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FINITE TYPE RULED HYPERSURFACES IN LORENTZ-MINKOWSKI SPACE

Dong-Soo Kim* and Young Ho Kim

Abstract. In this article, we study ruled hypersurfaces in a Lorentz-Minkowski space which has finite type immersion. As a result, we give a complete classification of ruled hypersurfaces of finite type immersion into a Lorentz-Minkowski space $\mathbb{L}^m$.

1. Introduction

In late 1970's B.-Y. Chen ([2, 3]) introduced the notion of finite type immersion into a Euclidean space. A lot of works have been done in this field of study since then. He also extended the notion of finite type immersion of submanifolds into a pseudo-Euclidean space in 1980's. It can be defined formally in the following: A pseudo-Riemannian submanifold M of an m-dimensional pseudo-Euclidean space $\mathbb{E}^m_s$ with signature $(s, m-s)$ is said to be of *finite type* if its position vector field X can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $X = X_0 + \sum_{i=1}^{k} X_i$, where X_0 is a constant map, $X_1, \cdots, X_k$ non-constant maps such that $\Delta X_i = \lambda_i X_i, \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, k$. If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are different, then M is said to be of *k-type*.

Ruled surfaces in Euclidean space of finite type were studied by B.-Y. Chen et al. ([4]). On the other hand, many authors([6,7,8]) studied ruled surfaces of finite type in m-dimensional Lorentz-Minkowski space. In particular, F. Dillen et al.([6]) classified ruled surfaces of finite type in 3-dimensional Lorentz-Minkowski space as an open portion of minimal, circular or hyperbolic cylinders and isoparametric surfaces with null rulings.

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For ruled hypersurfaces in Euclidean space of finite type, F. Dillen classified as follows([5]):

Proposition. A ruled hypersurface M in Euclidean space is of finite type if and only if M is a part of a hyperplane, a circular cylinder, a helicoid H^2 in H^3, a cone H^3 with top P on a minimal ruled surface in a 3-sphere, centered at P, or a cylinder on H^2 or H^3.

Therefore, we may raise a natural question: What kind of ruled hypersurfaces have finite type immersion into a Lorentz-Minkowski space?

In this article, we study ruled hypersurfaces in a Lorentz-Minkowski m-space $\mathbb{L}^m$, and we give the complete classification theorem of such ruled hypersurfaces.

Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless otherwise mentioned.

2. Finite type ruled hypersurfaces

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space of signature $(s, m-s)$ with the metric $ds^2 = -dx_1^2 - \cdots - dx_s^2 + dx_{s+1}^2 + \cdots + dx_m^2$, where $(x_1, x_2, \cdots, x_m)$ denotes the standard coordinate system in $\mathbb{E}^m_s$. In particular, for $m \geq 2$, $\mathbb{E}^m_1$ is called a *Lorentz-Minkowski m-space.* For simplicity, we denote $\mathbb{E}^m_1$ by $\mathbb{L}^m$ from now on.

Let $x : M \longrightarrow \mathbb{E}^m_s$ be an isometric immersion of an n-dimensional pseudo-Riemannian submanifold M into $\mathbb{E}^m_s$. From now on, a submanifold in $\mathbb{E}^m_s$ always means pseudo-Riemannian, that is, the induced metric on the submanifold is non-degenerate.

For the components g_{ij} of the induced pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ on M from that of $\mathbb{E}^m_s$ we denote by (g^{ij}) (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix (g_{ij}). Then, the Laplacian Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x^j}\right). \tag{2.1}$$

Let M^m be a ruled hypersurface in $\mathbb{L}^{m+1}$ with nondegenerate rulings. Let $x(s)$ be an orthogonal trajectory of M^m. Then we may assume that $x(s)$ is parametrized by arc length. Let $\{e_2(s), \cdots, e_m(s)\}$ be a set of orthonormal vector fields along x such that $\{e_2(s), \cdots, e_m(s)\}$ spans the nondegenerate ruling of M^m through $x(s)$. As in the Euclidean case([1]), the set $\{e_2(s), \cdots, e_m(s)\}$ can be chosen such that for all i and j

$$\langle e_i'(s), e_j(s)\rangle = 0. \tag{2.2}$$

Hence we can give a parametrization of M^m by

$$X(s, t_2, \cdots, t_m) = x(s) + \sum_{i=2}^{m} t_i e_i. \tag{2.3}$$

If we define ϵ_1 and q by $\epsilon_1 = \langle x'(s), x'(s)\rangle$ and $q = |\langle X_s, X_s\rangle|$, respectively, then for sufficiently small t_i we have

$$q = 1 + 2\epsilon_1 \sum_{i=2}^{m} t_i \langle x', e_i'\rangle + \epsilon_1 \sum_{i=2}^{m} t_i t_j \langle e_i', e_j'\rangle, \tag{2.4}$$

which is a polynomial in $t = (t_2, \cdots, t_m)$ with functions in s as coefficients. The Laplacian Δ of M^m can be expressed as follows:

$$\Delta = -\sum_{i=2}^{m} \epsilon_i \frac{\partial^2}{\partial t_i^2} - \frac{\epsilon_1}{q}\frac{\partial^2}{\partial s^2} + \frac{\epsilon_1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{2q}\sum_{i=2}^{m} \epsilon_i \frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i}, \tag{2.5}$$

where for each i, ϵ_i denotes $\langle e_i, e_i\rangle = \pm 1$. The degree of q is at most 2. We divide by three cases according to the degree of q.

Case 1. Suppose that $\deg(q) = 0$. Then (2.4) shows that for each j, $e_j'(s)$ is orthogonal to itself and is contained in the normal space $N_{x(s)}M$ of M^m at $x(s)$. Since the normal space is nondegenerate and of 1-dimensional, it follows that each e_j is constant. Hence we see that M^m is a cylinder over $x(s)$. Since $x(s)$ is a plane curve which is also of finite type([8, Theorem 3.1]) , M^m is one of the following:

$$E_s^m, E_s^{m-1} \times S^1(r), E^{m-1} \times S_1^1(r), E_s^{m-1} \times H^1(r), \tag{2.6}$$

where $s = 0, 1$.

Case 2. Suppose that $\deg(q) = 1$. Then (2.4) shows that $\langle e_i', e_j'\rangle = 0$ for all i, j. In particular, $\langle e_i', e_i'\rangle = 0$ for all i. If $e_i' = 0$ for all i, then it follows from (2.4) that $\deg(q) = 0$, which is a contradiction. Hence we may assume that $e_2' \neq 0$. Then for each $i = 3, \cdots, m$, e_i' is a null vector which is orthogonal to e_2'. Thus for some function $a_i(s)$, we have $e_i'(s) = a_i(s)e_2'(s)$. Since for each $i = 2, \cdots, m$, e_i is orthogonal to the null vector e_2', we see that $\epsilon_i = 1$. It follows from (2.4) and (2.5) that

$$\Delta X = \frac{P_1(t)}{q^2(t)}, \tag{2.7}$$

where $P_1(t)$ is a polynomial of degree ≤ 2. The proof of the following lemma is straightforward.

Lemma 1. If P is a polynomial in $t = (t_2, \cdots, t_m)$ with functions in s as coefficients and $\deg(P) = d$, then we have for $n = 1, 2, \cdots$

$$\Delta(\frac{P(t)}{q^n}) = \frac{\tilde{P}(t)}{q^{n+3}}, \tag{2.8}$$

where $\tilde{P}$ is a polynomial in t with functions in s as coefficients and $\deg(\tilde{P}) \leq d + 2$.

If M^m is of k-type, then there exist constants $c_1, \cdots, c_k$ such that([2, p.256])

$$\Delta^k X + c_1 \Delta^{k-1} X + \cdots + c_{k-1} \Delta X + c_k (X - X_0) = 0. \tag{2.9}$$

We know that $X - X_0$ is a linear function in t with functions in s as coefficients. By applying Lemma 1, for $r = 1, 2, \cdots, k$ we get

$$\Delta^r X = \frac{P_r(t)}{q^{3r-1}}, \quad \deg(P_r) \leq 2r. \tag{2.10}$$

Hence, by counting the degree of each term in (2.9), we see that $c_k = 0$. Since the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ are different, we get $c_{k-1} \neq 0$. Thus the sum in (2.9) never vanish, unless $\Delta X = 0$. Therefore M^m is a minimal hypersurface.

Case 3. Finally, suppose that $\deg(q) = 2$. It follows from (2.4) and (2.5) that

$$\Delta X = \frac{P_1(t)}{q^2(t)}, \tag{2.11}$$

where $P_1(t)$ is a polynomial of degree ≤ 3. The proof of the following lemma is also straightforward.

Lemma 2. If P is a polynomial in $t = (t_2, \cdots, t_m)$ with functions in s as coefficients and $\deg(P) = d$, then we have for $n = 1, 2, \cdots$

$$\Delta(\frac{P(t)}{q^n}) = \frac{\tilde{P}(t)}{q^{n+3}}, \tag{2.12}$$

where $\tilde{P}$ is a polynomial in t with functions in s as coefficients and $\deg(\tilde{P}) \leq d + 4$.

If M^m is of k-type, then there exists constants $c_1, \cdots, c_k$ such that

$$\Delta^k X + c_1 \Delta^{k-1} X + \cdots + c_{k-1} \Delta X + c_k (X - X_0) = 0. \tag{2.13}$$

By applying Lemma 2, for $r = 1, 2, \cdots, k$ we get

$$\Delta^r X = \frac{P_r(t)}{q^{3r-1}}, \quad \deg(P_r) \leq 4r - 1. \tag{2.14}$$

Hence, as in Case 2, we see that c_k must vanish and $c_{k-1} \neq 0$. Thus the sum in (2.13) never be zero, unless $\Delta X = 0$. Therefore M^m is a minimal hypersurface.

Summarizing above, we get the following classification theorem:

Theorem 3. *Let M^m be a ruled hypersurface in a Lorentz-Minkowski space $\mathbb{L}^{m+1}$ with nondegenerate rulings. If M^m is of finite type, then M^m is either minimal or an open part of one of the following:*

$$E_s^m, E_s^{m-1} \times S^1(r), E^{m-1} \times S_1^1(r), E_s^{m-1} \times H^1(r), \tag{2.15}$$

where $s = 0, 1$.

Remark. For finite type ruled surfaces in a Lorentz-Minkowski space $\mathbb{L}^{m+1}$ with degenerate rulings, see [8].

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Dong-Soo Kim
Department of Mathematics,
Chonnam National University,
Kwangju 500-757, Korea
E-mail: dosokim@chonnam.ac.kr

Young Ho Kim
Department of Mathematics,
Kyungpook National University,
Taegu 702-701, Korea
E-mail: yhkim@knu.ac.kr