** Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be bounded operators acting on $\mathcal{H}$. If $Re\,\langle X_i f, X_j g\rangle \geq 0$ for $i < j$ and all $f,\ g$ in $\mathcal{H}$, then the following are equivalent:

(1) range$Y_i^* \subseteq$ rangeX_i^* for each $i = 1, 2, \cdots, n$.
(2) There exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.
(3) $\sup \left\{ \dfrac{\| \sum_{i=1}^n Y_i f_i \|}{\| \sum_{i=1}^n X_i f_i \|} :\ f_i \in \mathcal{H} \right\} < \infty$.

By the above Theorem, we can get the following Corollary.

Corollary 7. Let $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ be bounded operators on a Hilbert space $\mathcal{H}$. If the range of X_k is dense in $\mathcal{H}$ for some k and $Re\,\langle X_i f, X_j g\rangle \geq 0$ for $i < j$ and all $f,\ g$ in $\mathcal{H}$, then the following are equivalent:

(1) range$Y_i^* \subseteq$ rangeX_i^* for each $i = 1, 2, \cdots, n$ and $< X_k f, Y_k g > = < Y_k f, X_k g >$ for all $f,\ g$ in $\mathcal{H}$.
(2) There is a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

Theorem 8. Let $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ be bounded operators on a Hilbert space $\mathcal{H}$. If the range of X_k is dense in $\mathcal{H}$ for some k and $Re < Y_i f, Y_j g > \leq Re < X_i f, X_j g >$ for $i < j$ and $f,\ g$ in $\mathcal{H}$, then the following are equivalent:

(1) $\sup \left\{ \dfrac{\|Y_i f\|}{\|X_i f\|} : f \in \mathcal{H} \right\} < \infty$ for each $i \leq n$ in $\mathbb{N}$ and $< X_k f, Y_k g > = < Y_k f, X_k g >$ for all $f,\ g$ in $\mathcal{H}$.
(2) There is a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

If we observe the proof of the above theorems, then we have the following theorem by using Theorem 2.4[5].

Theorem 9. Let X_i and Y_i be bounded operators on a Hilbert space $\mathcal{H}$ for $i \in \mathbb{N}$. If there exists a self-adjoint operator A such that $AX_i = Y_i$ for $i = 1, \cdots$, then $\sup\left\{ \frac{\| \sum_{i=1}^{n} Y_i f_i \|}{\| \sum_{i=1}^{n} X_i f_i \|} : f_i \in \mathcal{H} \text{ and } n \in \mathbb{N} \right\} < \infty$ and $< X_i f, Y_i g > = < Y_i f, X_i g >$ for any i in $\mathbb{N}$ and all $f,\ g$ in $\mathcal{H}$.

Proof If there exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$, then $\sup\left\{ \frac{\| \sum_{i=1}^{n} Y_i f_i \|}{\| \sum_{i=1}^{n} X_i f_i \|} : f_i \in \mathcal{H} \text{ and } n \in \mathbb{N} \right\} < \infty$ by Theorem 2.4[5]. Also since A is self-adjoint, $< X_i f, Y_i g > = < X_i f, AX_i g > = < A^* X_i f, X_i g > = < AX_i f, X_i g > = < Y_i f, X_i g >$ for $i = 1, 2, \cdots$. □

Theorem 10. Let X_i and Y_i be bounded operators on a Hilbert space $\mathcal{H}$ for $i \in \mathbb{N}$. Assume that the range of X_k is dense in $\mathcal{H}$ for a certain k in $\{1, 2, \cdots\}$. If $\sup\left\{ \frac{\| \sum_{i=1}^{n} Y_i f_i \|}{\| \sum_{i=1}^{n} X_i f_i \|} : f_i \in \mathcal{H}, n \in \mathbb{N} \right\} < \infty$ and if $< X_k f, Y_k g > = < Y_k f, X_k g >$ for the given k and all $f,\ g$ in $\mathcal{H}$, then there exists a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.

Proof. If $\sup\left\{ \frac{\| \sum_{i=1}^{n} Y_i f_i \|}{\| \sum_{i=1}^{n} X_i f_i \|} : f_i \in \mathcal{H}, n \in \mathbb{N} \right\} < \infty$, then there exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$ by Theorem 2.4[5]. If f and g are vectors in $\mathcal{H}$, then there are sequences $\{f_n\}, \{g_n\}$ in $\mathcal{H}$ such that $X_k f_n \longrightarrow f$ and $X_k g_n \longrightarrow g$ since $\overline{\text{range} X_k} = \mathcal{H}$. Since $< X_k f, Y_k g > = < Y_k f, X_k g >$,

$$
\begin{aligned}
< Af,\ g > &= < A \lim_{n\to\infty} X_k f_n, \lim_{n\to\infty} X_k g_n > \\
&= < \lim_{n\to\infty} AX_k f_n, \lim_{n\to\infty} X_k g_n > \\
&= < \lim_{n\to\infty} Y_k f_n, \lim_{n\to\infty} X_k g_n > \\
&= \lim_{n\to\infty} < Y_k f_n, X_k g_n > \\
&= \lim_{n\to\infty} < X_k f_n, Y_k g_n > \\
&= < \lim_{n\to\infty} X_k f_n, \lim_{n\to\infty} AX_k g_n > \\
&= < f, Ag > .
\end{aligned}
$$

Hence $A = A^*$. □

Theorem 11. Let X_i and Y_i be bounded operators acting on a Hilbert space $\mathcal{H}$ for $i = 1, 2, \cdots$. If for some k, the range of X_k is dense in $\mathcal{H}$ and $Re < Y_i f, Y_j g > \leq Re < X_i f, X_j g >$ for $i < j$ and $f,\ g$ in $\mathcal{H}$, then the following are equivalent:

(1) There exists $\lambda(\geq 0)$ such that $Y_i^* Y_i \leq \lambda^2 X_i^* X_i$ for each $i \in \mathbb{N}$ and $< X_k f, Y_k g > = < Y_k f, X_k g >$ for all $f,\ g$ in $\mathcal{H}$.

(2) There is a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.

Theorem 12. Let X_i and Y_i be bounded operators acting on a Hilbert space $\mathcal{H}$ for $i = 1, 2, \cdots$. If $Re < X_i f_i, X_j f_j > \geq 0$ for $i < j$, $\sup_{i,j \in \mathbb{N}} \left\{ \frac{Re< Y_i f, Y_j g>}{Re< X_i f, X_j g>} : f,\ g \in \mathcal{H} \right\} = k < \infty$ and if for some k, the range of X_k is dense in $\mathcal{H}$ and $< X_k f, Y_k g > = < Y_k f, X_k g >$ for all $f,\ g$ in $\mathcal{H}$, then there exists a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$ such that $AX_i = Y_i$ for any i in $\mathbb{N}$.

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Sung Kon Kwak
Department of Mathematics,
Daegu University,
Daegu, Korea
E-mail: sgkwak@daegu.ac.kr

Ki Sook Kim
Department of Mathematics,
Daegu University,
Daegu, Korea
E-mail: kks209@chollian.net