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THE PARITIES OF CONTINUED FRACTION

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Abstract. Let T be Gauss transformation on the unit interval defined by $T(x) = \{\frac{1}{x}\}$ where $\{x\}$ is the fractional part of x. Gauss transformation is closely related to the continued fraction expansions of real numbers. We show that almost every x is mod M normal number of Gauss transformation with respect to intervals whose endpoints are rational or quadratic irrational. Its connection to Central Limit Theorem is also shown.

1. Introduction

Let $X = \{x \,|\, 0 \le x < 1\}$ be the compact group of real numbers modulo 1, and let $\theta \in X$ be irrational. The numbers $j\theta, j = 0, \pm 1, \cdots$, comprise a dense subgroup of X. For each interval $I \subset X$ and $n > 0$ define $S_n = S_n(\theta, I)$ to be the numbers of integers $j, 1 \le j \le n$, such that $j\theta \in I$. By the Kronecker-Weyl theorem, we have $\lim_{n\to\infty} \frac{S_n}{n} = \mu(I)$ where μ is Lebesgue measure on X. Veech[7] is interested in the behavior of the sequence (d_n) of parities of (S_n). That is, d_n is 0 or 1 as S_n is even or odd, i.e., he investigates the existence of the limit

$$\mu_\theta(I) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n,$$

and he shows that a necessary and sufficient condition for $\mu_\theta(I)$ to exist for every interval $I \subset X$ is that θ has bounded partial quotients and show that (d_n) is evenly distributed if the length of the interval is not an integral multiple of θ modulo 1.

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Choe, Hamachi and Nakada[2] consider the case when T is a transformation on $[0,1)$ defined by $x \mapsto \{2x\}$ and shows that if I is an interval then (d_n) is evenly distributed almost everywhere with respect to Lebesgue measure except for $I = [1/6, 5/6]$.

Let $(X, \mathcal{B}, \mu)$ be a probability space. A transformation $T : X \to X$ is said to be μ-preserving if $\mu(T^{-1}E) = \mu(E)$. Sometimes we say that μ is a T-invariant measure. A transformation T is called ergodic if the constant function is the only T-invariant function and it is called weakly mixing if the constant function is the only eigenfunction. The Birkhoff's Ergodic Theorem says that if T is ergodic then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_E(T^i(x)) = \mu(E)$$

where E is a measurable subset of X and $\mathbf{1}_E$ is the indicator function of E[8].

Gauss transformation on the unit interval is defined by $T(x) = \{1/x\}$ with its invariant measure $d\mu = \dfrac{1}{\ln 2} \cdot \dfrac{1}{1+x}\, dx$, which is a weakly mixing transformation[8], where $\{x\}$ is the fractional part of x.

Since $x = 1/([1/x] + T(x))$ and $T(x) = 1/([1/T(x)] + T^2(x))$, we have

$$x = \cfrac{1}{[1/x] + \cfrac{1}{(1/[T(x)] + T^2(x))}}$$

where $[x]$ is the integral part of x.

Continuing this procedure indefinitely, we obtain

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where $a_1 = [1/x]$, $a_2 = [1/T(x)]$, and so on. Hence $a_n(x) = a_1(T^{n-1}(x))$. We write $x = [a_1(x), a_2(x), \cdots]$ and say that $[a_1(x), a_2(x), \cdots]$ is a continued fraction expansion of x. For $k \in \mathbb{N}$, $a_n(x) = k$ if and only if $T^{n-1}(x) \in (\frac{1}{k+1}, \frac{1}{k}]$. Thus the relative frequency of k in $[a_0, a_1, \cdots]$, i.e.,

$$\lim_{n\to\infty} \frac{\operatorname{card}\{i; a_i(x) = k\}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{1}{k+1}, \frac{1}{k}]}(T^i(x)) = \mu\left(\frac{1}{k+1}, \frac{1}{k}\right]$$

almost everywhere by applying the Birkhoff's Ergodic Theorem to Gauss transformation and an indicator function $\mathbf{1}_{(\frac{1}{k+1},\frac{1}{k}]}$.

In this paper, we are interested in the uniform distribution of the sequence $(d_n) \in \{0, \cdots, M-1\}$ defined by

$$d_n(x) \equiv \sum_{k=0}^{n-1} \mathbf{1}_E(T^k(x)) \pmod{M}$$

for Gauss transformations and E is a finite union of intervals whose endpoints are either rational or quadratic irrational numbers. If $(d_n(x))$ is uniformly distributed, then we say that x is *mod M normal number* with respect to given measurable set E.

2. Mod M normal numbers of Gauss transformation

To investigate the parity of the sequence $(d_n(x))$ for given measurable set E, we need the following Lemma. From now on, let's denote the unit circle group as $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Lemma 2.1. *Let G be a finite subgroup of the unit circle group $\mathbb{T}$ generated by $\exp(\frac{2\pi i}{M})$, E be a measurable subset of X and $\phi(x)$ be a measurable function defined by $\phi(x) = \exp(\frac{2\pi i}{M}\mathbf{1}_E(x))$. If the skew product transformation T_ϕ on $X \times G$ defined by $(x, g) \mapsto (T(x), \phi(x)g)$ is ergodic then the sequence*

$$d_n(x) \equiv \sum_{k=0}^{n-1} \mathbf{1}_E(T^k(x)) \pmod{M}$$

is uniformly distributed almost everywhere. Hence almost every x is mod M normal number with respect to given measurable set E.

Proof. Weyl's criterion on uniform distribution[4] says that the sequence $\exp(\frac{2\pi i}{M}d_n(x))$ is uniformly distributed if and only if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \exp^k\left(\frac{2\pi i}{M}d_n(x)\right) = 0$$

for all $1 \le k \le M-1$. Consider the skew product transformation T_ϕ on $X \times G$ defined by $T_\phi(x, g) = (Tx, \phi(x)g)$. Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \exp^k\left(\frac{2\pi i}{M}d_n(x)\right) \cdot z^k = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} U_{T_\phi}^n f(x, z)$$

where U_{T_ϕ} is an isometry on $L^2(X \times G)$ induced by T_ϕ and $f(x,z) = z^k$. Hence if T_ϕ is ergodic, then $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \exp^k(\frac{2\pi i}{M} d_n(x)) = 0$ by an application of Birkhoff's Ergodic theorem to $f(x,z) = z^k$. Hence the conclusion follows $\square$

Let $(Y, \mathcal{C}, \mu)$ be a probability space, $f \in L^1(Y, \mathcal{C}, \mu)$ and $\mathcal{B} \subset \mathcal{C}$ a sub σ-algebra. Put $\nu(B) = \int_B f\, d\mu$ for $B \in \mathcal{B}$. The Radon-Nikodym Theorem implies that there is a function $g \in L^1(Y, \mathcal{B}, \mu)$ such that $\nu(B) = \int_B g\, d\mu$ for $B \in \mathcal{B}$. We use the notation $E(f|\mathcal{B})$ for g, and call it the *conditional expectation* of f with respect to $\mathcal{B}$. Let S be a transformation defined on Y and $\mathcal{B}$ be *exhaustive* i.e., $S^{-1}\mathcal{B} \subset \mathcal{B}$ and $S^n\mathcal{B} \uparrow \mathcal{C}$ as $n \to +\infty$.

Lemma 2.2. *Let S be a measure preserving transformation on $(Y, \mathcal{C}, \mu)$, and $\mathcal{B}$ be an exhaustive σ-algebra, and let $f : Y \to \mathbb{T}$ be a $\mathcal{B}$-measurable map to the circle group $\mathbb{T}$. If $q : Y \to \mathbb{T}$ is a $\mathcal{C}$-measurable solution to the equation $f \cdot q \circ S = \lambda q$ where $\lambda \in \mathbb{C}$ and $|\lambda| = 1$, then q is $\mathcal{B}$-measurable.*

Proof. We follow the idea of Parry in [5]. Let $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$ an increasing sequence of sub σ-algebra of $\mathcal{C}$. A sequence $f_1, f_2, \cdots$ of functions in $L^1(Y)$ such that f_n is measurable with respect to $\mathcal{B}_n$ for $n = 1, 2, \cdots$ is called a *martingale* if $E(f_{n+1}|\mathcal{B}_n) = f_n$ a.e. for $n = 1, 2, \cdots$. Martingale Theorem says that every L^1-bounded martingale (i.e., $\sup_n \int_Y |f_n|\, d\mu < \infty$) converges a.e. and in L^1. Let $\mathcal{B}_n = S^n\mathcal{B}$ and $f_n = E(f|S^n\mathcal{B})$. Then by the properties of the conditional expectation operator, $E(f_{n+1}|\mathcal{B}_n) = f_n$ a.e. for $n = 1, 2, \cdots$, i.e., $E(f|S^n\mathcal{B})$ is martingale with respect to the sequence of sub σ-algebra $\{S^n\mathcal{B}\}$. Since $\mathcal{B}$ is exhaustive, Martingale Theorem says that $E(f|S^n\mathcal{B})$ converges to f a.e. and in $L^1(Y, \mathcal{C}, \mu)$ for $f \in L^1(Y, \mathcal{C}, \mu)$.

Applying the conditional expectation operator $E(\cdot|\mathcal{B})$ to the equation

$$f \cdot q \circ S = \lambda\, q \qquad (*)$$

then $f \cdot E(q \circ S|\mathcal{B}) = \lambda\, E(q|\mathcal{B})$ or $f \cdot E(q|S\mathcal{B}) \circ S = \lambda\, E(q|\mathcal{B})$. Multiplying this with $(*)$ inverted we have

$$\overline{q(y)} \cdot E(q|\mathcal{B})(y) = \overline{q(Sy)} \cdot E(q|S\mathcal{B}) \circ S(y) \quad \text{a.e.}$$

so that $\int_Y \overline{q} \cdot E(q|\mathcal{B})\, d\mu = \int_Y \overline{q} \cdot E(q|S\mathcal{B})\, d\mu$. By exactly the same argument, using $S^n\mathcal{B}$ in place of $\mathcal{B}$, we have $\int_Y \overline{q} \cdot E(q|S^n\mathcal{B})\, d\mu = \int_Y \overline{q} \cdot E(q|S^{n+1}\mathcal{B})\, d\mu$ so that $\int_Y \overline{q} \cdot E(q|\mathcal{B})\, d\mu = \int_Y \overline{q} \cdot E(q|S^n\mathcal{B})\, d\mu$. Taking limits, we get

$$\int_Y \overline{q} \cdot E(q|\mathcal{B})\, d\mu = \int_Y |q|^2\, d\mu.$$

Thus $E(q|\mathcal{B}) = q$ a.e., and q is $\mathcal{B}$-measurable. □

Let $T : (X, \mathcal{B}) \to (X, \mathcal{B})$ be a measurable transformation which is not invertible. We define a natural extension S_T of T as follows: Let

$$X_T = \{(x_0, x_1, x_2, \cdots) : x_n = T(x_{n+1}), x_n \in X, n = 0, 1, 2, \cdots\},$$

and let $S_T : X_T \to X_T$ be defined by

$$S_T((x_0, x_1, x_2, \cdots)) = (T(x_0), x_0, x_1, x_2, \cdots).$$

S_T is one to one on X_T. If T preserves a measure μ, then we can define a measure $\overline{\mu}$ on X_T by defining $\overline{\mu}$ on the cylinder sets

$$C(A_0, A_1, \cdots, A_k) = \{(x_0, x_1, x_2, \cdots) : x_0 \in A_0, x_1 \in A_1, \cdots, x_k \in A_k\}$$

where $A_i \in \mathcal{B}$ for $0 \le i \le k$ as follows:

$$\overline{\mu}(C(A_0, A_1, \cdots, A_k)) = \mu(T^{-k}A_0 \cap T^{-k+1}A_1 \cap \cdots \cap A_k).$$

Recall that if T preserves the measure μ, then S_T preserves the measure $\overline{\mu}$. Furthermore (T, μ) is ergodic if and only if $(S_T, \overline{\mu})$ is ergodic and (T, μ) is weakly mixing if and only if $(S_T, \overline{\mu})$ is weakly mixing. Note also that the natural extension is unique up to isomorphism.

Now consider the case when $X = \prod_{k=0}^{\infty}\{0, 1, 2, \cdots\}$ and T is a shift map, i.e.,

$$T((x_0, x_1, x_2, \cdots)) = (x_1, x_2, x_3, \cdots).$$

Then T is noninvertible. We will construct a natural extension of T. By the definition of natural extension, we define

$$X_T = \{\overline{x} = (y_0, y_1, y_2, \cdots) : y_i = (x_0^i, x_1^i, x_2^i, \cdots), T(y_{i+1}) = y_i,\ i \ge 0\}.$$

By virtue of the condition $T(y_{i+1}) = y_i$, $i = 0, 1, \cdots$, the sequence y_i are of the form

$$\begin{aligned}
y_0 &= (x_0, x_1, x_2, \cdots),\\
y_1 &= (x_{-1}, x_0, x_1, x_2, \cdots),\\
y_2 &= (x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots),\\
&\ \ \vdots\\
y_n &= (x_{-n}, x_{-n+1}, \cdots, x_{-1}, x_0, x_1, x_2, \cdots).
\end{aligned}$$

It is natural then to write the double sequence $\overline{x} = (y_0, y_1, y_2, \cdots)$ as one two-sided sequence:

$$\overline{x} = (\cdots, x_{-n}, x_{-n+1}, \cdots, x_{-1}, x_0, x_1, x_2, \cdots).$$

Hence the natural extension of one-sided shift map is just a two-sided shift map.

Recall that if a measure preserving transformation S on (Y, ν) is an extension of T on (X, μ), then S is isomorphic to a skew product of T. That is, there exists a measurable space Z and a measure $\hat{\mu}$ on $X \times Z$ whose projection onto X is equal to μ, and a collection of measurable map $\{f_x : x \in X\}$ on Z such that the extension S of T is isomorphic to a skew product $\hat{T}$ on $(X \times Z, \hat{\mu})$ defined by

$$\hat{T}(x, z) = (T(x), f_x(z)).$$

Proposition 2.3. *Assume that C_k be a finite sequence of intervals with rational or quadratic irrational endpoints. Let $\phi(x)$ be a measurable function defined by $\phi(x) = \exp(2\pi i \sum_{k=1}^{n} c_k \mathbf{1}_{C_k}(x))$ and G be a compact subgroup generated by the range of $\phi(x)$ where $c_k \in \mathbb{R}$ for all k. Then for Gauss transformation T, the skew product transformation T_ϕ on $X \times G$ is weakly mixing.*

Proof. Recall that if T is a weakly mixing transformation and G is a compact abelian group, then the skew product transformation T_ϕ on $X \times G$ is weakly mixing if and only if for any character $\chi \in \hat{G}, \chi \neq 1$ and $\lambda \in \mathbb{C}$, there is no measurable function $g(x)$ of modulus 1 such that

$$\chi(\phi(x)) = \lambda g(x) g(T(x))^{-1}.$$

Thus it is enough to show that if $g(x)$ is the solution of the equation $f(x) = \lambda g(x) g(T(x))^{-1}$ for a function $f(x) = \chi(\phi(x))$, then $g(x)$ has to be a constant function, since G is a compact subgroup of $\mathbb{T}$ generated by the range of $\phi(x)$[3, 6].

Note that if $x \in (0, 1]$ is rational or quadratic irrational, then for Gauss transformation T, there exist N and p such that $T^{n+p}(x) = T^n(x)$ for all $n \geq N$.

Since $\phi(x) = \exp(2\pi i \sum_{k=1}^{n} c_k \mathbf{1}_{C_k}(x))$ and each C_k is an interval, there exist a_k and b_k such that $C_k = (a_k, b_k)$ for $1 \leq k \leq n$. Let $D = \{a_k : 1 \leq k \leq n\} \cup \{b_k : 1 \leq k \leq n\} \cup \{1/k : k \in \mathbb{N}\}$ and $\mathcal{P} = \{P_j : j \in \mathbb{N}\}$ be a countable partition of $(0, 1]$ induced by the set D. Note that $\mathcal{P}$ is a generating partition of $(0, 1]$ with respect to Gauss transformation. Since every element of D is either rational or quadratic irrational and the cardinality of quadratic irrational numbers in D is finite, there exist N such that $\bigcup_{i=1}^{n} T^i(D) = \bigcup_{i=1}^{m} T^i(D)$ for all $n > m \geq N$ and the cardinality of $\bigcup_{i=1}^{N} T^i(D)$ is finite. Let $Y = \prod_0^\infty \{1, 2, \cdots\}$. Consider the following commutative diagram

$$\begin{array}{ccc} (0,1] & \xrightarrow{T} & (0,1] \\ \psi \big\downarrow & & \big\downarrow \psi \\ Y & \xrightarrow{\tilde{T}} & Y \end{array}$$

where $(\psi(x))_i = j$ if $T^i x \in P_j$ for $i \in \mathbb{N}$. Then ψ is a measure theoretic isomorphism between $((0,1], T, \mu)$ and $(Y, \tilde{T}, \nu)$ where ν is the induced measure by ψ and $\tilde{T}$ is the left shift map on the one-sided shift space Y. Now we consider the natural extension $(Y_{\tilde{T}}, S_{\tilde{T}}, \overline{\nu})$ of $(Y, \tilde{T}, \nu)$. Let $\tilde{\mathcal{P}}$ be the partition of $Y_{\tilde{T}}$ induced by zero coordinate and $\mathcal{B} = \bigvee_{i=0}^{\infty} S_{\tilde{T}}^{-i}\tilde{\mathcal{P}}$. Note that since ψ is an isomorphism, $f(x) = \lambda g(x) g(Tx)^{-1}$ holds if and only if $f(\psi(x)) = \lambda g(\psi(x)) g(S_{\tilde{T}}(\psi(x)))^{-1}$. Hence $f(\psi(x))$ and $g(\psi(x))$ are $\mathcal{B}$-measurable. Now let $\mathcal{A} = \bigvee_{i=1}^{\infty} S_{\tilde{T}}^{i}\mathcal{P}$. Since $\psi(\mathcal{P}) = \tilde{\mathcal{P}}$, $f(S_{\tilde{T}}^{-1}(\psi(x)))$ is $\mathcal{A}$-measurable and $\mathcal{A}$ is exhaustive with respect to the right shift map $S_{\tilde{T}}^{-1}$. Since $f(\psi(x)) = \lambda g(\psi(x)) g(S_{\tilde{T}}(\psi(x)))^{-1}$ can be rewritten as $f(S_{\tilde{T}}^{-1}(\psi(x))) = \lambda g(S_{\tilde{T}}^{-1}(\psi(x))) g(\psi(x))^{-1}$, i.e.,

$$f^*(S_{\tilde{T}}^{-1}(\psi(x))) g(S_{\tilde{T}}^{-1}(\psi(x))) = \lambda^* g(\psi(x)),$$

$g(\psi(x))$ is also $\mathcal{A}$-measurable by applying Lemma 2.2 to the map $S_{\tilde{T}}^{-1}$. Hence $g(\psi(x))$ is $\mathcal{A} \cap \mathcal{B}$-measurable. Since $(Y, \tilde{T}, \nu)$ is isomorphic to Gauss transformation, the natural extension $(Y_{\tilde{T}}, S_{\tilde{T}}, \overline{\nu})$ of $(Y, \tilde{T}, \nu)$ can be regarded as an extension of Gauss transformation T. Thus $g(x)$ is measurable with respect to the sub σ-algebra generated by the set $\bigcup_{i=1}^{N} T^i D$ on $(0,1]$. Note that the cardinality of $\bigcup_{i=1}^{N} T^i D$ is finite. Hence there exists k such that both $f(x)$ and $g(x)$ are constant on $I = (\frac{1}{k+1}, \frac{1}{k}]$, and there exists x_0 such that both x_0 and $T(x_0)$ are in I. Hence $f(x) \equiv \lambda$ on I. Since $f(x) = \lambda g(x) g(T(x))^{-1}$, we have $g(T(x)) = f^{-1}(x) \lambda g(x)$ and specially $g(T(x)) = g(x)$ on I. Since $T(I) = (0,1]$, $g(x)$ has to be constant on $(0,1]$. Hence T_ϕ is weakly mixing. □

The following theorem is a direct consequence of Lemma 2.1 and Proposition 2.3, since weak mixing implies ergodicity.

Theorem 2.4. *Let T be Gauss transformation and E be a finite union of intervals whose endpoints are either rational or quadratic irrational. Then almost all points are mod M normal numbers with respect to E.*

3. Central limit theorem for Gauss transformation

In this section, we investigate the applicability of the Central Limit Theorem to a class of step functions. The following Lemma is well-known[1].

Lemma 3.1 (Central Limit Theorem). *Let T be a piecewise expanding transformation on $[0,1)$ with $g(x) \equiv \frac{1}{|T'(x)|}$ being a function of bounded variation and μ be the T-invariant absolutely continuous measure. Assume that (X, T, μ) is weakly mixing and $f(x)$ be a bounded variation function. If the equation*

$$f = C + g \circ T - g$$

does not have a solution $C \in \mathbb{R}$, then

$$\sigma^2 = \lim_{n\to\infty} \int \left(\frac{S_n f - n\mu(f)}{\sqrt{n}} \right)^2 d\mu > 0$$

and, for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} \mu\left\{ x : \frac{S_n f(x) - n\mu(f)}{\sigma\sqrt{n}} \leq z \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-t^2/2)dt,$$

where $S_n f(x) = \sum_{i=0}^{n-1} f(T^i(x))$ and $\mu(f) = \int_X f d\mu$

Theorem 3.2. *For Gauss transformation, if $f(x)$ is a step function whose discontinuity points are either rational or quadratic irrational, then it satisfies the Central Limit Theorem with respect to its invariant absolutely continuous measure.*

Proof. Recall that a function $g(x)$ is called a *multiplicative coboundary* if the equation $g(x)h(Tx) = h(x)$, $|h(x)| = 1$ has a solution. If the equation $f = C + g \circ T - g$ has a solution, then

$$\exp(2\pi i f(x)) = \exp(2\pi i C) \exp(2\pi i g \circ T(x)) \overline{\exp(2\pi i g(x))} \qquad (**)$$

has a solution and $h(x) = \overline{\exp(2\pi i C)} \exp(2\pi i f(x))$ is a multiplicative coboundary with cobounding function $h(x) = \overline{\exp(2\pi i g(x))}$. Hence if f does not have a solution of $(**)$ then f does not have a solution of the equation

$$f(x) = C + g \circ T(x) - g(x).$$

By Proposition 2.3 and Lemma 3.1, we can apply the Central Limit Theorem to a step function whose discontinuity points are either rational or quadratic irrational.

□

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