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THE SIMPLE FORMULA OF CONDITIONAL EXPECTATION ON ANALOGUE OF WIENER MEASURE

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Abstract. In this note, we establish the uniqueness theorem of conditional expectation on analogue of Wiener measure space for given distributions and prove the simple formula of conditional expectation on analogue of Wiener measure which is essentially similar to Park and Skoug's formula on the concrete Wiener measure.

1. Introduction

Conditional expectation is one of the fundamental notions in probability theory and is a most frequently used tool. The Wiener measure is a very useful concept in many probability and many mathematical physicists.

In 1974, Yeh derived the inversion formula for conditional expectation on the concrete Wiener measure space [9], but Yeh's formula is very complicated when the conditioning function is vector-valued. Park and Skoug devised a new formula for conditional expectation on the concrete Wiener measure space, the so-called simple formula for conditional Wiener integral [3].

In 2002, the author and Professor Im defined the analogue of Wiener measure m_φ on the space $C[a,b]$, the space of all real-valued continuous functions on $[a,b]$, associated with a probability measure φ on $\mathbb{R}$, which is a generalization of the concrete Wiener measure [2, 5, 6].

In this paper, we prove the simple formula for conditional expectation on analogue of Wiener measure and establish the uniqueness theorem

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for conditional expectation, in other words, our conditional expectation is not depending upon of the selection of the probability measure.

2. Preliminaries

Wiener suggested a measure space $(C_0[a,b],\omega)$ where $C_0[a,b]$ is the space of all continuous functions on a closed interval $[a,b]$ which vanish at origin, the so called Wiener space in 1923 [8]. The author introduced a generalized measure space compare with the concrete Wiener measure, similar to Wiener measure in [5] as follows; Let $C[a,b]$ be the space of all continuous functions on a closed interval $[a,b]$. Let t be a positive real number and let n be a non-negative integer. For $\vec{t}=(t_0,t_1,\cdots,t_n)$ with $a=t_0<t_1<\cdots<t_n\le b$, let $J_{\vec{t}}: C[a,b]\to\mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) \;=\; (x(t_0),x(t_1),\cdots,x(t_n))\ .$$

For B_j $(j=0,1,2,\cdots,n)$ in $\mathcal{B}(\mathbb{R})$ where $\mathcal{B}(\mathbb{R})$is the sets of all Borel subsets of the real number system $\mathbb{R}$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n}B_j)$ of $C[a,b]$ is called an interval and let $\mathcal{I}$ be the set of all intervals. Then $\mathcal{I}$ is a semi-algebra. Let $\mathcal{A}$ be the set of all the finite unions of disjoint intervals. Then $\mathcal{A}$ is an algebra containing $\mathcal{I}$. For a probability measure φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, we let

$$\begin{aligned}
&m_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n}B_j))\\
&=\int_{B_0}\Big[\int_{\prod_{j=1}^{n}B_j}W(n+1;\vec{t};u_0,u_1,\cdots,u_n)\;d\prod_{j=1}^{n}m_L(u_1,\cdots,u_n)\Big]\;d\varphi(u_0)
\end{aligned}$$

where

$$\begin{aligned}
&W(n+1;\vec{t};u_0,u_1,\cdots,u_n)\\
&=\;\Big(\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\Big)\exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\Big\}.
\end{aligned}$$

By [4], $\mathcal{B}(C[a,b])$, the set of all Borel subsets in $C[a,b]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique measure ω_φ on $(C[a,b],\mathcal{B}(C[a,b]))$ such that $\omega_\varphi(I)=m_\varphi(I)$ for all I in $\mathcal{I}$. This measure ω_φ is called analogue of Wiener measure associated with the measure φ. If φ is a Dirac measure δ_0 at the origin in $\mathbb{R}$, then ω_φ is the classical Wiener measure.

By the change of variables formula, we can easily prove the following theorem in [5].

Theorem 2.1. *(Integration formula for analogue of Wiener measure) If $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then the following equality holds.*

$$\begin{aligned}&\int_{C[a,b]} f(x(t_0), x(t_1), \cdots, x(t_n))\ d\omega_\varphi(x)\\ &\stackrel{*}{=} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}^n} f(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, u_1, \cdots, u_n)\\ &\qquad d(\prod_{j=1}^{n} m_L)((u_1, u_2, \cdots, u_n), u_0)\Big] d\varphi(u_0)\end{aligned}$$

where $\stackrel{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

For fixed natural number n, let $X : C[a,b] \to \mathbb{R}^n$ be a Borel measurable function and $Z : C[a,b] \to \mathbb{R}$ be integrable with respect to the Wiener measure m_φ. The set function P_X^φ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ defined by $P_X^\varphi(E) = m_\varphi(X^{-1}(E))$. Then P_X^φ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. The conditional expectation of Z given X, written $E(Z|X)$, is defined to be a real-valued Borel measurable and P_X^φ-integrable function ψ on $\mathbb{R}^n$ such that

$$\int_{X^{-1}(E)} Z(x)\ dm_\varphi(x) = \int_E \psi(\xi)\ dP_X(\xi)$$

for E in $\mathcal{B}(\mathbb{R}^n)$. The existence of such a function $E^\varphi(Z|X)$ follows from the Radon-Nikodym theorem.

For B in $\mathcal{B}(C[a,b])$ and for E in $\mathcal{B}(\mathbb{R}^n)$, we let

$$[V_X^\varphi(B)](E) = m_\varphi(B \cap X^{-1}(E)).$$

Clearly, $V_X^\varphi(B)$ is a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ for B in $\mathcal{B}(C[a,b])$ and V_X^φ is a measure-valued measure on $(C[a,b], \mathcal{B}(C[a,b]))$ in the total variation norm sense.

From Theorem 3.1 in [6],

Theorem 2.2. *Let $X : C[a,b] \to \mathbb{R}^n$ be a Borel measurable function and let $F : C[a,b] \to \mathbb{R}$ be V_X^φ-Bartle integrable function. Then for E*

in $\mathcal{B}(\mathbb{R}^n)$,

$$\big[(Ba) - \int_{C[a,b]} F(x)dV_X^{\varphi}(x)\big](E)$$
$$= \int_E E^{\varphi}(F|X)(\xi)dP_X^{\varphi}(\xi).$$

3. The main theorems

In this section, we prove the simple formula for conditional expectation on analogue of Wiener measure. Throughout in this section, let $a = t_0 < t_1 < \cdots < t_n = b$ be given, let

$$[y](s)$$
$$= \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(s)[y(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}}(y(t_j) - y(t_{j-1}))] + y(b)\xi_{\{b\}}(s)$$

for y in $C[a,b]$ and let

$$[u](s)$$
$$= \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(s)[u_{j-1} + \frac{s - t_{j-1}}{t_j - t_{j-1}}(u_j - u_{j-1})] + u_n\xi_{\{b\}}(s)$$

for $u_0, u_1, \cdots, u_n)$ in $\mathbb{R}^{n+1}$.

Theorem 3.1. *Let* φ *be a probability measure on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$*. Let* $a = t_0 < t_1 < \cdots < s_1 < t_{j-1} < s_2 < t_j < s < \cdots < t_n = b$ *and let* X*,* Y *and* Z *be three functions from* $C[a,b]$ *into* $\mathbb{R}$ *with* $X(y) = y(s) - [y](s)$*,* $Y(y) = y(s_1)$ *and* $Z(y) = y(s_2)$*, respectively. Then* X *and* Y *are stochastically independent and* X *and* Z *are stochastically independent.*

Proof. Let $A = \sqrt{\frac{(t_j-s)(s-t_{j-1})}{t_j-t_{j-1}}}$, Then by integration formula for analogue of Wiener measure

$$
\begin{aligned}
&E(e^{i\lambda X}) \\
&= \int_{C[a,b]} \exp\{i\lambda(y(s) - [y](s))\} dm_\varphi(y) \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\{i\lambda(\frac{t_j - s}{t_j - t_{j-1}} v_2 - \frac{s - t_{j-1}}{t_j - t_{j-1}} v_3)\} \\
&\quad \times \frac{1}{\sqrt{(2\pi)^3 (t_j - s)(s - t_{j-1})(t_{j-1} - a)}} \exp\{-\frac{1}{2}(\frac{(u_3 - u_2)^2}{t_j - s} + \frac{(u_2 - u_1)^2}{s - t_{j-1}} \\
&\quad + \frac{(u_1 - u_0)^2}{t_{j-1} - a})\} dm_L(u_3) dm_L(u_2) dm_L(u_1)\Big) d\varphi(u_0) \\
&= \exp\{-\frac{1}{2} A^2 \lambda^2\},
\end{aligned}
$$

$$
\begin{aligned}
&E(\exp\{i\lambda Y\}) \\
&= \int_{C[a,b]} \exp\{i\lambda y(s_1)\} dm_\varphi(y) \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \exp\{i\lambda u_1\} \frac{1}{\sqrt{2\pi(s_1 - a)}} \exp\{-\frac{(u_1 - u_0)^2}{2(s_1 - a)}\} \\
&\quad dm_L(u_1)\Big) d\varphi(u_0) \\
&= \exp\{-\frac{1}{2}(s_1 - a)\lambda^2\} \int_{\mathbb{R}} \exp\{i\lambda v_0\} d\varphi(v_0),
\end{aligned}
$$

$$
\begin{aligned}
&E(\exp\{i\lambda Z\}) \\
&= \int_{C[a,b]} \exp\{i\lambda y(s_2)\} dm_\varphi(y) \\
&= \exp\{-\frac{1}{2}(s_2 - a)\lambda^2\} \int_{\mathbb{R}} \exp\{i\lambda u_0\} d\varphi(u_0),
\end{aligned}
$$

$$
\begin{aligned}
&E(\exp\{i\lambda_1 X + i\lambda_2 Y\}) \\
&= \int_{C[a,b]} \exp\{i\lambda_1(y(s) - [y](s) + i\lambda_2 y(s_2))\} dm_\varphi(y) \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\{i\lambda_1(\frac{t_j - s}{t_j - t_{j-1}} u_3 - \frac{s - t_{j-1}}{t_j - t_{j-1}} u_4)\} \\
&\quad \times \exp\{i\lambda_2 v_1\} \frac{1}{\sqrt{(2\pi)^4 (t_j - s)(s - t_{j-1})(t_{j-1} - s_1)(s_1 - a)}} \\
&\quad \exp\{-\frac{1}{2}(\frac{(u_4 - u_3)^2}{t_j - s} + \frac{(u_3 - u_2)^2}{s - t_{j-1}} + \frac{(u_2 - u_1)^2}{t_{j-1} - s_1} + \frac{(u_1 - u_0)^2}{s_1 - a})\} \\
&\quad dm_L(u_4) dm_L(u_3) dm_L(u_2) dm_L(u_1)\Big) d\varphi(u_0) \\
&= \exp\{-\frac{1}{2} A^2 \lambda_1^2\} \exp\{-\frac{1}{2}(s_1 - a)\lambda_2^2\} \int_{\mathbb{R}} \exp\{i\lambda_2 u_0\} d\varphi(u_0)
\end{aligned}
$$

and

$$
\begin{aligned}
&E(\exp\{i\lambda_1 X + i\lambda_2 Z\}) \\
&= \int_{C[a,b]} \exp\{i\lambda_1(y(s) - [y](s) + i\lambda_2 y(s_2))\} dm_\varphi(y) \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\{i\lambda_1(\frac{t_j - s}{t_j - t_{j-1}} u_2 - \frac{s - t_{j-1}}{t_j - t_{j-1}} u_3) + i\lambda_2 u_4\} \\
&\quad \times \frac{1}{\sqrt{(2\pi)^4 (s_2 - t_j)(t_j - s)(s - t_{j-1})(t_{j-1} - a)}} \\
&\quad \exp\{-\frac{1}{2}(\frac{(u_4 - u_3)^2}{s_2 - t_{j-1}} + \frac{(u_3 - u_2)^2}{t_j - s} + \frac{(u_2 - u_1)^2}{s - t_{j-1}} + \frac{(u_1 - u_0)^2}{t_{j-1} - a})\} \\
&\quad dm_L(u_4) dm_L(u_3) dm_L(u_2) dm_L(u_1)\Big) d\varphi(u_0) \\
&= \exp\{-\frac{1}{2} A^2 \lambda_1^2\} \exp\{-\frac{1}{2}(s_2 - a)\lambda_2^2\} \int_{\mathbb{R}} \exp\{i\lambda_2 u_0\} d\varphi(u_0).
\end{aligned}
$$

Hence we have

$$E(\exp\{i\lambda_1 X + i\lambda_2 Y\}) = E(\exp\{i\lambda_1 X\}) E(\exp\{i\lambda_2 Y\})$$

and

$$E(\exp\{i\lambda_1 X + i\lambda_2 Z\}) = E(\exp\{i\lambda_1 X\}) E(\exp\{i\lambda_2 Z\}).$$

Hence, X and Y are stochastically independent and X and Z are stochastically independent. □

Theorem 3.2. *(The simple formula for conditional expectation) Let φ be a Borel probability measure on $\mathbb{R}$. Let $J_{\vec{t}} : C[a,b] \to \mathbb{R}^{n+1}$ be the* *function with* $J_{\vec{t}}(y) = (y(t_0), y(t_1), \cdots, y(t_n))$. *Let* F *be* m_φ*-integrable on* $C[a,b]$. *Then for* E *in* $\mathcal{B}(\mathbb{R}^{n+1})$,

$$(1) \qquad \big[(Ba) - \int_{C(\mathbb{B})} F(y) dV^{\varphi}_{J_{\vec{t}}}(y)\big](E)$$

$$(2) \qquad = \int_{J_{\vec{t}}^{-1}(E)} F(y) dm_\varphi(y)$$

$$= \int_E \big(\int_{C([a,b])} F(y - [y] + [\vec{u}]) dm_\varphi(y) \big) dP^{\varphi}_{J_{\vec{t}}}(\vec{u})$$

$$= \int_E E^{\varphi}(F|J_{\vec{t}}) dP^{\varphi}_{J_{\vec{t}}}(\vec{u}).$$

That is,

$$E^{\varphi}(F|J_{\vec{t}}) = E(F(y - [y] + [\vec{u}])).$$

Proof. Let A be in $\mathcal{B}(C[a,b])$ and let $F = \chi_A$. Then for E in $\mathcal{B}(\mathbb{R}^{n+1})$,

$$(3) \qquad \int_{J_{\vec{t}}^{-1}(E)} F(y) dm_\varphi(y)$$

$$= m_\varphi(A \cap J_{\vec{t}}^{-1}(E))$$

$$= \int_E E^{\varphi}(F|J_{\vec{t}})(\vec{u}) dP^{\varphi}_{J_{\vec{t}}}(\vec{u})$$

$$= \int_E E^{\varphi}(F(y - [y] + [\vec{u}])|J_{\vec{t}}) dP^{\varphi}_{J_{\vec{t}}}(\vec{u}).$$

Thus, the result holds for the characteristic function of A in $\mathcal{B}(C(\mathbb{B}^n))$. The general case follows by the usual arguments in Bartle integration theory. □

We know that for any bounded measurable function F on $C[a,b]$ and for any probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there is a conditional expectation $E^{\varphi}(F|J_{\vec{t}})$. What happen if the probability measure φ change ?

Theorem 3.3. *(The uniqueness theorem for giving distributions) For a bounded measurable function F on $C[a,b]$, there is a unique conditional expectation $E(F|J_{\vec{t}})$, independent of the selection of the distribution φ such that*

$$\begin{aligned}&\big[(Ba) - \int_{C[a,b]} F(x)dV^{\varphi}_{J_{\vec{t}}}(x)\big](E)\\&= \int_E E(F|J_{\vec{t}})(\vec{u})dP^{\varphi}_{J_{\vec{t}}}(\vec{u})\end{aligned}$$

for any E in $\mathcal{B}(\mathbb{R}^{n+1})$ and for any Borel probability measure φ on $\mathbb{R}$.

Proof. Let φ be a Borel probability measure on $\mathbb{R}$. Suppose B is in $\mathcal{I}$. Without loss of generality, we can write

$$B = J_{\vec{t}}^{-1}(\prod_{j=0}^{m} B_j)$$

where $B_0, B_1, \cdots, B_m$ are all Borel subsets of $\mathbb{R}$ and $\vec{s} = (s_0, s_1, \cdots, s_m)$ with $a = s_0 = t_0 < s_1 < s_2 < \cdots < s_{i_1} = t_1 < s_{i_1+1} < s_{i_1+2} < \cdots < s_{i_1+i_2} = t_2 < s_{i_1+i_2+1} < \cdots < s_{i_1+\cdots+i_n} = s_m = t_m = b$. Let $\overrightarrow{s \setminus t} = (s_0, s_1, \cdots, s_{i_1-1}, s_{i_1+1}, \cdots, s_{i_1+i_2-1}, s_{i_1+i_2+1}, \cdots, s_m)$. Set $\mathcal{M} = \{V \subset \mathcal{B}(C[a,b])|$ *there is a unique function* $E(\chi_V|J_{\vec{t}})$ *from* $\mathbb{R}^{n+1}$ *into* $\mathbb{R}$ *such that*

$$\begin{aligned}&\big[(Ba) - \int_{C[a,b]} \chi_V(x)dV^{\varphi}_{J_{\vec{t}}}(x)\big](E)\\&= \int_E E(\chi_V|J_{\vec{t}})(\vec{u})dP^{\varphi}_{J_{\vec{t}}}(\vec{u})\end{aligned}$$

holds for all Borel subsets E *of* $\mathbb{R}^{n+1}$ *and for all Borel probability measures* φ *on* $\mathbb{R}$ }. By Theorem 3.2,

$$
\begin{aligned}
&E(\chi_B|J_{\vec{t}})(\vec{u}) \\
&= \int_{C([a,b])} \chi_B(y - [y] + [\vec{u}])d\omega_\varphi(y) \\
&= \int_{C([a,b])} \prod_{k=0}^{m} \chi_{B_k}(y(s_k) - [y](s_k) + [\vec{u}](s_k))d\omega_\varphi(y) \\
&= \int_{C([a,b])} \prod_{\substack{k\neq i_j \\ j=0,1,\cdots,m}} \chi_{B_k}(y(s_k) - [y](s_k) + [\vec{u}](s_k)) \\
&\quad \times \prod_{j=0}^{m} \chi_{B_{i_j}}(y(s_{i_j}) - [y](s_{i_j}) + [\vec{u}](s_{i_j}))d\omega_\varphi(y) \\
&\overset{*}{=} \int_{\mathbb{R}^{m+1}} (\prod_{\substack{k\neq i_j \\ j=0,1,\cdots,m}} \chi_{B_k}(u_k))(\prod_{j=0}^{m} \chi_{B_{i_j}}(u_{i_j})) \\
&\quad \times W(m+1, \vec{s}; u_0, u_1, \cdots, u_m)d(\varphi \times \prod_{j=1}^{m} m_L)(u_0, (u_1, \cdots, u_m)) \\
&= (\prod_{j=0}^{m} \chi_{B_{i_j}}(u_{i_j}))\omega(J^{-1}_{\vec{s\setminus t}}(\prod_{\substack{k\neq i_j \\ j=0,1,\cdots,m}} B_k))
\end{aligned}
$$

where ω is a concrete Wiener measure on $C_0[a,b]$. If $i_{j-1} < k < i_j$, one can change the variables, using Theorem 2.1, $y(s_k) - [y](s_k) + [\vec{u}](s_k)$ replaced by $u_k - [u_{j-1} + \frac{s_{i_j} - t_{j-1}}{t_j - t_{j-1}}(u_j - u_{j-1})] + [u_{j-1} + \frac{s_{i_j} - t_{j-1}}{t_j - t_{j-1}}(u_j - u_{j-1})] = u_k$. If $k = i_j$ for some $j = 0, 1, 2, \cdots, m$ then $y(s_k) - [y](s_k) + [\vec{u}](s_k) = [\vec{u}](s_k)$. So, we obtain the equality $\overset{*}{=}$ in above. Here, we know that φ doesn't appear in the right side term of equalities in above. So, we have $\mathcal{I} \subset \mathcal{M}$. From the finite additivity of Bartle integral and conditional expectation, we obtain $\mathcal{A} \subset \mathcal{M}$. If $\langle V_n \rangle$ is an increasing sequence in $\mathcal{M}$ such that $lin_{n\to\infty} V_n = V$ then by the bounded convergence theorem for

Bartle integral and conditional expectation in [7], for E in $\mathcal{B}(\mathbb{R}^{n+1})$

$$\begin{aligned}&\big[(Ba) - \int_{C[a,b]} \chi_V(y)dV^{\varphi}_{J_{\vec{t}}}(y)\big](E)\\ &= \int_E E(\chi_V|J_{\vec{t}})(\vec{u})dP^{\varphi}_{J_{\vec{t}}}(\vec{u})\end{aligned}$$

holds, so $\mathcal{M}$ is a monotone class. Thus, the smallest monotone class $M(\mathcal{A})$ containing $\mathcal{A}$, contained in $\mathcal{M}$. By [1], we have $M(\mathcal{A}) = \mathcal{B}(C[a,b]) \subset \mathcal{M}$, that is, $\mathcal{B}(C[a,b]) = \mathcal{M}$. Therefore, we proved our theorem for $F = \chi_V$ and V in $\mathcal{B}(C[a,b])$. The general case, F is a bounded Borel measurable function, we can prove our theorem by general arguments of Bartle integral and conditional expectation. □

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