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FIXED POINT THEORY FOR INWARD SET VALUED MAPS IN HYPERCONVEX METRIC SPACES

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Abstract. In this paper, we first introduce inwards set valued maps in hyperconvex metric spaces. Then we present fixed point theory for continuous condensing inward set valued maps.

1. Introduction and Preliminaries

Let X and Y be topological spaces with $A \subseteq X$ and $B \subseteq Y$. Let $F : X \multimap Y$ be a multimap with nonempty values. The image of A under F is the set $F(A) = \bigcup_{x \in A} F(x)$ and the inverse image of B under F is $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. Now F is said to be:

(i) lower semicontinuous if, for each open set $B \subseteq Y$, $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is open in X,

(ii) upper semicontinuous, if for each closed set $B \subseteq Y$, $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in X,

(iii) continuous if, F is both lower semicontinuous and upper semicontinuous.

Let (M, d) be a metric space and $B(x, r) = \{y \in M : d(x, y) \leq r\}$, denotes the closed ball with center x and radius r. Let

$$co(A) = \bigcap \{B \subseteq M : B \text{ is a closed ball in } M \text{ such that } A \subset B\}.$$

If $A = \mathrm{co}(A)$, we say that A is admissible subset of M. Note, $\mathrm{co}(A)$ is admissible and the intersection of any family of admissible subsets of M is admissible. Note an admissible set is bounded. The following definition of a hyperconvex metric space is due to Aronszajn and Pantichpakdi [3].

Definition 1.1. A metric space (M, d) is said to be a *hyperconvex metric space* if for any collection of points x_α of M and any collection r_α of non-negative real numbers with $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, we have

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset.$$

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The simplest examples of hyperconvex spaces are finite dimensional real Banach space and l_∞ endowed with the maximum norm. It is well known that for any hyperconvex metric space M there exists an index set I and a natural isometric embedding from M into $l_\infty(I)$. Also there exists a nonexpansive retraction $r : l_\infty(I) \to M$. Henceforth let $r : l_\infty(I) \to M$ denotes an arbitrary nonexpansive retraction and (note M is isometrically embedded in $l_\infty(I)$) $conv(a,b) = \{(1-t)a + tb : 0 \le t \le 1\}$. For each $a, b \in M$ we have

$$\begin{aligned}
&r(conv(a,b)) \\
&\subseteq r\left(\bigcap\{B \subseteq l_\infty(I) : B \text{ is a closed ball in } l_\infty(I) \text{ such that } conv(a,b) \subseteq B\}\right) \\
&\subseteq \bigcap\{B \subseteq M : B \text{ is a closed ball in } M \text{ such that } \{a,b\} \subseteq B\} \\
&= co(a,b).
\end{aligned}$$

Thus,

$$r(conv(a,b)) \subseteq co(a,b).$$

Definition 1.2. A subset E of a metric space M is said to be *proximinal* if the intersection $E \cap B(x, d(x,E))$ is nonempty for each $x \in M$.

Lemma 1.3 ([5], page 398)**.** *If E is admissible subset of a hyperconvex metric space M, then E is proximinal in M.*

The following best approximation theorem, which will be used in the next section, is due to Kirk et al. [6].

Theorem 1.4. *Let X be a compact admissible subset of a hyperconvex metric space (M,d). Suppose that $F : X \multimap M$ is continuous with admissible values. Then, there exists a point $x_0 \in X$, such that*

$$d(x_0, F(x_0)) = \inf_{x \in X} d(x, F(x_0)).$$

Moreover, if $x_0 \notin F(x_0)$, x_0 must be a boundary point of X.

2. Fixed point theory

In his thesis [4], Halpern initiated the study of fixed point theorems for continuous single valued outward mappings in the setting of topological vector spaces. Now, we introduce inward set valued maps in hyperconvex metric spaces.

Definition 2.1. Let C be a subset of hyperconvex metric space (M,d). For each $x \in M$, let the inward set of C at x, $I_C(x)$, be defined by

$$\begin{aligned}
I_C(x) = \cup_{y \in C} \mathrm{co}\{x,y\} \cup \{z \in M : \exists\, r : l_\infty(I) \to M : \\
(r(\mathrm{conv}\{x,z\}) \backslash \{x\}) \cap C \neq \emptyset\}.
\end{aligned}$$

Note $C \subseteq I_C(x)$ for each $x \in M$. A set valued map $F : C \multimap M$ is said to be *inward* if, for each $x \in C$,

$$F(x) \cap I_C(x) \neq \emptyset.$$

Remark 2.2. For the topological vector space X, the inward set of C at x is defined by

$$\begin{aligned} I_C(x) &= \{y \in X : \ x + \lambda(y - x) \in C, \text{ for some } \lambda > 0\} \\ &= \bigcup_{y \in C} \operatorname{conv}\{x, y\} \cup \{z \in X : (\operatorname{conv}\{x, z\})\backslash\{x\}) \cap C \neq \emptyset\}. \end{aligned}$$

The following is a new best approximation theorem in hyperconvex metric spaces.

Theorem 2.3. *Let X be a nonempty, compact admissible subset of hyperconvex metric space (M, d). Suppose that $F : X \multimap M$ is continuous set valued map with admissible values. Then either*

(i) *there exists an $x_0 \in X$ such that $x_0 \in F(x_0)$; or*

(ii) *there exists an $x_0 \in X$ such that $x_0 \in \partial X$ and*

$$0 < d(x_0, F(x_0)) \leq d(y, F(x_0)), \ \forall \ y \in I_X(x_0).$$

Proof. By Theorem 1.4, there is an $x_0 \in X$ such that

$$d(x_0, F(x_0)) = \inf_{x \in X} d(x, F(x_0)). \tag{2.1}$$

If $d(x_0, F(x_0)) = 0$, then x_0 is a fixed point of F, suppose $d(x_0, F(x_0)) > 0$. We now show that

$$d(x_0, F(x_0)) \leq d(y, F(x_0)), \ \forall \ y \in I_X(x_0).$$

On the contrary, suppose there exists an $z \in I_X(x_0)\backslash X$ such that $d(z, F(x_0)) < d(x_0, F(x_0))$. Then, there exists an $x \in X$ such that $x \in r(\operatorname{conv}\{x_0, z\})\backslash\{x_0\}$ where $r : l_\infty(I) \to M$ is a nonexpansive retraction. Since $r(w) = w$ for each $w \in M$ (note M is isometrically embedded in $l_\infty(I)$), then $r(x_0) = x_0 \neq x$. Hence $x = r((1-t)x_0 + tz)$ for some $0 < t \leq 1$. Since $F(x_0)$ is admissible, then by Lemma 1.3, (pick $x = x_0$ and $x = z$ in Definition 1.2), there exist $y_1, y_2 \in F(x_0)$ such that

$$d(x_0, F(x_0)) = d(x_0, y_1) \text{ and } d(z, F(x_0)) = d(z, y_2).$$

We have

$$\begin{aligned} d(x_0, F(x_0)) &= (1-t)d(x_0, F(x_0)) + td(x_0, F(x_0)) \\ &> (1-t)d(x_0, F(x_0)) + td(z, F(x_0)) \\ &= (1-t)d(x_0, y_1) + td(z, y_2). \end{aligned}$$

Since M is isometrically embeded in l_∞ and r is nonexpansive, then

$$\begin{aligned} (1-t)d(x_0, y_1) + td(z, y_2) &= (1-t)\|x_0 - y_1\|_\infty + t\|z - y_2\|_\infty \\ &= \|((1-t)x_0 + tz) - ((1-t)y_1 + ty_2)\|_\infty \\ &\geq d(r((1-t)x_0 + tz), r((1-t)y_1 + ty_2)) \\ &= d(x, r((1-t)y_1 + ty_2)). \end{aligned}$$

Since $F(x_0)$ is admissible we have

$$r((1-t)y_1 + ty_2)) \in co(y_1, y_2) \subseteq F(x_0),$$

and so

$$d(x, r((1-t)y_1 + ty_2)) \geq d(x, F(x_0)).$$

Therefore

$$d(x, F(x_0)) < d(x_0, F(x_0)),$$

which contradicts (2.1). □

As a corollary, we get the following fixed point theorem for inward set valued maps.

Corollary 2.4. *Let X be a nonempty, compact admissible subset of a hyperconvex metric space (M, d). Suppose that $F : X \multimap M$ is continuous set valued map with admissible values. In addition, suppose*

$$F(x) \cap I_X(x) \neq \emptyset, \ \forall x \in X.$$

Then F has a fixed point.

Proof. On the contrary, we suppose that F is fixed point free. From Theorem 2.3, there exists an $x_0 \in X$ such that

$$0 < d(x_0, F(x_0)) \leq d(y, F(x_0)), \ \forall \ y \in I_X(x_0).$$

Take $y_0 \in F(x_0) \cap I_X(x_0)$, and note

$$0 < d(x_0, F(x_0)) \leq d(y_0, F(x_0)) = 0,$$

and is a contradiction. □

The following Lemma is a particular case of Lemma 2.1, in [2], and we give its proof for completeness.

Lemma 2.5. *Let X be a nonempty admissible subset of a hyperconvex metric space M and $F : X \multimap M$. If $\emptyset \neq Q \subseteq X$, then either*

(i) *$M = co(F(M \cap X) \cup Q) = co(F(X) \cup Q)$ or;*
(ii) *there exists an admissible set $K = K(F, Q)$ with $Q \subseteq K \subseteq M$ and*

$$K = co(F(K \cap X) \cup Q).$$

Proof. Suppose $M \neq co(F(X) \cup Q)$. Let

$$\mathcal{F} = \{A \subseteq M : \ A \text{ is admissible and } co(F(X \cap A) \cup Q) \subseteq A\}.$$

Let $A_0 = co(F(X) \cup Q)$. Then A_0 is admissible and

$$co(F(X \cap A_0) \cup Q) \subseteq co(F(X) \cup Q) = A_0,$$

so $A_0 \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Define a partial order by inverse inclusion, that is, for A, $B \in \mathcal{F}$, $A \leq B \Leftrightarrow B \subseteq A$. Let $\mathcal{C}$ be any chain in $\mathcal{F}$. Put $N = \bigcap_{A \in \mathcal{C}} A$. Since each $A \in \mathcal{C}$ is admissible and contains Q, we infer that N is admissible and contains Q, for all $A \in \mathcal{C}$, and it follows from

$$F(N \cap X) \cup Q \subseteq F(A \cap X) \cup Q$$

that $co(F(N \cap X) \cup Q) \subseteq co(F(A \cap X) \cup Q) \subseteq A$ and so

$$co(F(N \cap X) \cup Q) \subseteq \bigcap_{A \in \mathcal{C}} A = N.$$

Thus $N \in \mathcal{F}$ and N is an upper bound of $\mathcal{C}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element, say K. We claim that $co(F(K \cap X) \cup Q) = K$. In fact, put $K_0 = co(F(K \cap X) \cup Q)$. It is obvious that K_0 is admissible and contains Q. Furthermore, since

$$co(F(K_0 \cap X) \cup Q) \subseteq co(F(K \cap X) \cup Q) = K_0,$$

we have $K_0 \in \mathcal{F}$ and $K_0 \geq K$. By the maximality of K, we conclude that $K = K_0$, that is

$$co(F(K \cap X) \cup Q) = K.$$

□

The proof of the following theorem, follows along the lines of Theorem 2.1 in [1].

Theorem 2.6. *Let X be a nonempty admissible subset of a hyperconvex metric space M. Assume $F : X \multimap M$, is a continuous set valued map with admissible values and*

$$F(x) \cap I_X(x) \neq \emptyset, \quad \forall\, x \in X. \tag{2.2}$$

Then F has a fixed point provided the following condition holds:

$$\begin{aligned} &\textit{for any } x_0 \in X, \textit{ and } A \subseteq M \textit{ with } A = co(F(A \cap X) \cup \{x_0\}) \\ &\textit{we have that } \overline{A} \textit{ is compact.} \end{aligned} \tag{2.3}$$

Proof. Putting $Q = \{x_0\}$ in Lemma 2.5, we obtain that either $M = co(F(M \cap X) \cup \{x_0\}) = co(F(X) \cup \{x_0\})$ or there exists an admissible set $K \subseteq M$ with $K = co(F(K \cap X) \cup \{x_0\})$. In the first case, (2.3) implies that M is compact. Since X is a closed subset of M, then X is compact and the conclusion follows from Corollary 2.4. In the second case, since (2.3) holds, then $K = \overline{K}$ is compact. We now concentrate our study on $F|_{K \cap X}$. Notice $F|_{K \cap X} : K \cap X \multimap M$ is continuous with $K \cap X$ compact (note X is closed since it is admissible) and admissible. Now, we claim

$$F(x) \cap I_{K \cap X}(x) \neq \emptyset, \qquad \forall\, x \in K \cap X. \tag{2.4}$$

If our claim is true, then Corollary 2.4, implies $F|_{K \cap X}$ has a fixed point, so we are finished. It remains to prove the claim. Let $x \in K \cap X$, we have

$$F(x) \subseteq F(K \cap X) \subseteq K.$$

From (2.2), there is a $y \in F(x) \cap I_X(x)$. If $y \in X$ (note then $y \in K$ by above so $y \in K \cap X$) , then $y \in F(x) \cap I_{K\cap X}(x) \neq \emptyset$. If $y \notin X$ there is a nonexpansive retraction $r : l_\infty(I) \to M$ an $z \in X$ with $z \in r(\mathrm{conv}\{x, y\})\backslash\{x\}$. Now, since K is admissible we have

$$z \in r(\mathrm{conv}\{x, y\}) \subseteq co(x, y) \subseteq K.$$

Consequently,

$$z \in r(\mathrm{conv}\{x, y\})\backslash\{x\} \cap (K \cap X),$$

so $y \in I_{K\cap X}(x)$ and (2.4) holds. □

Theorem 2.7. *Let X be a nonempty admissible subset of a hyperconvex metric space M. Assume $F : X \multimap M$, is a continuous set valued map with admissible values and*

$$F(x) \cap I_X(x) \neq \emptyset, \quad \forall\, x \in X.$$

Then, F has a fixed point provided that the following condition holds:

$$\text{(2.5)}\qquad \begin{aligned} &\textit{whenever } x_0 \in X,\ A \subseteq M,\ F(A \cap X) \subseteq A \textit{ and}\\ &A \setminus co(F(A \cap X)) \subseteq \{x_0\} \textit{ we have that } \overline{A} \textit{ is compact.}\end{aligned}$$

Proof. Choose $x_0 \in X$ and let $A = \bigcup_{i\geq 0} F^{(i)}(x_0)$ where $F^{(0)}(x_0) = \{x_0\}$ and $F^{(i+1)}(x_0) = F(F^{(i)}(x_0) \cap X)$. Then $F(A \cap X) \subseteq A$ and $A \setminus co(F(A \cap X)) \subseteq \{x_0\}$, so $\overline{A}$ is compact. Define $G : \overline{A \cap X} \multimap \overline{A \cap X}$ by $G(x) = F(x) \cap \overline{A \cap X}$. Since F is continuous and $\overline{A}$ is compact, it is easy to see that $G(x) \neq \emptyset, \forall\, x \in \overline{A}$. Put

$$\mathcal{A} = \{Y :\ Y \text{ is a nonempty closed subset of } \overline{A \cap X} \text{ and } \ G(Y) \subseteq Y\}.$$

Since $\overline{A \cap X} \in \mathcal{A}$, $\mathcal{A} \neq \emptyset$. Define a partial order $\leq$ on $\mathcal{A}$ by $A \leq B \Leftrightarrow B \subseteq A$. Let $\mathcal{C}$ be any chain in $\mathcal{A}$ and put $N = \bigcap_{L\in\mathcal{C}} L$. Now N is an upper bound of $\mathcal{C}$ and so, by Zorn's Lemma, $\mathcal{A}$ has a maximal element, say Q. Since F is continuous, so is G, and this with the compactness of $\overline{A}$ guarantees that $G(Q)$ is compact. Putting $Y = G(Q)$ and noting that $G(Y) = G(G(Q)) \subseteq G(Q) = Y$, the maximality of Q gives us that $Q = Y$. Thus

$$Q = G(Q) = F(Q) \cap \overline{A \cap X} \subseteq F(Q).$$

If $M = co(F(M \cap X) \cup Q) = co(F(X) \cup Q)$ then M is compact by (2.5). Hence X is compact and conclusion follows from Corollary 2.4. If $M \neq co(F(X) \cup Q)$, let $K = K(F, Q)$ be the admissible subset of M described in Lemma 2.5, so $K = co(F(K \cap X) \cup Q)$. Since $Q \subseteq F(Q) = F(Q \cap X) \subseteq F(K \cap X)$ (note, $Q \subseteq X$), we have $K = co(F(K \cap X))$ and so we have shown that there exists an admissible subset $K \subseteq M$ such that $K\backslash co(F(K \cap X)) = \emptyset \subseteq \{x_0\}$. Now (2.5) implies that $K = \overline{K}$ is compact subset of M. Now, the same reasoning as in Theorem 2.6 gives the conclusion. □

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