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THE NONEXISTENCE OF WARPING FUNCTIONS ON SPACE-TIMES

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Abstract. In this paper, when N is a compact Riemannian manifold of class (A), we consider the nonexistence of some warping functions on space-times $M = [a, \infty) \times_f N$ with prescribed scalar curvatures.

1. Introduction

One of the well-known problems in differential geometry is that of whether a given smooth function on a compact Riemannian manifold is necessarily the scalar curvature of some metric. In order to study these kinds of problems, we need some analytic methods in differential geometry, because they have the forms of differential equations.

By the results of Kazdan and Warner ([7, 8, 9]), if N is a compact Riemannian $n-$manifold without boundary, $n \geq 3$, then N belongs to one of the following three catagories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.
(B) A Smooth function on N is the scalar curvature of some Rimannian metric on N if and only if the function is either identically zero or strictly negative somewhere.
(C) Any smooth function on N is the scalar curvature of some Rimannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

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In [7, 8, 9], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

Ironically, even though there exists some obstruction of positive or zero scalar curvature on a Riemannian manifold, results of [2], say, Theorem 3.1, Theorem 3.5 and Theorem 3.7 of [2] show that there exists no obstruction of positive scalar curvature on a Lorentzian warped product manifold, but there may exist some obstruction of negative or zero scalar curvature.

And also in [2], authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature.

In recent work, some authors have considered the problem of scalar curvature functions on a warped product manifold and obtained partial results about the existence and nonexistence of a warped metric with some prescribed scalar curvature function (cf. [3], [4],[5],[6], [7]).

In this paper, we consider the solution of some partial differential equations on a warped product manifold. That is, we express the scalar curvature of a warped product manifold $M = [a, \infty) \times_f N$ in terms of its warping function f and the scalar curvatures of N. Using key lemma, we treat the nonexistence of a warping function f on a warped product manifold $M = [a, \infty) \times_f N$. In other words, it is shown that if the fiber manifold N belongs to class (A), then M does not admit a Lorentzian metric with some positive scalar curvature near the end outside a compact set.

In this paper, we extend the results of Proposition 2.4 and Theorem 3.1 in [2]. That is, we show that if $R(t, x) = R(t) \in C^\infty([a, \infty))$ is a positive function such that

$$\frac{4n}{n+1}\frac{C}{t^2} \geq R(t) \geq 0 \quad for \quad t \geq t_0,$$

where $t_0 > a$ and $1 \leq C$ is a positive constant, then equation (2.4) has no positive solution on $[a, \infty)$.

2. Main Results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be smooth function, where a is a positive number. The Lorentzian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a,\infty)\times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g. \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N,g). Then the scalar curvature $R(t,x)$ of g' is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{2.2}$$

for $t \in [a,\infty)$ and $x \in N$ (for details, cf. [1] or [2]). If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u'' - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

If $R(t,x) = R(t)$ is the function of only t-variable, then we have the following proposition:

Proposition 2.1. If N belongs to class (B) or (C), i.e., $R(g) \le 0$, then there is no positive solution to equation (2.3) with

$$R(t) \le -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \ge t_0,$$

where $c > 1$ and $t_0 > a$ are positive constants.

Proof. See Proposition 2.4 and Theorem 3.1 in [3]. □

In particular, if $R(g) \le 0$, then using Lorentzian warped product it is impossible to obtain a Lorentzian metric of uniformly negative scalar curvature less than some negative constant outside a compact subset. The best we can do is when $u(t) = t^{\frac{1}{2}}$, or $f(t) = t^{\frac{1}{n+1}}$, where the scalar curvature is negative but goes to zero at infinity.

However, the results of [2] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature even when the fiber manifold has constant scalar curvature. Except the case that both $R(t,x)$ and $R(g)$ are constants, when $R(g)$ is a negative constant and $R(t,x)$ is a prescribed function, the result of equation (2.3) is little known. That is, when N belongs to class (B) or (C), there are many results about the existence and the nonexistence of equation (2.3). But in case that N belongs to class (A), there is little known result.

We assume that the fiber manifold N of $M = [a, \infty) \times_f N$ has a negative scalar curvature, where a is a positive number. If we let $u(t) = t^\alpha$, where $\alpha > 1$ is a constant, then we have

$$R(t, x) \leq \frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2}, \quad t > a.$$

If N belongs to class (A), then, by the results of Kazdan- Warner's papers ([8, 9, 10]), there exists a metric with a negative constant scalar curvature. So we may assume that $R(g)$ is a negative constant.

If N has a negative constant scalar curvature $-\frac{4n}{n+1}k^2$, where k is a constant and $R(t, x) = R(t)$ is the function of only t-variable, then equation (2.3) becomes

$$\frac{4n}{n+1}u''(t) - R(t)u(t) - \frac{4n}{n+1}k^2 u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.4}$$

In order to prove the nonexistence of some warped product metric, we need the following Lemma ([3]).

Lemma 2.2. Let $u(t)$ be a positive smooth function on $[a, \infty)$. If $u(t)$ satisfies

$$\frac{u''(t)}{u(t)} \leq \frac{C}{t^2}$$

for some constant $C \geq 1$, then there exists $t_0 > a$ such that for all $t > t_0$

$$u(t) \leq C_0 t^\epsilon$$

for some positive constant C_0 and $\epsilon > 1$.

Proof. Since $C \geq 1$, we can choose $\epsilon > 1$ such that $\epsilon(\epsilon - 1) = C$. Then from the hypothesis, we have

$$t^\epsilon u''(t) \leq \epsilon(\epsilon-1)t^{\epsilon-2}u(t).$$

Upon integration from $t_1(\geq a)$ to $t(> t_1 \geq a)$, and using integration by parts, we obtain

$$t^\epsilon u'(t) - \epsilon t^{\epsilon-1}u(t) - t_1^\epsilon u'(t_1) + \epsilon t_1^{\epsilon-1}u(t_1) + \epsilon(\epsilon-1)\int_{t_1}^{t} s^{\epsilon-2}u(s)ds \leq C\int_{t_1}^{t} s^{\epsilon-2}u(s)ds.$$

Therefore we have

$$t^\epsilon u'(t) - \epsilon t^{\epsilon-1}u(t) \leq t_1^\epsilon u'(t_1) - \epsilon t_1^{\epsilon-1}u(t_1) \tag{2.5}$$

We consider two following cases:

[Case 1] There exists $t_1 \geq a$ such that $u'(t_1) \leq 0$.

If there is a number $t_1 \geq a$ such that $u'(t_1) \leq 0$, then we have

$$t^{\epsilon}u'(t) - \epsilon t^{\epsilon-1}u(t) \leq 0.$$

This gives

$$(\ln u(t))' \leq \epsilon(\ln t)'.$$

Hence

$$u(t) \leq c_1 t^{\epsilon}$$

for all $t > t_1$, where c_1 is a positive constant.

[Case 2] There does not exist $t_1 \geq a$ such that $u'(t_1) \leq 0$.

In other words, if $u'(t) > 0$ for all $t \geq a$, then $u(t) \geq c'$ for some positive constant c'. Let c_2 be a positive constant such that

$$t_1^{\epsilon}u'(t_1) - \epsilon t_1^{\epsilon-1}u(t_1) \leq c_2,$$

then equation (2.5) gives

$$t^{\epsilon}u'(t) - \epsilon t^{\epsilon-1}u(t) \leq c_2$$

for all $t > t_1$. Thus

$$\frac{u'(t)}{u(t)} \leq \frac{\epsilon}{t} + \frac{c_2}{u(t)t^{\epsilon}} \leq \frac{\epsilon}{t} + \frac{c_2}{c't^{\epsilon}}.$$

Integrating from t_1 to t we have

$$\ln\frac{u(t)}{u(t_1)} \leq \epsilon\ln\left(\frac{t}{t_1}\right) + \frac{c_2}{(\epsilon-1)c't_1^{\epsilon-1}} \leq \epsilon\ln\left(\frac{c_3 t}{t_1}\right),$$

as $\epsilon > 1$. Here c_3 is a positive constant such that $\ln c_3 \geq \frac{c_2}{\epsilon(\epsilon-1)c't_1^{\epsilon-1}}$. Hence we again obtain the inequality

$$u(t) \leq bt^{\epsilon}$$

for some positive constant b and for all $t \geq t_1$.

Thus from two cases we always find $t_0 > a$ and a constant $c_0 > 0$ such that

$$u(t) \leq c_0 t^{\epsilon}$$

for all $t \geq t_0$. □

If $R(t,x) = R(t)$ is the bounded function of only t-variable, our first main theorem is as follows:

Theorem 2.3. Suppose that $R(g) = -\frac{4n}{n+1}k^2$ for $n > 3$ and $R(t,x) = R(t) \in C^\infty([a,\infty))$. Assume that for $t > t_0$, there exists a positive solution $u(t)$ of equation (2.4) with $M > R(t) \geq 0$ for some positive constant M. Then $u(t) \geq t^\alpha$ for large t and all $\alpha > 0$.

Proof. Suppose $u(t) > 0$ satisfies equation (2.4), i.e.,

$$\frac{4n}{n+1}u''(t) = R(t)u(t) + \frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}} \tag{2.6}$$

Since $R(t) \geq 0$ and $\frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}} > 0$, integrating equation (2.6) from $\tau(\geq a)$ to t, we have

$$u'(t) - u'(\tau) > 0$$

for all $t(> \tau)$.

Here we have two following cases:

[Case 1] There exists $\tau(\geq a)$ such that $u'(\tau) \geq 0$. Then $u'(t) \geq 0$, so $u(t)$ is an increasing function. Thus $u(t) \geq u(\tau) > 0$. Therefore from equation (2.6) we have

$$u''(t) \geq \frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}} \geq c_0$$

for large t and some positive constant c_0. Hence we have

$$u(t) \geq \frac{c_0}{2}t^2 + c_1t + c_2 \tag{2.7}$$

for some constants c_1, c_2. Again substituting equation (2.7) to equation (2.6), we have

$$u''(t) > \frac{4n}{n+1}k^2(\frac{c_0}{2}t^2 + c_1t + c_2)^{1-\frac{4}{n+1}}$$

$$u(t) > c_3t^{2+2(1-\frac{4}{n+1})}$$

for some positive constant c_3. Reiterating this method, we complete the theorem.

[Case 2] There does not exist $\tau(\geq a)$ such that $u'(\tau) \geq 0$. In other words, we have $u'(t) < 0$ for all $t(\geq a)$. Then $u(t)$ is a decreasing function. If $u(t) \geq c_0$ for some positive constant c_0, then by [Case 1] our theorem holds. Otherwise, since $u(t)$ is a decreasing function, $u(t) \to +0$ as $t \to \infty$. From equation (2.6) $R(t)$ is bounded and $1 - \frac{4}{n+1} > 0$, so $u''(t) \to +0$ as $t \to \infty$ and $u'(t)$ is an increasing function.

Put $u(t) = e^{-g(t)}$. Then $u'(t) = e^{-g(t)}g'(t) < 0$, so $g'(t) > 0$ and $g(t)$ is an increasing function. Thus we have

$$u^{''}(t) = e^{-g(t)}(g^{'}(t))^2 - g^{''}(t)) \to +0$$

as $t \to \infty$, so $g^{''}(t) > 0$ because $\frac{(g'(t))^2}{e^{g(t)}} \approx \frac{2g^{'}(t)g^{''}(t)}{e^{g(t)}g^{'}(t)} = \frac{2g^{''}(t)}{e^{g(t)}}$ by L'Hospital's Theorem. Therefore from equation (2.6) we have

$$(g^{'}(t))^2 - g^{''}(t) = \frac{n+1}{4n}R(t) + k^2 e^{\frac{4}{n+1}g(t)}$$

Since $g^{''}(t) > 0$ and $R(t) \geq 0$, we have

$$(g^{'}(t))^2 \geq k^2 e^{\frac{4}{n+1}g(t)}$$

And since $g^{'}(t) > 0$, we have

$$g^{'}(t) \geq k e^{\frac{2}{n+1}g(t)} \tag{2.8}$$

Since $g(t)$ is increasing, from equation (2.8) $g^{'}(t) \geq c_0$ for some positive constant c_0. Hence we have

$$g(t) \geq c_0 t + c_1, \tag{2.9}$$

for some constant c_1. Again, substituting equation (2.9) to equation (2.8), we have

$$g^{'}(t) \geq k e^{\frac{2}{n+1}(c_0 t + c_1)}$$

and, integrating the above equation, we have

$$g(t) \geq a_1 e^{\frac{2}{n+1}c_0 t} \tag{2.10}$$

for some positive constants a_1 and c_0. Again substituting equation (2.10) to equation (2.8), we have

$$g^{'}(t) \geq k e^{\frac{2}{n+1}a_1 e^{\frac{2}{n+1}c_0 t}}$$

and, integrating the above equation, we have

$$g(t) \geq a_2 e^{\frac{2}{n+1}a_1 e^{\frac{2}{n+1}c_0 t}}$$

for some positive constant a_2. And again, iterating this way, we have

$$g(t) \geq a_n e^{\frac{2}{n+1}a_{n-1} e^{\frac{2}{n+1}\cdot^{\cdot^{\cdot e^{a_1 e^{\frac{2}{n+1}c_0 t}}}}}},$$

which is impossible.

From [Case 1] and [Case 2], we complete the theorem. □

If $R(t,x)$ is also the function of only t-variable, our second main theorem is as follows:

Theorem 2.4. Suppose that $R(g) = -\frac{4n}{n+1}k^2$. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a function such that

$$0 \leq R(t) \leq \frac{4n}{n+1}\frac{c}{t^2} \quad \text{for} \quad t > t_0,$$

where $t_0 > a$ and $1 \leq c$ is a constant. Then equation (2.4) has no positive solution on $[a,\infty)$.

Proof. Assume that for $t > t_0$, there exists a solution $u(t)$ of equation (2.4) with $0 \leq R(t) \leq \frac{4n}{n+1}\frac{c}{t^2}$. Then by Theorem 2.3 u(t) is an increasing function such that $u(t) \geq t^\alpha$ for large t and all $\alpha > 0$. From equation (2.6), we have

$$\frac{4n}{n+1}\frac{u''(t)}{u(t)} = R(t) + \frac{4n}{n+1}k^2 u(t)^{-\frac{4}{n+1}} \leq \frac{C}{t^2}$$

for some constant $C \geq 1$ and large t. Hence the Lemma 2.2 implies that for all large t

$$u(t) \leq C_0 t^\epsilon$$

for some positive constant C_0 and $\epsilon > 1$, which is a contradiction to the fact that $u(t) \geq t^\alpha$ for large t and all $\alpha > 0$. Therefore equation (2.4) has no positive solution on $[a,\infty)$. □

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