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INTERVAL-VALUED FUZZY WEAKLY α-CONTINUOUS MAPPINGS

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Abstract. In this paper, we introduce the concepts of IVF weakly α-continuous mappings and investigate some characterizations for them on the interval-valued fuzzy topological spaces.

1. Introduction

Zadeh [5] introduced the concept of fuzzy set and investigated basic properties. Gorzalczany [1] introduced the concept of interval-valued fuzzy set which is a generalization of fuzzy sets. In [4], Mondal and Samanta introduced the concepts of interval-valued fuzzy topology, continuity and compactness and studied some topological properties. In [2], Jun et al. introduced the concepts of IVF α-open sets and IVF α-open mappings and studied some results about them.

In this paper, we introduce the concepts of IVF weakly α-continuous mappings and investigate some properties for them.

2. Preliminaries

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$ and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0,0], \mathbf{1} = [1,1]$, and $\mathbf{a} = [a,a]$ for $a \in (0,1)$. We also note that

1. $(\forall M, N \in D[0,1])(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$.
2. $(\forall M, N \in D[0,1])(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

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For every $M \in D[0,1]$, the complement of M, denoted by M^c, is defined by $M^c = \mathbf{1} - M = [1 - M^U, 1 - M^L]$. Let X be a nonempty set. A mapping $A : X \to D[0,1]$ is called an *interval-valued fuzzy set* (simply, IVF set) in X. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^L$ and $A(x)^U$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$. In particular, for any $a \in [a,b]$, the IVF set whose value is $\mathbf{a} = [a,a]$ for all x X is denoted by simply $\widetilde{a}$. For a point $p \in X$ and for $[a,b] \in D[0,1]$ with $b > 0$, the IVF set which takes the value $[a,b]$ at p and $\mathbf{0}$ elsewhere in X is called an *interval-valued fuzzy point* (simply, IVF point) and is denoted by $[a,b]_p$. In particular, if $b = a$, then it is also denoted by a_p. Denoted by D^X the set of all IVF sets in X.

For every $A, B \in D^X$, we define

$$A = B \Leftrightarrow (\forall x \in X)([A(x)]^L = [B(x)]^L \text{ and } [A(x)]^U = [B(x)]^U),$$

$$A \subseteq B \Leftrightarrow (\forall x \in X)([A(x)]^L \subseteq [B(x)]^L \text{ and } [A(x)]^U \subseteq [B(x)]^U).$$

The complement A^c of A is defined by

$$[A^c(x)]^L = 1 - [A(x)]^U \text{ and } [A^c(x)]^U = 1 - [A(x)]^L$$

for all $x \in X$.

For a family of IVF sets $\{A_i : i \in J\}$ where J is an index set, the union $G = \cup_{i \in J} A_i$ and $F = \cap_{i \in J} A_i$ are defined by

$(\forall x \in X)$ $([G(x)]^L = \sup_{i \in J}[A_i(x)]^L, [G(x)]^U = \sup_{i \in J}[A_i(x)]^U)$,

$(\forall x \in X)$ $([F(x)]^L = \inf_{i \in J}[A_i(x)]^L, [F(x)]^U = \inf_{i \in J}[A_i(x)]^U)$, respectively.

Let $f : X \to Y$ be a mapping and let A be an IVF set in X. Then the image of A under f, denoted by $f(A)$, is defined as follows.

$$[f(A)(y)]^L = \begin{cases} \sup_{f(x)=y}[A(x)]^L, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

$$[f(A)(y)]^U = \begin{cases} \sup_{f(x)=y}[A(x)]^U, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

for all $y \in Y$.

Let B be an IVF set in Y. Then the inverse image of B under f, denoted by $f^{-1}(B)$, is defined as follows.

$$(\forall x \in X)([f^{-1}(B)(x)]^L = [B(f(x))]^L, [f^{-1}(B)(x)]^U = [B(f(x))]^U).$$

Definition 2.1 ([4]). A family τ of IVF sets in X is called an *interval-valued fuzzy topology* (simply, IVFT) on X if it satisfies:

1. $\mathbf{0}, \mathbf{1} \in \tau$.
2. $A, B \in \tau \Rightarrow A \cap B \in \tau$.
3. For $i \in J$, $A_i \in \tau \Rightarrow \cup_{i \in J} A_i \in \tau$.

Every member of τ is called an IVF open set. An IVF set A is called an IVF closed set if the complement of A is an IVF open set. And (X, τ) is called an *interval-valued fuzzy topological space* (simply, IVFTS).

In an IVF topological space (X, τ), for an IVF set A in X, the *IVF closure* [4], denoted by $cl(A)$, is defined by

$$cl(A) = \cap\{B \in IVF(X) : B^c \in \tau \text{ and } A \subseteq B\}$$

and the *IVF interior* of A [4], denoted by $int(A)$, is defined by

$$int(A) = \cup\{B \in IVF(X) : B \in \tau \text{ and } B \subseteq A\}.$$

Let A be an IVF set in an IVFTS (X, τ). Then A is said to be *IVF α-open* [2] (resp., *IVF semiopen* [2], *IVF preopen* [2], *IVF regular open* [3] and *IVF β-open* [3]) if $A \subseteq int(cl(int(A)))$ (resp., $A \subseteq cl(int(A))$, $A \subseteq int(cl(A))$, $A = int(cl(A))$ and $A \subseteq cl(int(cl(A)))$).

Let (X, τ_1) and (Y, τ_2) be two IVFTS's. Then $f : X \to Y$ is said to be *continuous* [4] (resp., *IVF α-continuous* [2]) if for every IVF open set B in Y, $f^{-1}(B)$ is IVF open (resp., IVF α-open) in X. And f is said to be *IVF weakly continuous* [3] if for every IVF point M_x and each IVF open set V containing $f(M_x)$, there exists IVF open set U containing M_x such that $f(U) \subseteq cl(V)$.

An IVF set A in an IVF topological space X is said to be *IVF compact* [4] if every IVF open cover $\mathcal{A} = \{A_i : i \in J\}$ of B has a finite IVF subcover.

Theorem 2.2 ([4]). Let A be an IVF set in an IVF topological space (X, τ). Then
(1) A is IVF closed iff $A = cl(A)$,
(2) $cl(A \cup B) = cl(A) \cup cl(B)$,
(3) $cl(cl(A)) = cl(A)$,
(4) $int(A) = \mathbf{1} - cl(\mathbf{1} - A)$ and $cl(A) = \mathbf{1} - int(\mathbf{1} - A)$.

Theorem 2.3 ([3]). Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then f is IVF weakly continuous if and only if $f^{-1}(B) \subseteq int(f^{-1}(cl(B)))$ for each IVF open set B of Y.

3. IVF weakly α-continuous mappings

Definition 3.1. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then f is said to be IVF weakly α-continuous if for every IVF point M_x and each IVF open set V containing $f(M_x)$, there exists IVF α-open set U containing M_x such that $f(U) \subseteq cl(V)$.

Lemma 3.2. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). If f is IVF α-continuous, then it is IVF weakly continuous.

Proof. Let G be IVF open in Y. Since f is IVF α-continuous, $f^{-1}(G)$ and $f^{-1}(cl(G))$ are IVF α-open and IVF α-closed in X, respectively. Thus we have

$$f^{-1}(G) \subseteq int(cl(int(f^{-1}(G)))) \subseteq cl(int(cl(f^{-1}(cl(G))))) \subseteq f^{-1}(cl(G)).$$

This implies $f^{-1}(G) \subseteq int(f^{-1}(cl(G)))$. Hence by Theorem 2.3, f is IVF weakly continuous. □

Example 3.3. Let $X = I$ and let A and B be IVF sets defined as follows.

$$A(x) = [x, 1], \text{ for all } x \in I;$$

$$B(x) = [\frac{3}{4}x, \frac{1}{2}x], \text{ for all } x \in I.$$

Define IVF topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X as follows.

$$\mathcal{T}_1 = \{\mathbf{0}, A, \mathbf{1}\}$$

and

$$\mathcal{T}_2 = \{\mathbf{0}, B, \mathbf{1}\}.$$

Consider the identity mapping $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$. Then f is an IVF weakly continuous mapping but it is not IVF α-continuous.

Remark 3.4. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Obviously every IVF weakly continuous mapping is IVF weakly α-continuous but the converse need not be true as shown in the next example.

Example 3.5. Let $X = I$ and let A,B, C and D be IVF sets defined as follows.

$$A(x) = \widetilde{(\frac{1}{3})};\ B(x) = \widetilde{(\frac{8}{9})};\ C(x) = \widetilde{(\frac{2}{9})};$$

$$D(x) = -\frac{1}{18}(5x - 14), \text{ for all } x \in I.$$

Define IVF topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X as follows.

$$\mathcal{T}_1 = \{\mathbf{0}, A, B, \mathbf{1}\}$$

and

$$\mathcal{T}_2 = \{\mathbf{0}, C, D, \mathbf{1}\}.$$

Consider the identity mapping $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$. Then f is an IVF weakly α-continuous mapping but it is not IVF weakly-continuous.

By Definition 3.1, Lemma 3.2 and Remark 3.4, we obtain the following diagram.

IVF continuous $\Rightarrow$ IVF α-continuous $\Rightarrow$ IVF weakly continuous $\Rightarrow$ IVF weakly α-continuous

From Examples 3.3 and 3.5, it follows that non of the implications in the above diagram is reversible.

Theorem 3.6. Let (X, τ) be an IVFTS and A an IVF set in X.
(1) $A \cap int(cl(int(A)))$ is IVF α-open;
(2) $A \cup cl(int(cl(A)))$ is IVF α-closed.

Proof. (1) From Theorem 2.2 and $int(A) \subseteq int(cl(int(A)))$, it follows

$$int(A \cap int(cl(int(A))) = int(A) \cap int(cl(int(A))) = int(A).$$

This implies $int(cl(int(A))) = int(cl(int(A \cap int(cl(int(A))))))$, and so

$$A \cap int(cl(int(A))) \subseteq int(cl(int(A \cap int(cl(int(A)))))).$$

Hence $A \cap int(cl(int(A)))$ is IVF α-open.
(2) It is similar to (1).

□

Theorem 3.7. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then the following statements are equivalent:

1. f is IVF weakly α-continuous.
2. $f^{-1}(V) \subseteq int(cl(int(f^{-1}(cl(V)))))$ for each IVF open set V in Y.
3. $cl(int(cl(f^{-1}(int(F))))) \subseteq f^{-1}(F)$ for each IVF closed set F in Y.
4. $cl(int(cl(f^{-1}(int(cl(A)))))) \subseteq f^{-1}(cl(A))$ for each $A \in D^Y$.
5. $f^{-1}(int(A)) \subseteq int(cl(int(f^{-1}(cl(int(A))))))$ for each $A \in D^Y$.
6. $cl(int(cl(f^{-1}(V)))) \subseteq f^{-1}(cl(V))$ for each IVF open set V in Y.

Proof. (1) $\Rightarrow$ (2) Let $V \in \tau_2$. Since f is IVF weakly α-continuous, for each $M_x \in f^{-1}(V)$, there exists an IVF α-open set U_{M_x} containing M_x such that $f(U_{M_x}) \subseteq cl(V)$. Thus

$$M_x \in U_{M_x} \subseteq f^{-1}(cl(V)).$$

Since U_{M_x} is α-open, we have $M_x \in int(cl(int(f^{-1}(cl(V)))))$. Hence $f^{-1}(V) \subseteq int(cl(int(f^{-1}(cl(V)))))$.

(2) $\Rightarrow$ (1) Let M_x be an IVF point in X and V an IVF open set containing $f(M_x)$. Then since $M_x \in f^{-1}(V) \subseteq int(cl(int(f^{-1}(cl(V)))))$, $f^{-1}(V) \subseteq f^{-1}(cl(V)) \cap int(cl(int(f^{-1}(cl(V))))) \subseteq f^{-1}(cl(V))$. Set $U = f^{-1}(cl(V)) \cap int(cl(int(f^{-1}(cl(V)))))$; then by Theorem 3.6, U is an IVF α-open set such that $M_x \in U \subseteq f^{-1}(cl(V))$. Hence f is IVF weakly α-continuous.

(2) $\Rightarrow$ (3) Let F be any IVF closed set of Y. Then $\mathbf{1} - F$ is an IVF open set in Y and

$$\begin{aligned} f^{-1}(\mathbf{1} - F) &\subseteq int(cl(int(f^{-1}(cl(\mathbf{1} - F))))) \\ &= int(cl(int(f^{-1}(\mathbf{1} - int(F))))) \\ &= int(cl(int(\mathbf{1} - f^{-1}(int(F))))) \\ &= \mathbf{1} - cl(int(cl(f^{-1}(int(F))))). \end{aligned}$$

Hence we have $cl(int(cl(f^{-1}(int(F))))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) For $A \in D^Y$, by (3),

$$cl(int(cl(f^{-1}(int(cl(A)))))) \subseteq f^{-1}(cl(A)).$$

(4) $\Rightarrow$ (5) For $A \in D^Y$, by (4),

$$\begin{aligned} f^{-1}(int(A)) &= \mathbf{1} - (f^{-1}(cl(\mathbf{1} - A))) \\ &\subseteq \mathbf{1} - cl(int(cl(f^{-1}(int(cl(\mathbf{1} - A)))))) \\ &= int(cl(int(f^{-1}(cl(int(A)))))). \end{aligned}$$

(5) $\Rightarrow$ (6) Let V be any IVF open set of Y. Then by (5),

$$\begin{aligned} \mathbf{1} - f^{-1}(cl(V)) &= f^{-1}(int(\mathbf{1} - V)) \\ &\subseteq int(cl(int(f^{-1}(cl(int(\mathbf{1} - V)))))) \\ &= int(cl(int(\mathbf{1} - (f^{-1}(int(cl(V))))))) \\ &= \mathbf{1} - cl(int(cl(f^{-1}(int(cl(V)))))) \\ &\subseteq \mathbf{1} - cl(int(cl(f^{-1}(V)))). \end{aligned}$$

This implies $cl(int(cl(f^{-1}(V)))) \subseteq f^{-1}(cl(V))$.

(6) $\Rightarrow$ (2) Let V be an IVF open set in Y. From $V \subseteq int(cl(V))$ and (6), it follows

$$\begin{aligned} f^{-1}(V) &\subseteq f^{-1}(int(cl(V))) \\ &= \mathbf{1} - f^{-1}(cl(\mathbf{1} - cl(V))) \\ &\subseteq \mathbf{1} - cl(int(cl(f^{-1}(\mathbf{1} - cl(V))))) \\ &= int(cl(int(f^{-1}(cl(V))))). \end{aligned}$$

Hence we have (2). □

Theorem 3.8. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then the following statements are equivalent:

1. f is IVF weakly α-continuous.
2. $cl(int(cl(f^{-1}(int(cl(G)))))) \subseteq f^{-1}(cl(G))$ for each IVF open set G in Y.
3. $cl(int(cl(f^{-1}(int(cl(V)))))) \subseteq f^{-1}(cl(V))$ for each IVF preopen set V in Y.
4. $cl(int(cl(f^{-1}(int(K))))) \subseteq f^{-1}(K)$ for each IVF regular closed set K in Y.
5. $cl(int(cl(f^{-1}(int(cl(G)))))) \subseteq f^{-1}(cl(G))$ for each IVF β-open set G in Y.
6. $cl(int(cl(f^{-1}(int(cl(G)))))) \subseteq f^{-1}(cl(G))$ for each IVF semiopen set G in Y.

Proof. (1) $\Rightarrow$ (2) Let G be an IVF open set of Y. Then from Theorem 3.7 (3), it follows $cl(int(cl(f^{-1}(int(cl(G)))))) \subseteq f^{-1}(cl(G))$.

(2) $\Rightarrow$ (3) Let V be an IVF preopen of Y and $A = int(cl(V))$. Then since A is an IVF open set, from (2), it follows

$$cl(int(cl(f^{-1}(int(cl(A)))))) \subseteq f^{-1}(cl(A)).$$

From $V \subseteq int(cl(V))$, it follows $cl(A) = cl(V)$, and so

$$cl(int(cl(f^{-1}(int(cl(V)))))) \subseteq f^{-1}(cl(V)).$$

(3) $\Rightarrow$ (4) Let K be an IVF regular closed set of Y. Since $int(K)$ is an IVF preopen set, by (3),

$$cl(int(cl(f^{-1}(int(cl(int(K))))))) \subseteq f^{-1}(cl(int(K))).$$

Hence (4) is obtained from $int(K) = int(cl(int(K)))$.

(4) $\Rightarrow$ (5) Let G be an IVF β-open set. Then $cl(G)$ is an IVF regular closed set. Thus (5) is obtained.

(5) $\Rightarrow$ (6) Obvious.

(6) $\Rightarrow$ (1) Let V be an IVF open set. Then V is also an IVF semiopen set, from (6) and $V \subseteq int(cl(V))$, it follows

$$cl(int(cl(f^{-1}(V)))) \subseteq cl(int(cl(f^{-1}(int(cl(V)))))) \subseteq f^{-1}(cl(V)).$$

Hence by Theorem 3.7, f is an IVF weakly α-continuous mapping. □

Definition 3.9 ([3]). Let (X, τ) be an IVFTS. An IVF set A in X is said to be *almost IVF compact* (resp., *nearly IVF compact*) if for every IVF open cover $\mathcal{A} = \{A_i \in D^X : i \in J\}$ of A, there exists $J_0 = \{1, 2, \cdots, n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} cl(A_i)$ (resp., $A \subseteq \cup_{i \in J_0} int(cl(A_i)))$.

Theorem 3.10. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an IVF weakly α-continuous mapping on two IVFTS's. If A is a nearly IVF compact set, then $f(A)$ is an almost IVF compact set.

Proof. Let $\{B_i \in D^Y : i \in J\}$ be an IVF open cover of $f(A)$ in Y. Then $\{f^{-1}(B_i) : i \in J\}$ is an IVF open cover of A in X. From IVF nearly compactness, there exists $J_0 = \{1, 2, \cdots, n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} int(cl(f^{-1}(B_i)))$. Since f is IVF weakly α-continuous, from Theorem 3.7 (6), it follows

$$\begin{aligned} \cup_{i \in J_0} int(cl(f^{-1}(B_i))) &\subseteq \cup_{i \in J_0} cl(int(cl(f^{-1}(B_i)))) \\ &\subseteq \cup_{i \in J_0} f^{-1}(cl(B_i)) \\ &= f^{-1}(\cup_{i \in J_0} cl(B_i)). \end{aligned}$$

It implies $f(A) \subseteq \cup_{i \in J_0} cl(B_i)$, and hence $f(A)$ is almost IVF compact. □

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