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IDEALS AND DIRECT PRODUCT OF ZERO SQUARE RINGS

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Abstract. We consider associative ring R (not necessarily commutative). In this paper the concepts: zero square ring of type-1/type-2, zero square ideal of type-1/type-2, zero square dimension of a ring R were introduced and obtained several important results. Finally, some relations between the zero square dimension of the direct sum of finite number of rings; and the sum of the zero square dimension of individual rings; were obtained. Necessary examples were provided.

1. Introduction

This section contains some definitions and results from the literature that are useful in the later sections. Throughout this paper R stands for an associative ring (not necessarily commutative). Stanley [3] calls a ring R a zero square if $x^2 = 0$ for all $x \in R$. Zero square rings were also studied by Vasantha Kandaswamy [9, 10]. As it was discussed by Stanley (i) every zero square ring is anti commutative (that is, $xy = -yx$ for all x, y); and (ii) a zero square ring R is commutative if and only if $2R^2 = 0$.

The concept finite dimension in modules was introduced by Goldie [1] and later it was studied by Reddy and Satyanarayana [4], Satyanarayana [5], Satyanarayana, Syam Prasad, Nagaraju [6]. This dimension concept explains about the dimension related to one sided ideals, in case of associative (not necessarily commutative) rings. Satyanarayana, Nagaraju, Murugan, Godloza [8] introduced the concept of dimension related to two sided ideals in associative rings, and it is also observed that the dimension of a ring with respect to two sided ideals is different from the dimension of the module R (when the given ring R is considered as a module over itself).

Let I, J be two ideals of R such that $I \subseteq J$. (i) We say that I is *essential* (or *ideal essential*) in J if it satisfies the following condition: K is an ideal of R, $K \subseteq J$, $I \cap K = (0)$ imply $K = (0)$. (ii) If I is essential in J and $I \neq J$, then

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we say that J is a *proper essential extension* of I. If I is essential in J, then we denote this fact by $I \leq_e J$. A non-zero ideal I of R is said to be *uniform* if B is a non-zero ideal of R, and $B \subseteq I$ implies $B \leq_e I$.

We say that R has *finite dimension on ideals* (FDI, in short) if R do not contain infinite number of non-zero ideals whose sum is direct.

Theorem 1.1 (Corollary 3.5 [8]). *If R is a ring with FDI, then the following* (i)–(ii) *are true:*

(i) *(Existence) There exist uniform (two sided) ideals $U_1, U_2, ..., U_n$ in R whose sum is direct and essential in R;*
(ii) *(Uniqueness) If V_i, $1 \leq i \leq k$, are uniform ideals of R whose sum is direct and essential in R, then $k = n$.*

The number n of the above Theorem is independent of the choice of the uniform ideals, and this number n is called the *dimension* of R (it is denoted by $\dim R$).

Theorem 1.2 (Lemma 1.7(ii) [8]). *If R_i, $1 \leq i \leq k$ are rings and I_i is an ideal of R_i for $1 \leq i \leq k$, then the following two conditions are equivalent:*

(i) $I_i \leq_e R_i$, $1 \leq i \leq k$;
(ii) $I_1 \oplus I_2 \oplus ... \oplus I_k \leq_e R_1 \oplus R_2 \oplus ... \oplus R_k$.

From Theorems 1.1 and 1.2, we get the following theorem.

Theorem 1.3. *If R_i, $1 \leq i \leq k$ are rings with FDI, then* $\dim(R_1 \oplus R_2 \oplus ... \oplus R_k) = \dim R_1 + \dim R_2 + ... + \dim R_k$.

For other preliminary concepts we refer Lambek [2].

The ideal generated by an element $x \in R$ is denoted by $< x >$. We do not present the proofs of some results in this paper when they are simple or parallel to those results in the literature on ring theory.

In Section-2, we defined and studied the concepts zero square ring of type-1/type-2. Zero square ring of type-2 is same as the zero square ring studied by the earlier authors. We presented some illustrations. Every zero square ring of type-1 is a zero square ring of type-2, but the converse need not be true, in general. In Section-3, we defined and studied zero square ideal of type-1/type-2. We observed that the class of all zero square rings R of type-1 for which $R^2 \nsubseteq I$ for all non-zero ideals I of R, is homomorphically closed. In Section-4, we proved that the direct product of zero square rings R_i, $1 \leq i \leq k$ of type-1 is also a zero square ring of type-1, but the converse need not be true, in general. We obtained some important consequences. In Section-5, we introduced zero square dimension of type-1/type-2. We considered a class of rings R and obtained some relations between the concepts dimension of R, zero square dimension of type-1/type-2. Finally, we applied this result for the direct sum of rings.

2. Zero Square Rings

Definition 2.1. (i) A ring R is said to be a *zero square ring of type-1* if $x^2 = 0$ for all $x \in R$, and there exists two elements a, $b \in R$ such that $ab \neq 0$.

(ii) A ring R is said to be a *zero square ring of type-2* if $x^2 = 0$ for all $x \in R$.

Zero square rings of type-2 are same as the zero square rings studied by the earlier authors like Stanley. Every zero square ring of type-1 is a zero square ring of type-2.

Example 2.2. (i) Every null ring (that is $R^2 = 0$) is a zero square ring of type-2, but not of type-1.

(ii) Let $(G, +)$ be a group (not necessarily Abelian). Define a multiplicative operation on G by $a.b = 0$ for all a, $b \in G$, where 0 is the additive identity. Then $(G, +, .)$ is a null ring. So $(G, +, .)$ is a zero square ring of type-2, but not of type-1. We can conclude that every group can be made into a zero square ring of type-2.

(iii) Suppose that R is a non-zero Boolean ring. Then $x^2 = x$ for all x $\in R$. So R is a non-null ring and for any $x \neq 0$, we have $x^2 \neq 0$. Hence every non-zero Boolean ring can neither a zero square ring of type-1 nor a zero square ring of type-2.

(iv) Let S be a non null ring (that is, $S^2 \neq 0$). Write $R = S \times S \times S$. Define addition on R component wise. Define multiplication on R by $(x_1, y_1, z_1).(x_2, y_2, z_2) = (0, 0, x_1y_2 - x_2y_1)$. Stanley [3] mentioned that $R^2 \neq 0$ (that is R is not a null ring) and $a^2 = 0$ for all $a \in R$. Hence R is a zero square ring of type-1.

Theorem 2.3. *Suppose R is a zero square ring of type-2, and A is a module. Then*

(i) *$aR \neq A$ for all $0 \neq a \in A$.*
(ii) *If A is irreducible, then $AR = 0$.*

Proof. (i) Let R be a zero square ring, A a module, and $0 \neq a \in A$. Suppose $aR = A$. Then $a \in A = aR \Rightarrow a = ar$ for some $r \in R \Rightarrow a = ar = (ar)r = ar^2 = a0 = 0$, a contradiction.
(ii) Suppose $AR \neq 0$. Then there exist $s \in R$, $a \in A$ such that $as \neq 0 \Rightarrow 0 \neq as \in aR$. Since A is irreducible and $aR \neq 0$, we have that $aR = A$, a contradiction. Hence $AR = 0$. □

Corollary 2.4. *A primitive ring cannot be a zero square ring of type-2.*

Proof. Since R is primitive, it has a faithful irreducible module A. Let $0 \neq r \in R$. Since A is faithful we have $Ar \neq 0$. Now $0 \neq Ar \subseteq AR \Rightarrow 0 \neq AR$. By Theorem 2.3(ii), $AR = 0$, a contradiction. □

Corollary 2.5. *If R is a zero square ring of type-2, then $rR \neq R$ for all non zero $r \in R$.*

Proof. Since every ring is a module over itself, the result follows from Theorem 2.3(i). □

Corollary 2.6. *Let R be a zero square ring of type-2.*

(i) *If I is a non-zero right ideal of R, then I can not be a monogenic right ideal; and*
(ii) *If I is a non-zero left ideal of R, then I can not be a monogenic left ideal.*

Proof. (i) In a contrary way, suppose that I is a monogenic right ideal. Then there exist $0 \neq a \in I$ such that $aR = I$, a contradiction (to Theorem 2.3(i)) because every one sided ideal may be considered as a module over R.

The proof for (ii) is similar to (i). □

Corollary 2.7. *If R is a non-zero zero-square ring of type-2, then*

(i) *$Rr \neq R$ for all $r \in R$; and*
(ii) *$rR \neq R$ for all $r \in R$.*

Proof. The proof follows by taking R instead of I in Corollary 2.6. □

3. Zero Square Ideals

Definition 3.1. A proper ideal I of R is said to be a *zero square ideal of type-1* (respectively, *type-2*) if the quotient ring R/I is a zero square ring of type-1 (respectively of type-2).

Remark 3.2. (i) If R is a zero square ring of type-2, then every ideal I of R is a zero square ideal of type-2. The converse of this statement is not true. For this observe the following Example 3.3.

(ii) If R is a zero square ring of type-2, then every ideal of R is also a zero square ring of type-2.

Example 3.3. Consider Z_2, the ring of integers modulo 2. This Z_2 is not a zero square ring of type-2. Let G be a non-zero additive group and define $a.b = 0$ for all a, $b \in G$. Now $(G, +, .)$ is a zero square ring of type-2. Write $R = Z_2 \oplus G$, the direct sum of rings Z_2 and G. Now $I = Z_2$ is an ideal of R; for any $x + I \in R/I$, we get that $(x + I)^2 = 0 + I$; and hence I is a zero square ideal of type-2. Since $1 = 1 + 0 \in Z_2 + G = R$ and $1^2 = 1 \neq 0$, it follows that R is not a zero square ring of type-2.

Remark 3.4. Let I, J be two ideals of a ring R. If I, J are two zero square ideals of type-2, then $I \cap J$ is also a zero square ideal of type-2.

[*Verification.* Let $x \in R/(I \cap J)$. Now $x + I \in R/I \Rightarrow x^2 + I = 0 + I \Rightarrow x^2 \in I$. Similarly $x^2 \in J$ it follows that $x^2 \in I \cap J \Rightarrow x^2 + (I \cap J) = 0 + (I \cap J) \Rightarrow (x + (I \cap J))^2 = 0$ in $R/(I \cap J)$. Hence $R/(I \cap J)$ is a zero square ring of type-2. Therefore $I \cap J$ is a zero square ideal of type-2.]

Note 3.5. A class $\mathbb{B}$ of rings is said to be homomorphically closed if every homomorphic image of R is in $\mathbb{B}$ for all R in $\mathbb{B}$.

Theorem 3.6. *The class $\mathbb{B}$ of all zero square rings of type-2 is homomorphically closed.*

Proof. Let $R \in \mathbb{B}$. We know that every homomorphic image of R is isomorphic to R/I for some ideal I of R. Let I an ideal of R. Take $x + I \in R/I$. Now $(x + I)^2 = x^2 + I = 0 + I$ (since R is a zero square ring of type-2). So R/I is a zero square ring of type-2 and hence $R/I \in \mathbb{B}$. □

Remark 3.7. Suppose I is an ideal of R, I is a zero square ideal of type-2 and also a zero square ring of type-2, then $x^4 = 0$ for all $x \in R$.

[*Verification*: $x \in R \Rightarrow x + I \in R/I \Rightarrow (x + I)^2 = 0 + I$ (since I is a zero square ideal of type-2) $\Rightarrow x^2 \in I \Rightarrow (x^2)^2 = 0$ (since I is a zero square ring of type-2) $\Rightarrow x^4 = 0$].

Theorem 3.8. *Let R be a zero square ring of type-2 and I an ideal of R. Then the following two conditions are equivalent:*

(i) *$R^2 \nsubseteq I$; and*

(ii) *I is a zero square ideal of type-1.*

Proof. (i) $\Rightarrow$ (ii): By Remark 3.2, we get that I is a zero square ideal of type-2. Since $R^2 \nsubseteq I$ there exist x, $y \in R$ with $xy \notin I$ and so $(x + I)(y + I) \neq 0 + I$ in R/I. Therefore R/I is a zero square ring of type-1 and so I is a zero square ideal of type-1.

(ii) $\Rightarrow$ (i): Since R/I is a zero square ring of type-1, there exist two non-zero elements $c+I$ and $d+I$ in R/I whose product is non-zero in R/I. This means that $cd \notin I$ and so $R^2 \nsubseteq I$. □

Corollary 3.9. (i) *Let I and J be ideals of a zero square ring R of type-2 with $I \subseteq J$. If J is a zero square ideal of type-1, then I is also a zero square ideal of type-1.*

(ii) *Intersection of any collection of zero square ideals of type-1 is also a zero square ideal of type-1.*

Corollary 3.10. *Let $\aleph$ be the class of all zero square rings R of type-1 for which $R^2 \nsubseteq I$ for all non-zero ideals I of R. Then the class $\aleph$ is homomorphically closed.*

Proof. Let $R \in \aleph$ and $h : R \to R^1$ be an epimorphism. Then $R/I \cong R^1$, where $I = kerh$, an ideal of R.

Case (i): Suppose h is an isomorphism. Then $I = 0$. Since R is a zero square ring of type-1, there exists $x, y \in R$ such that $xy \neq 0$. So $R^2 \neq 0$ and $R^2 \nsubseteq I$.

Case (ii): Suppose h is not an isomorphism. Then $I \neq 0$. By the assumed condition $R^2 \nsubseteq I$. Now by Theorem 3.8, I is a zero square ideal of type-1 and hence $R^1 \cong R/I \in \aleph$. □

Corollary 3.11. *In a zero square ring R of type-2, (i) every semi-prime ideal S of R is a zero square ideal of type-1; and (ii) every prime ideal P of R is a zero square ideal of type-1.*

Proof. (i) Suppose S is not a zero square ideal of type-1. Then by Theorem 3.8 we get that $R^2 \subseteq S$. Since S is semi-prime ideal, we have that $S = R$, a contradiction.

(ii) follows because every prime ideal is a semi-prime ideal. □

Definition 3.12. A ring R is said to be a *strong zero square ring of type-1* if every ideal of R is a zero square ideal of type-1.

Remark 3.13. (i) If R is a strong zero square ring of type-1, then R is a zero square ring of type-1.

(ii) The converse of (i) is not true, in general. Observe the Example 3.14.

Example 3.14. Let S be a zero square ring of type-1. Let $(G, +)$ be a group. Define multiplication on G by $a.b = 0$ for all a, $b \in G$. Then $(G, +, .)$ is a ring. Write $R = S \oplus G$, the direct sum of rings S and G. It is clear that S is an ideal of R. Now we wish to show that R is a zero square ring of type-1, but the ideal S of R is not a zero square ideal of type-1. Since S (as a ring) is a zero square ring of type-1, there exist $x, y \in S$ such that $xy \neq 0$. Now x, y are also elements of R with $xy \neq 0$. It is clear that $a^2 = 0$ for all $a \in R$. This shows that R is a zero square ring of type-1. Let $u, v \in R$ with $u = s_1 + g_1$, $v = s_2 + g_2$, s_1, $s_2 \in S$, g_1, $g_2 \in G$. It is clear that $uv = (s_1 + g_1)(s_2 + g_2) = s_1s_2 + g_1g_2 = s_1s_2 + 0 = s_1s_2 \in S$. Thus $R^2 \subseteq S$. By Theorem 3.8, it follows that S is not a zero square ideal of type-1. Hence R is a zero square ring of type-1, but it is not a strong zero square ring of type-1.

We can restate the Corollary 3.10 as follows:

Corollary 3.15. *The class of all strong zero square rings of type-1, is homomorphically closed.*

Notation 3.16. Let R be a ring. Write $ZS1(R)$ = the intersection of all non-zero zero square ideals (of R) of type-1; and $ZS2(R)$ = the intersection of all non-zero zero square ideals (of R) of type-2. If there are no non-zero zero square ideals of type-1 (respectively, type-2) in R, then we define $ZS1(R) = R$ (respectively, $ZS2(R) = R$).

Remark 3.17. If R is a zero square ring of type-2, then we have the following:

(i) By Theorem 3.8, we get that if R is a zero square ring of type-2, then $ZS1(R) = \bigcap \{I \ / \ I \text{ is a non-zero ideal of } R \text{ with } R^2 \nsubseteq I\}$;

(ii) If $ZS2(R) = 0$ (respectively, $ZS1(R) = 0$), then it follows that R is a sub-direct product of the zero square rings R/I, where I runs over all non-zero zero square ideals of type-2 (respectively, type-1) in R. If $ZS2(R) \neq 0$ (respectively, $ZS1(R) \neq 0$), then $ZS2(R)$ (respectively, $ZS1(R)$) is the smallest non-zero zero square ideal of type-2 (respectively, type-1), among all non-zero zero square ideals of type-2 (respectively, type-1).

(iii) In Example 3.3, $R = Z_2 \oplus G$ is not a zero square ring of type-2. In this case $ZS2(R) = Z_2$. Note that $(0) \neq ZS2(R) \neq R$.

(iv) If R is a zero square ring of type-2 and R contains a zero square ideal I of type-1, then by Corollary 3.9, we get that $ZS1(R) \subseteq I$.

(v) If $R^2 = 0$, then R contains no zero square ideals of type-1 and so $ZS1(R) = R$.

Theorem 3.18. *If there exists a chain $R = I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq ? \supsetneq I_k = (0)$ of ideals of R such that I_{s+1} is a zero square ideal of type-2 in the ring I_s for $0 \leq s < k$, then R is a nil ideal of R. In particular, $x^{(2^k)} = 0$ for all $x \in R$.*

Proof. Let $x \in R = I_0$. Since I_1 is zero square ideal of type-2 in the ring I_0 and $x \in I_0$ we have that $(x + I_1)^2 = 0$ in I_0/I_1. So $x^2 \in I_1$. Since $x^2 \in I_1$ and I_2 is zero square ideal of type-2 in the ring I_1, it follows that $(x^2 + I_2)^2 = 0$ in I_1/I_2 and so $x^4 \in I_2$. If we continue this process, eventually, we get that $x^{(2^k)} \in (0)$. Thus $x^{(2^k)} = 0$ and this is true for all $x \in R$. Therefore R is a nil ideal of R. □

Corollary 3.19. *Let $I_1, \cdots, I_k$ be as in the above Theorem 3.18. For any ideal I of R, I and R/I are also nil.*

4. Zero Square Rings and Direct Products

If $R_1, R_2, \cdots, R_k$ are rings, then the ring $R_1 \times R_2 \times \cdots \times R_k$, the direct product of R_i, $1 \leq i \leq k$ is denoted by $\prod_{i=1}^{k} R_i$. For any ring R, let us write $R^k = \prod_k R$ for the direct product of k copies of R.

A straight forward verification provides the following Theorem.

Theorem 4.1. (i) *If $R_i, 1 \leq i \leq k$ are zero square rings of type-1, then $\prod_{i=1}^{k} R_i$ is also a zero square ring of type-1;*

(ii) *Each R_i, $1 \leq i \leq k$ are zero square ring of type-2 if and only if $\prod_{i=1}^{k} R_i$ is a zero square ring of type-2.*

Remark 4.2. The converse of the above Theorem 4.1(i) is not true, in general. For this let us observe the following example.

Example 4.3. Write $(R, +) = (Z_2, +)$, the additive group of integers modulo 2. Consider the zero product on R (that is $xy = 0$ for all x, $y \in R$). Then R is ring which is not a zero square ring of type-1. Let M be a zero square ring of type-1. Consider the ring $R \times M$ which is the direct product of R and M. Now $R \times M$ is a zero square ring of type-1, where as R is not a zero square ring of type-1.

Theorem 4.4. *Let R_i, $1 \leq i \leq k$ be rings. The direct product $\prod_{i=1}^{k} R_i$ is a zero square rings of type-1 if and only if there exists a non-empty subset I*

$\subseteq \{1, 2, \cdots, k\}$ *such that* R_i *is a zero square rings of type-1 for all* $i \in I$ *and* R_j *is a zero square ring of type-2 but not of type-1 for all* $j \in \{1, 2, \cdots, k\} \setminus I$.

Proof. Suppose that $\prod_{i=1}^{k} R_i$ is a zero square ring of type-1. Let $s \in \{1, 2, \cdots, k\}$ and $x_s \in R_s$. Consider the element $(0, \cdots, 0, x_s, 0, \cdots, 0) \in \prod_{i=1}^{k} R_i$, the s^{th} co-ordinate is x_s and zero else where. Now $0 = (0, \cdots, 0, x_s, 0, \cdots, 0)^2 = (0, \cdots, 0, x_s^2, 0, \cdots, 0)$ and $x_s^2 = 0$. Thus $a^2 = 0$ for all $a \in R_s$, and this is true for all $1 \leq s \leq k$. So each R_s is a zero square ring of type-2. Write $I = \{s/1 \leq s \leq k$ and there exist elements x, y in R_s such that $xy \neq 0\}$. Now it is clear that R_i, is a zero square ring of type-1 for all $i \in I$. Since $\prod_{i=1}^{k} R_i$ is a zero square ring of type-1, there exist at least two elements $(x_1, x_2, \cdots, x_k)$, $(y_1, y_2, \cdots, y_k)$ in $\prod_{i=1}^{k} R_i$ with $(x_1y_1, x_2y_2, \cdots, x_ky_k) \neq 0$. Thus there exist t $(1 \leq t \leq k)$ such that $x_ty_t \neq 0$. Now $t \in I$ and so $I \neq \phi$. It is clear that for all $j \in J = \{1, 2, \cdots, k\} - I$, we have that $xy = 0$ for all x, $y \in R_j$. Hence R_j is not a zero square ring of type-1, for all $j \in J$.

Converse: Since I is non-empty, there exists $i \in I$ such that R_i is a zero square ring of type-1. So there exist x_i, $y_i \in R_i$ with $x_iy_i \neq 0$. Now $(0, \cdots, x_i, \cdots, 0)$, $(0, \cdots, y_i, \cdots, 0) \in \prod_{i=1}^{k} R_i$ and the product of these elements is non-zero. By Theorem 4.1, $\prod_{i=1}^{k} R_i$ is a zero square ring of type-1. Hence $\prod_{i=1}^{k} R_i$ is a zero square ring of type-1. □

Corollary 4.5. *For any positive integer* k*, we have that* R *is a zero square ring of type-2 (respectively, type-1) if and only if* R^k *is a zero square ring of type-2 (respectively, type-1).*

5. Zero Square Dimension

Definition 5.1. Let R has FDI. We define the *zero square dimension* of R (denoted by $ZSd(R)$) as follows:

$$ZSd(R) = \{s \,|\, \text{there exist uniform ideals } U_i,\ 1 \leq i \leq s \text{ in } R \text{ such that the sum } U_1 + U_2 + \cdots + U_s \text{ is direct and each } U_i \text{ is a zero square ring of type-2}\}.$$

Lemma 5.2. (i) *If* R *has* FDI*, and* R *is a zero square ring of type-2, then* $ZSd(R) = \dim R$.

(ii) *If* R_i*,* $1 \leq i \leq n$ *are rings with* FDI *and each* R_i *is a zero square ring of type-2, then* $ZSd(\prod_{i=1}^{n} R_i) = \sum_{i=1}^{n} ZSd(R_i)$.

Proof. (i) Suppose $k = \dim R$. Since $k = \dim R$, there exist uniform ideals $U_1, U_2, \cdots, U_k$ in R such that $U_1 \oplus U_2 \oplus \cdots \oplus U_k \leq_e R$. Since R is a zero square ring of type-2, by Remark 3.2(ii), each U_i is also zero square ring of type-2. By Definition 5.1, $ZSd(R) = k$. Hence $ZSd(R) = \dim R$.

(ii) By Theorem 4.1(ii), $\prod_{i=1}^{n} R_i$ is also a zero square ring of type-2. Now $ZSd(\prod_{i=1}^{n} R_i) = \dim(\prod_{i=1}^{n} R_i)$ (by (i)) $= \sum_{i=1}^{n} \dim(R_i)$ (by Theorem 1.3) $= \sum_{i=1}^{n} ZSd(R_i)$ (by (i)). □

Lemma 5.3. *Suppose R has FDI and satisfies the condition $\langle xy\rangle = \langle x\rangle\langle y\rangle$ for all x, $y \in R$ with $xy \neq 0$. If R is zero square ring of type-1, then there exists a uniform ideal U in R such that U itself a zero square ring of type-1.*

Proof. Since R has FDI, by Theorem 1.1, $\dim R = k$, and there exist uniform ideals $I_1, I_2, \cdots, I_k$ such that $I_1 \oplus I_2 \oplus \cdots \oplus I_k \leq_e R$. Write $E = I_1 \oplus I_2 \oplus \cdots \oplus I_k$. Since R is a zero square ring of type-1, there exist $x, y \in R$ with $xy \neq 0$. Since $0 \neq xy \in \langle xy\rangle$, and E is essential ideal in R, it follows that $\langle xy\rangle \cap E \neq 0$. Now $\langle x\rangle\langle y\rangle \cap E \neq 0 \Rightarrow$ there exists $x^1 \in \langle x\rangle$, $y^1 \in \langle y\rangle$ such that $0 \neq x^1y^1 \in E$. So $E = I_1 \oplus I_2 \oplus \cdots \oplus I_k$ is a zero square ring of type-1. By Theorem 4.4, there exists $t \in \{1, 2, \cdots, k\}$ such that I_t is a zero square ring of type-1. □

Definition 5.4. Let R has FDI and $\dim R = k$. If R contains no uniform ideal which is a zero square ring of type-1, then we define the zero square-1 dimension of R ($ZS1d(R)$, in short) is equal to zero. We write $ZS1d(R) = 0$. If R contains a uniform ideal which is a zero square ring of type-1, then we define the zero square-1 dimension of R as follows: $ZS1d(R) = \max\{t/U_1, U_2, \cdots, U_t, U_{t+1}, \cdots, U_k$ are uniform ideals of R, whose sum is direct and essential in R (that is, $U_1 \oplus U_2 \oplus \cdots \oplus U_k \leq_e R$), $U_1, U_2, \cdots, U_t$ are zero square rings of type-1, $U_{t+1}, \cdots, U_k$ are not zero square rings of type-1$\}$.

Note 5.5. (i) If R has FDI, R is a zero square ring of type-1 and satisfies the condition $\langle xy\rangle = \langle x\rangle\langle y\rangle$ for all $x, y \in R$ with $xy \neq 0$. By Lemma 5.3, there exist uniform ideals $U_1, U_2, \cdots, U_k$ in R whose sum is direct and essential in R. Also at least one of the U_i's is a zero square ring of type-1. Thus, in this case, $ZS1d(R) \geq 1$.

(ii) If R is a zero square ring of type-2 but not of type-1, then there exist no uniform ideal in R which is a zero square ring of type-1. So in this case $ZS1d(R) = 0$.

Theorem 5.6. *If R_1, R_2 are rings with FDI and $R = R_1 \oplus R_2$, the direct sum of rings, then $ZS1d(R_1 \oplus R_2) \geq ZS1d(R_1) + ZS1d(R_2)$.*

Proof. Suppose $ZS1d(R_1) = n$ and $ZS1d(R_2) = m$. Then there exists uniform ideals $I_1, I_2, \cdots, I_k$ of R_1 such that $I_1 \oplus I_2 \oplus \cdots \oplus I_k \leq_e R_1$, I_i, $1 \leq i \leq n$ are zero square rings of type-1. Similarly there exists uniform ideals $J_1, J_2, \cdots, J_s$ of R_2 such that $J_1 \oplus J_2 \oplus \cdots \oplus J_s \leq_e R_2$, J_i, $1 \leq i \leq m$ are zero square rings of type-1. Since $R = R_1 \oplus R_2$, we have that the ideals of R_1 and the ideals of R_2 are also ideals of R. Now $I_1 \oplus I_2 \oplus \cdots \oplus I_n \oplus J_1 \oplus J_2 \oplus \cdots \oplus J_m \oplus I_{n+1} \oplus I_{n+2} \oplus \cdots \oplus I_k \oplus J_{m+1} \oplus \cdots \oplus J_s \leq_e R$ (by Theorem 1.2); $I_1 \oplus I_2 \oplus \cdots \oplus I_n \oplus J_1 \oplus J_2 \oplus \cdots \oplus J_m$ is a sum of $(n+m)$ uniform ideals which are zero square rings of type-1. So by Definition 5.4, it follows that $ZS1d(R_1 \oplus R_2) \geq n + m = ZS1d(R_1) + ZS1d(R_2)$. □

Corollary 5.7. *If R_i, $1 \leq i \leq k$ are rings with FDI, then $ZS1d(R_1 \times R_2 \times \cdots \times R_k) \geq \sum_{i=1}^{k} ZS1d(R_i)$.*

Definition 5.8. Let R be a ring with FDI. We define $ZS2d(R)$, the *zero square-2 dimension* of R as follows:

$$ZS2d(R) = \min\{t / U_1, U_2, \cdots, U_k \text{ are uniform ideals of } R \text{ such that } U_1 \oplus U_2 \oplus \cdots \oplus U_k \leq_e R,\ U_1, U_2, \cdots, U_t \text{ are zero square rings of type-2 but not of type-1}\}.$$

Note 5.9. Suppose R has FDI, $\dim R = k$ and R is a zero square ring of type-2 but not of type-1. Then by Note 5.5 (ii), $ZS1d(R) = 0$. Since every representation $E = U_1 \oplus U_2 \oplus \cdots \oplus U_k$ that is equal to a direct sum of uniform ideals with $E \leq_e R$, contains exactly k uniform ideals, we have that $ZS2d(R) = k$. So in this case, $ZS1d(R) = 0$ and $ZS2d(R) = \dim R$.

Theorem 5.10. (i) *If R has FDI and R is a zero square ring of type-1, then* $\dim(R) = ZSd(R) = ZS1d(R) + ZS2d(R)$.

(ii) *If R_i, $1 \leq i \leq k$ are rings with FDI, and also zero square rings of type-1, then* $\dim(R_1 \times R_2 \times \cdots \times R_k) = ZSd(R_1 \times R_2 \times \cdots \times R_k) \geq \sum_{i=1}^{k} ZS1d(R_i) + \sum_{i=1}^{k} ZS2d(R_i)$.

Proof. (i) By Lemma 5.2(i), $\dim(R) = ZSd(R)$. Suppose $\dim(R) = k$ and $ZS1d(R) = n$. Then there exist uniform ideals $I_1, I_2, \cdots, I_k$ in R such that $I_1 \oplus I_2 \oplus \cdots \oplus I_k \leq_e R$ and I_i, $1 \leq i \leq n$ are zero square rings of type-1, n is maximum among such n. Also $I_{n+1}, \cdots, I_k$ are uniform ideals of R ($k - n$ in number) which are zero square-rings of type-2 (but not of type-1). So $ZS2d(R) \leq k - n$. Suppose $m = ZS2d(R)$. Then there exist uniform ideals $U_1, U_2, \cdots, U_k$ in R such that $U_1 \oplus U_2 \oplus \cdots \oplus U_k \leq_e R$, U_i, $1 \leq i \leq m$ are zero square-rings of type-2 (but not type-1) and m is the minimum among these numbers. This means that the remaining $k - m$ uniform ideals $U_{m+1}, \cdots, U_k$ are zero square rings of type-1 (we get this because of the hypothesis that R is a zero square ring of type-2). By the Definition 5.4, we conclude that $k - m \leq n$, which imply that $m \geq k - n$. Hence $ZS2d(R) = m = k - n = \dim R$ - $ZS1d(R)$. Finally we got that $\dim R = ZSd(R) = ZS1d(R) + ZS2d(R)$.

Proof for (ii) follows by using (i), Theorem 5.6 and mathematical induction. □

Corollary 5.11. (i) *If R_1, R_2 are zero square rings of type-2 with FDI, then* $ZS2d(R_1 \oplus R_2) \leq ZS2d(R_1) + ZS2d(R_2)$

(ii) *If R_i, $1 \leq i \leq k$ are zero square rings with FDI, then* $ZS2d(R_1 \times R_2 \times \cdots \times R_k) \leq \sum_{i=1}^{k} ZS2d(R_i)$.

Proof. (i) $ZS1d(R_1 \oplus R_2) + ZS2d(R_1 \oplus R_2) = ZSd(R_1 \oplus R_2)$ (by Theorem 5.10) $= ZSd(R_1) + ZSd(R_2)$ (by Lemma 5.2(ii)) $= ZS1d(R_1) + ZS2d(R_1) + ZS1d(R_2) + ZS2d(R_2)$ (by Theorem 5.10) $\leq ZS1d(R_1 \oplus R_2) + ZS2d(R_1) + ZS2d(R_2)$ (by Theorem 5.6). Therefore $ZS2d(R_1 \oplus R_2) \leq ZS2d(R_1) + ZS2d(R_2)$.

Proof for (ii) follows by using (i) and mathematical induction. □

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